DTFT (3A)

- Discrete Time Fourier Transform

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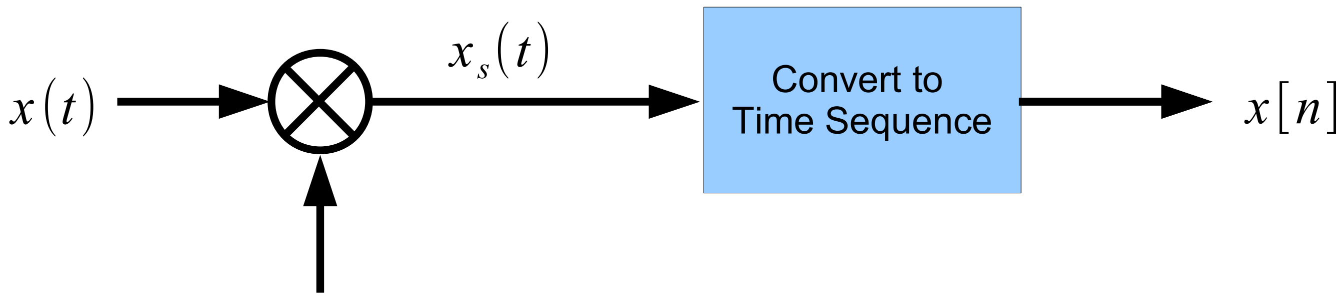
Sampling and Reconstruction

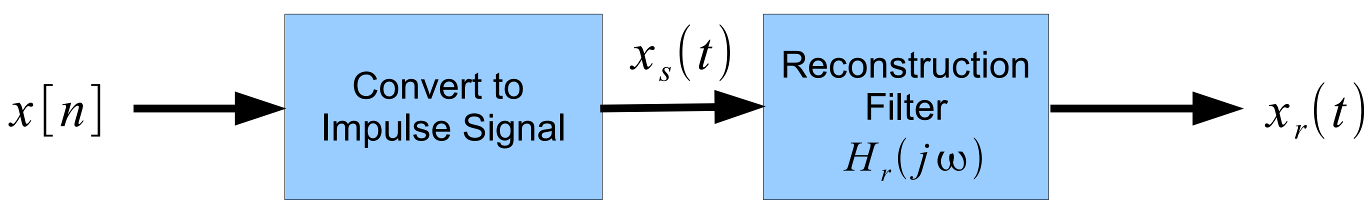
Ideal Sampling



$$p(t) = \sum_{n=-\infty}^{+\infty} \delta(t - nT_s)$$

T_s **Sampling Period**

Ideal Reconstruction



CTFS of Impulse Train (1)

$$p(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT_s)$$

Fourier Series →

$$p(t) = \sum_{k=-\infty}^{+\infty} \frac{1}{T_s} e^{+jk\omega_s t}$$

Fourier Series Expansion

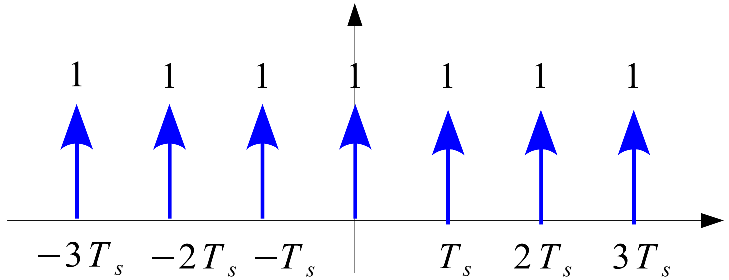
$$p(t) = \sum_{k=-\infty}^{+\infty} a_k \, e^{+jk\omega_s t}$$

Fourier Series Coefficients

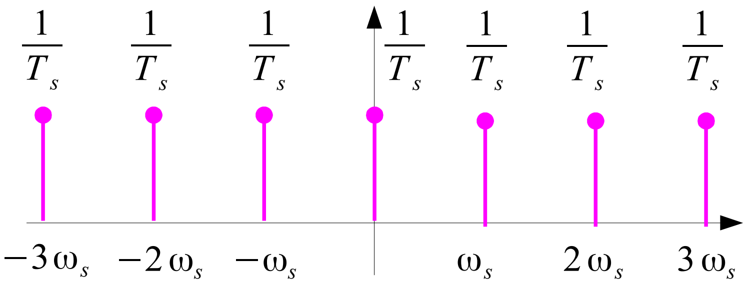
$$a_k = \frac{1}{T_s} \int_{-T_s/2}^{+T_s/2} \delta(t)\, e^{-jk\omega_s t}\, dt$$

