Complex Integration (2A)

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Contour Integrals

$f(z)$ defined at points of a smooth curve C

a smooth curve C is defined by

$$x = x(t) \qquad a \le t \le b$$
$$y = y(t)$$

The contour integral of **f** along C

$$\int_C f(z)\,dz = \int_C (u + iv)(dx + i\,dy) = \int_C u\,dx - v\,dy + i\int_C v\,dx + u\,dy$$

$$= \int_a^b [u\,x'(t) - v\,y'(t)]\,dt + i\int_C [v\,x'(t) + u\,y'(t)]\,dt$$

$$= \int_a^b (u + iv)(x'(t) + i\,y'(t))\,dt$$

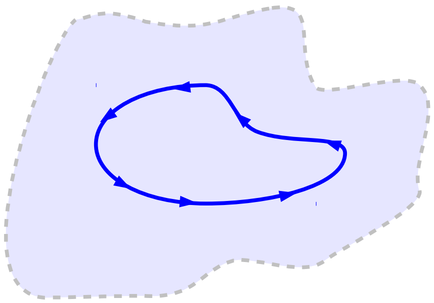
$$\int_C f(z)\,dz = \int_a^b f(z(t))\,z'(t)\,dt$$

$$z(t) = x(t) + i\,y(t)$$
$$z'(t) = x'(t) + i\,y'(t)$$
$$a \le t \le b$$

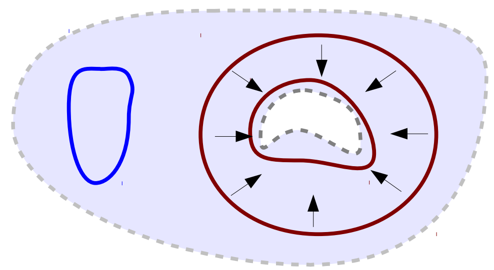
Connected Region

Connected Domains

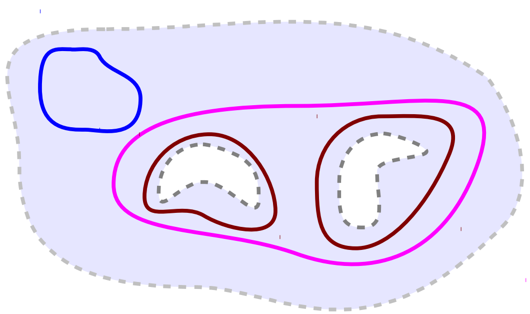
Simply Connected



Doubly Connected

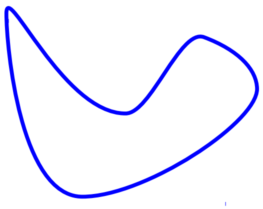


Triply Connected

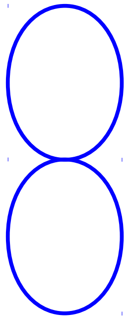


Closed Paths

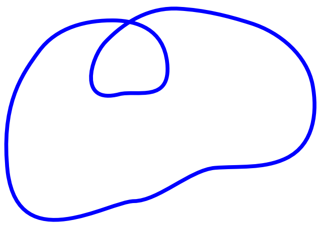
simple closed path



not simple closed path



not simple closed path



Cauchy's Theorem

$f(z)$: **analytic** in a **simply connected domain** D

$f'(z)$: **continuous** in a **simply connected domain** D

⇒ for every **simple closed contour** C in D

$$\oint_C f(z)\,dz = 0$$

$$\int_C f(z)\,dz = \int_C (u+iv)(dx+i\,dy) = \int_C u\,dx - v\,dy + i\int_C v\,dx + u\,dy$$

$$= \iint_D \left(-\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) dA + i\iint_D \left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right) dA = 0$$

$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \qquad \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$$

Cauchy-Goursat Theorem

$f(z)$: **analytic** in a **simply connected domain** D

⇒ for every **simple closed contour** C in D

$$\oint_C f(z)\,dz = 0$$

$f'(z)$: ~~**continuous**~~ in a simply connected domain D

simple closed curve

a continuously turning tangent

except possibly at a finite number of points

allow a finite number of corners (not smooth)

Cauchy's Integral Formula (1)

$f(z)$: **analytic** on and inside simple close curve C

$$f(a) = \frac{1}{2\pi i}\oint \frac{f(z)}{z-a}dz$$

the value of $f(z)$ at a point $z = a$ inside C

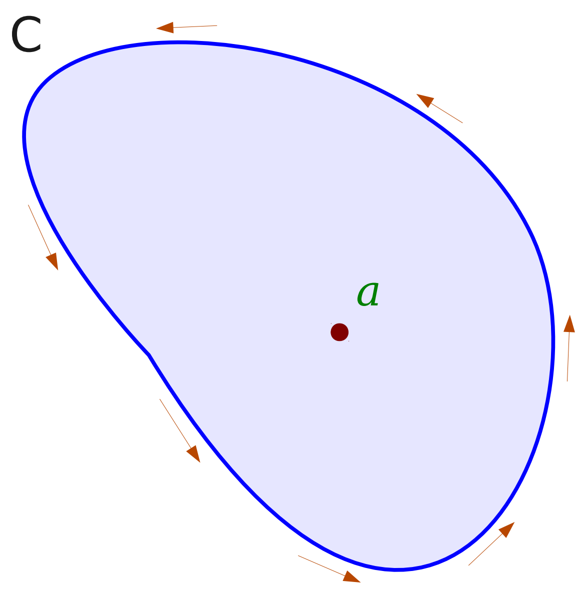
$$f(z) = \frac{1}{2\pi i}\oint \frac{f(w)}{w-z}dw$$