$$= \frac{1}{T_s} \int_{-T_s/2}^{+T_s/2} \delta(t)\, e^{-jk\omega_s 0}\, dt$$

$$= \frac{1}{T_s} \int_{-T_s/2}^{+T_s/2} \delta(t)\, dt = \frac{1}{T_s}$$



$$\omega_s = \frac{2\pi}{T_s}$$



CTFS of Impulse Train (2)

$$p(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT_s)$$

Fourier Series

$$p(t) = \sum_{k=-\infty}^{+\infty} \frac{1}{T_s} e^{+jk\omega_s t}$$

$$p(t) = \sum_{k=-\infty}^{+\infty} a_k e^{+jk\omega_s t}$$

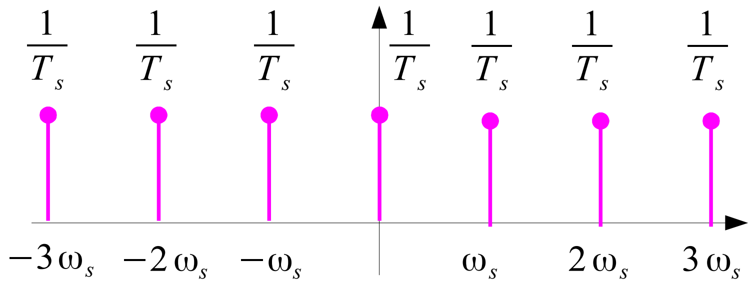
Fourier Transform

$$P(j\omega) = \sum_{k=-\infty}^{+\infty} 2\pi a_k \delta(\omega - k\omega_s)$$

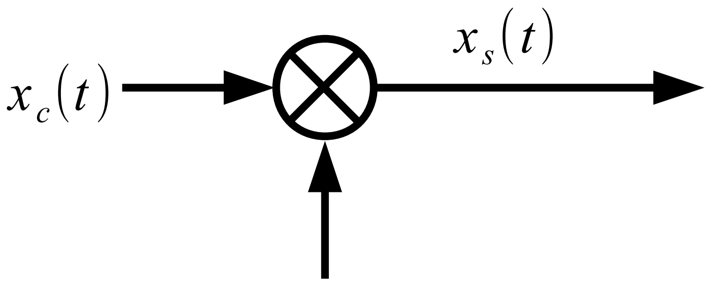
Fourier Transform of impulse train

$$P(j\omega) = \sum_{k=-\infty}^{+\infty} \frac{2\pi}{T_s} \delta(\omega - k\omega_s)$$

$$\omega_s = \frac{2\pi}{T_s}$$



Ideal Sampling



$$p(t) = \sum_{n=-\infty}^{+\infty} \delta(t - nT_s)$$

$$x_s(t) = x_c(t) \sum_{n=-\infty}^{+\infty} \delta(t - nT_s)$$

$$= \sum_{n=-\infty}^{+\infty} x_c(nT_s)\,\delta(t - nT_s)$$

$$x_s(t) = x_c(t) \sum_{k=-\infty}^{+\infty} \frac{1}{T_s} e^{jk\omega_s t}$$

$$\omega_s = \frac{2\pi}{T_s}$$

CTFT Frequency Shift Property

Continuous Time Fourier Transform

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} X(j\omega)\, e^{+j\omega t}\, d\omega \quad \Longleftrightarrow \quad X(j\omega) = \int_{-\infty}^{+\infty} x(t)\, e^{-j\omega t}\, dt$$

Frequency Shift Property

$$x_c(t) \quad \Longleftrightarrow \quad X_c(j\omega)$$

$$x_c(t)\, e^{jk\omega_s t} \quad \Longleftrightarrow \quad X_c(j(\omega - k\omega_s))$$

$$x_s(t) = x_c(t) \sum_{k=-\infty}^{+\infty} \frac{1}{T_s} e^{jk\omega_s t} \quad \Longleftrightarrow \quad X_s(j\omega) = \frac{1}{T_s} \sum_{k=-\infty}^{+\infty} X_c(j(\omega - k\omega_s))$$

$$\omega_s = \frac{2\pi}{T_s} \qquad\qquad \omega_s = \frac{2\pi}{T_s}$$

CTFT Delay Property

Continuous Time Fourier Transform

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} X(j\omega)\, e^{+j\omega t}\, d\omega \quad \Longleftrightarrow \quad X(j\omega) = \int_{-\infty}^{+\infty} x(t)\, e^{-j\omega t}\, dt$$

Fourier Transform of an Impulse

$$\delta(t - t_d) \quad \Longleftrightarrow \quad e^{j\omega t_d}$$

$$\delta(t - nT_s) \quad \Longleftrightarrow \quad e^{-j\omega n T_s}$$

$$x_s(t) = x_c(t) \sum_{n=-\infty}^{+\infty} \delta(t - nT_s)$$

$$X_s(j\omega) = \sum_{n=-\infty}^{+\infty} x_c(nT_s)\, e^{-j\omega n T_s}$$

$$x_s(t) = \sum_{n=-\infty}^{+\infty} x_c(nT_s)\, \delta(t - nT_s) \quad \Longleftrightarrow \quad X_s(j\omega) = \sum_{n=-\infty}^{+\infty} x[n]\, e^{-j\omega n T_s}$$

CTFT of a Sampled Signal

Continuous Time Fourier Transform

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} X(j\omega)\, e^{+j\omega t}\, d\omega \quad \Longleftrightarrow \quad X(j\omega) = \int_{-\infty}^{+\infty} x(t)\, e^{-j\omega t}\, dt$$

$$x_s(t) = x_c(t) \sum_{n=-\infty}^{+\infty} \delta(t - nT_s)$$

$$x_s(t) = \sum_{n=-\infty}^{+\infty} x_c(nT_s)\,\delta(t - nT_s)$$

CTFT $\Longleftrightarrow$

$$X_s(j\omega) = \sum_{n=-\infty}^{+\infty} x_c(nT_s)\, e^{-j\omega n T_s}$$

$$X_s(j\omega) = \sum_{n=-\infty}^{+\infty} x[n]\, e^{-j\omega n T_s}$$

CTFS ||

DTFT ||

$$x_s(t) = x_c(t) \sum_{k=-\infty}^{+\infty} \frac{1}{T_s} e^{jk\omega_s t}$$

$$\omega_s = \frac{2\pi}{T_s}$$

CTFT $\Longleftrightarrow$

$$X_s(j\omega) = \frac{1}{T_s}\sum_{k=-\infty}^{+\infty} X_c(j(\omega - k\omega_s))$$

$$\omega_s = \frac{2\pi}{T_s}$$

z-Transform of a Sampled Signal

$$X_s(j\omega) = \sum_{n=-\infty}^{+\infty} x_c(nT_s)\, e^{-j\omega n T_s}$$

$$X_s(j\omega) = \sum_{n=-\infty}^{+\infty} x[n]\, e^{-j\omega n T_s}$$

$\Leftrightarrow$ =

$$X_s(j\omega) = \frac{1}{T_s}\sum_{k=-\infty}^{+\infty} x_c(j(\omega - k\omega_s))$$

$$\omega_s = \frac{2\pi}{T_s}$$

CTFT of a sampled signal

$$\sum_{n=-\infty}^{+\infty} x[n]\, e^{-j\omega n T_s} = \frac{1}{T_s}\sum_{k=-\infty}^{+\infty} X_c(j(\omega - k\omega_s)) = \frac{1}{T_s}\sum_{k=-\infty}^{+\infty} X_c(j(\omega - \frac{2\pi k}{T_s}))$$

Z-Transform of a sampled signal

$$X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n} \qquad x[n] = x_c(nT_s)$$

$$X(z)\Big|_{z = e^{j\omega T_s}} = X(e^{j\omega T_s})$$

evaluated at $z = e^{j\omega T_s}$

z-Transform and Normalized Frequency

$$X_s(j\omega) = \sum_{n=-\infty}^{+\infty} x_c(nT_s)\, e^{-j\omega n T_s}$$

$$X_s(j\omega) = \sum_{n=-\infty}^{+\infty} x[n]\, e^{-j\omega n T_s}$$

=

$$X_s(j\omega) = \frac{1}{T_s} \sum_{k=-\infty}^{+\infty} x_c(j(\omega - k\omega_s))$$

$$\omega_s = \frac{2\pi}{T_s}$$

$$X(z)\Big|_{z=e^{j\omega T_s}} = X(e^{j\omega T_s}) = \sum_{n=-\infty}^{+\infty} x[n]\, e^{-j\omega n T_s}$$

z-Transform

$$\hat{\omega} = \omega T_s$$

Normalized Frequency

$$X(z)\Big|_{z=e^{j\hat{\omega}}} \quad X(e^{j\hat{\omega}}) = \sum_{n=-\infty}^{+\infty} x[n]\, e^{-j\hat{\omega} n}$$