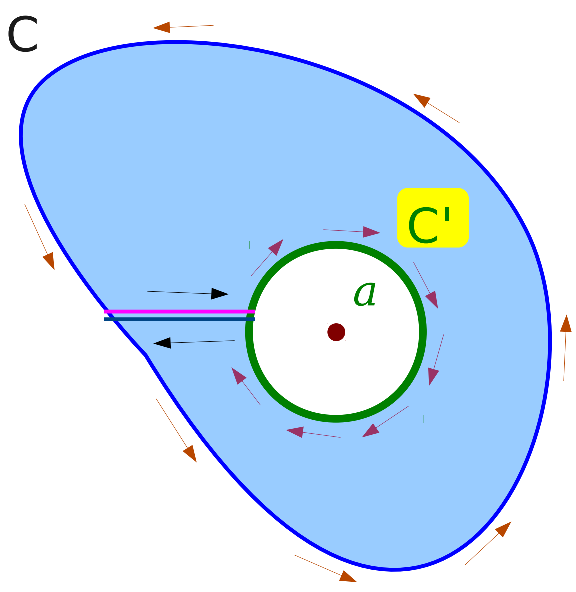
Cauchy's Integral Formula (2)

$f(z)$: **analytic** on and inside simple close curve C

$$f(a) = \frac{1}{2\pi i} \oint \frac{f(z)}{z-a} dz$$

the value of $f(z)$ at a point $z = a$ inside C





$$\oint_C f(z)\, dz = 0$$

$$\oint_{ccw\ C} \frac{f(z)\, dz}{z-a} + \oint_{cw\ C'} \frac{f(z)\, dz}{z-a} = 0$$

Cauchy's Integral Formula (3)

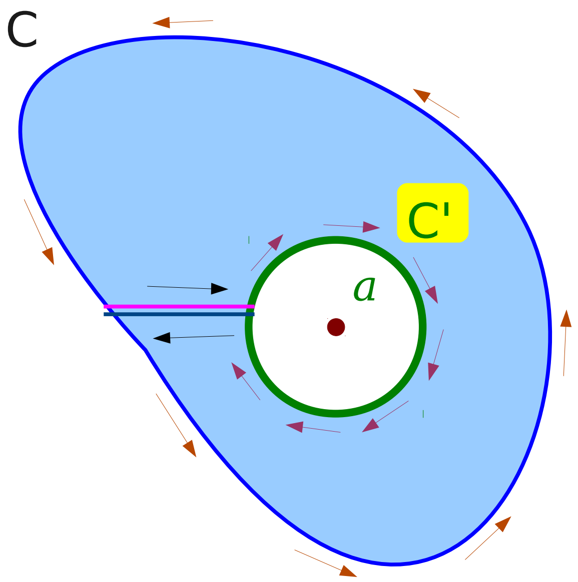
$f(z)$: **analytic** on and inside simple close curve C

$$f(a) = \frac{1}{2\pi i} \oint \frac{f(z)}{z-a} dz$$

the value of $f(z)$ at a point $z = a$ inside C

$$\oint_{ccw\ C} \frac{f(z)\,dz}{z-a} + \oint_{cw\ C'} \frac{f(z)\,dz}{z-a} = 0$$