Discrete Time Fourier Transform

DTFT and CTFT

$$X_s(j\omega) = \sum_{n=-\infty}^{+\infty} x_c(nT_s)\, e^{-j\omega n T_s}$$

$$X_s(j\omega) = \sum_{n=-\infty}^{+\infty} x[n]\, e^{-j\omega n T_s}$$

$=$

$$X_s(j\omega) = \frac{1}{T_s}\sum_{k=-\infty}^{+\infty} x_c(j(\omega - k\omega_s))$$

$$\omega_s = \frac{2\pi}{T_s}$$

$$X(e^{j\hat{\omega}}) = \sum_{n=-\infty}^{+\infty} x[n]\, e^{-j\hat{\omega} n}$$

DTFT of a sampled signal

$$X(e^{j\hat{\omega}})\Big|_{\hat{\omega} = \omega T_s} = X(e^{j\omega T_s}) = \frac{1}{T_s}\sum_{k=-\infty}^{+\infty} X_c(j(\omega - k\omega_s))$$

$$= \frac{1}{T_s}\sum_{k=-\infty}^{+\infty} X_c\left(j\left(\omega - \frac{2\pi k}{T_s}\right)\right)$$

CTFT of a sampled signal

DTFT and CTFT

Continuous Time Fourier Transform CTFT

$$X(j\omega) = \int_{-\infty}^{+\infty} x(t)\, e^{-j\omega t}\, dt \quad \Longleftrightarrow \quad x(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} X(j\omega)\, e^{+j\omega t}\, d\omega$$

Discrete Time Fourier Transform DTFT

$$X(e^{j\hat{\omega}}) = \sum_{n=-\infty}^{+\infty} x[n]\, e^{-j\hat{\omega} n} \quad \Longleftrightarrow \quad x[n] = \frac{1}{2\pi}\int_{-\pi}^{+\pi} X(e^{j\hat{\omega}})\, e^{+j\hat{\omega} n}$$

$$X_s(j\omega) = \sum_{n=-\infty}^{+\infty} x_c(nT_s)\, e^{-j\omega n T_s}$$

$$= \sum_{n=-\infty}^{+\infty} x[n]\, e^{-j\omega n T_s}$$

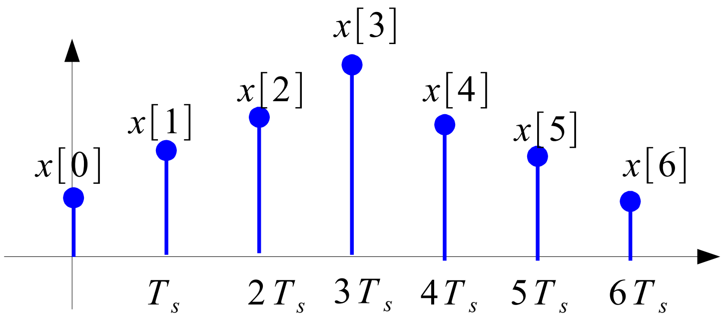
$=$

$$X_s(j\omega) = \frac{1}{T_s}\sum_{k=-\infty}^{+\infty} X_c(j(\omega - k\omega_s))$$

$$\omega_s = \frac{2\pi}{T_s}$$

DTFT and Periodic Frequency

A general formula for the CTFT
of any <u>periodic</u> function
for which a CTFS exists



Period

$$T_s \Rightarrow \omega_s = \frac{2\pi}{T_s}$$

Dual of Fourier Series Expansion*

$$X_s(j\omega) = \sum_{n=-\infty}^{+\infty} x[n]\, e^{+jnT_s\omega}$$

Repetition of Fourier Transform

$$X_s(j\omega) = \frac{1}{T_s} \sum_{k=-\infty}^{\infty} X_c(j(\omega - k\omega_s))$$

Dual of Fourier Series Coefficients*

$$x[n] = \frac{1}{2\pi} \int_{-\pi}^{+\pi} X(e^{j\hat{\omega}})\, e^{+jk\hat{\omega}_s n}\, d\hat{\omega}$$

Sampling in Time ← **Periodic Frequency**

Dual CTFS

References

[1] http://en.wikipedia.org/
[2] J.H. McClellan, et al., Signal Processing First, Pearson Prentice Hall, 2003