$$\oint_{ccw\ C} \frac{f(z)\,dz}{z-a} = \oint_{ccw\ C'} \frac{f(z)\,dz}{z-a}$$

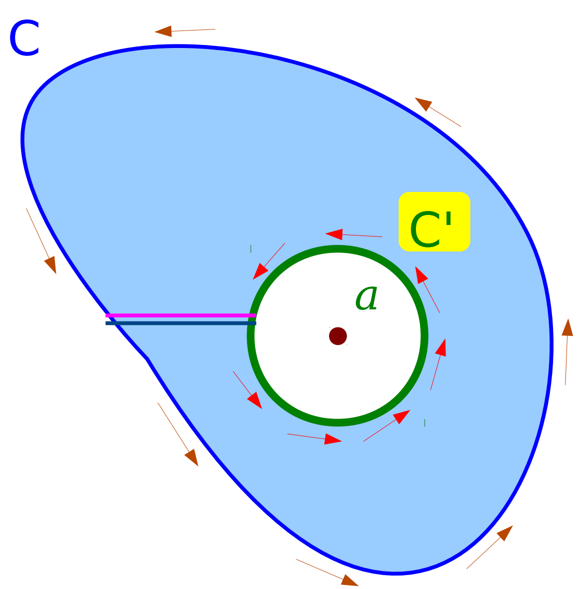


$$\oint_{ccw\ C} \frac{f(z)\,dz}{z-a} = \oint_{ccw\ C'} \frac{f(z)\,dz}{z-a}$$

$$= 2\pi i\ f(a)$$

along C' $\quad z - a = \rho e^{i\theta}$

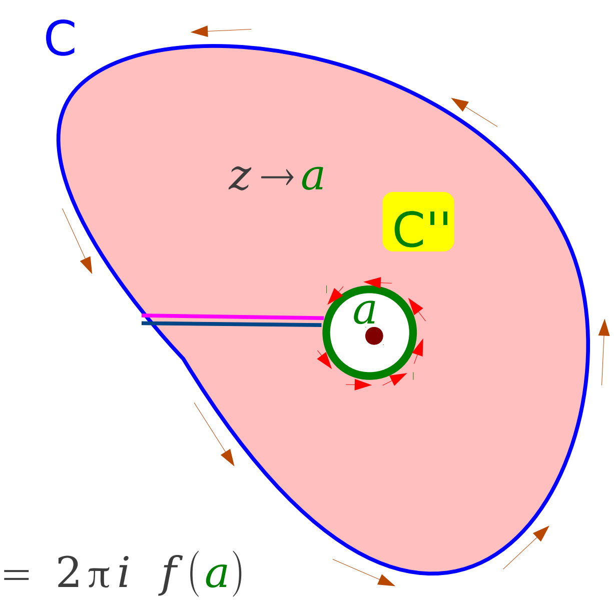
as $\quad z \rightarrow a$ ➡ $\rho \rightarrow 0$



$$z = a - \rho e^{i\theta}$$

$$dz = i\rho e^{i\theta} d\theta$$

$$\frac{dz}{z-a} = \frac{i\rho e^{i\theta} d\theta}{\rho e^{i\theta}}$$



$$\oint_{ccw\ C} \frac{f(z)\,dz}{z-a} = \int_0^{2\pi} f(z)\,i\,d\theta = 2\pi i\ f(a)$$

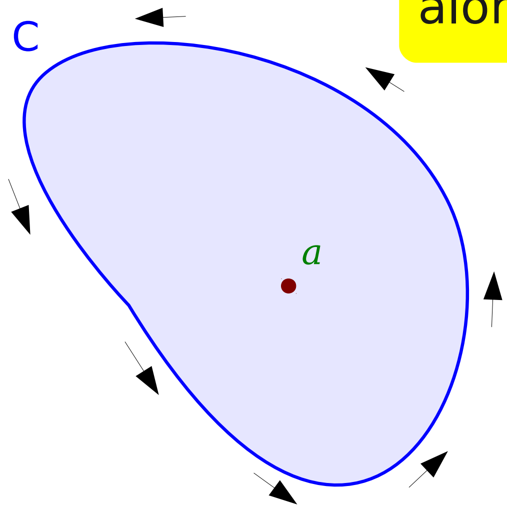
Cauchy's Integral Formula (5)

$$\frac{dz}{(z-a)^2} = \frac{i\rho e^{i\theta} d\theta}{(\rho e^{i\theta})^2}$$

$$\oint_{ccw\ C} \frac{f(z)\,dz}{(z-a)^2} = \int_0^{2\pi} \frac{f(z)i}{\rho e^{i\theta}} d\theta$$

$$= \int_0^{2\pi} \frac{f(z)}{\rho} i e^{-i\theta} d\theta = \left[-\frac{f(z)}{\rho} e^{-i\theta}\right]_0^{2\pi}$$

$$= -\frac{f(z)}{\rho}(e^{-i2\pi} - e^{-i0}) = 0$$



along C' $z - a = \rho e^{i\theta}$

$$z = a - \rho e^{i\theta}$$

$$dz = i\rho e^{i\theta} d\theta$$

$$dz = i\rho e^{i\theta} d\theta$$

$$\oint_{ccw\ C} f(z)\,dz = \int_0^{2\pi} f(z) i\rho e^{i\theta} d\theta$$

$$= \left[f(z)\rho e^{i\theta}\right]_0^{2\pi}$$

$$= f(z)\rho(e^{-i2\pi} - e^{-i0}) = 0$$

$$(z-a)\,dz = \rho e^{i\theta} i\rho e^{i\theta} d\theta$$

$$\oint_{ccw\ C} (z-a) f(z)\,dz = \int_0^{2\pi} f(z) i(\rho e^{i\theta})^2 d\theta$$

$$= \int_0^{2\pi} f(z)\rho^2 i e^{i2\theta} d\theta = \left[f(z)\frac{\rho}{2} e^{i2\theta}\right]_0^{2\pi}$$

$$= f(z)\frac{\rho}{2}(e^{-i4\pi} - e^{-i0}) = 0$$

References

[1] http://en.wikipedia.org/
[2] http://planetmath.org/
[3] M.L. Boas, “Mathematical Methods in the Physical Sciences”