Row Reduction

Young Won Lim
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Please send corrections (or suggestions) to youngwlim@hotmail.com.

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Linear Equations

$$
\begin{aligned}
a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\
a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\
\vdots \qquad\quad \vdots \qquad\qquad\qquad \vdots \qquad &\quad\ \vdots \\
a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m
\end{aligned}
$$

$$
\begin{pmatrix}
a_{11} & a_{12} & \cdots & a_{1n} \\
a_{21} & a_{22} & \cdots & a_{2n} \\
\vdots & \vdots & & \vdots \\
a_{m1} & a_{m2} & \cdots & a_{mn}
\end{pmatrix}
\begin{pmatrix}
x_1 \\ x_2 \\ \vdots \\ x_n
\end{pmatrix}
\begin{matrix} = \\ = \\ \\ = \end{matrix}
\begin{pmatrix}
b_1 \\ b_2 \\ \vdots \\ b_m
\end{pmatrix}
$$

Linear Equations

$$
\begin{array}{ccccccccc}
a_{11} x_1 & + & a_{12} x_2 & + & \cdots & + & a_{1n} x_n & = & b_1 \\
a_{21} x_1 & + & a_{22} x_2 & + & \cdots & + & a_{2n} x_n & = & b_2 \\
\vdots & & \vdots & & & & \vdots & & \vdots \\
a_{m1} x_1 & + & a_{m2} x_2 & + & \cdots & + & a_{mn} x_n & = & b_m
\end{array}
$$

$$
\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}
\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}
\begin{matrix} = \\ = \\ \\ = \end{matrix}
\begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix}
$$

$$
\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \end{bmatrix}
\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{bmatrix} b_1 \end{bmatrix}
$$

$$\sum_{j=1}^{n} a_{1j} \cdot x_j = b_1$$

$$
\begin{bmatrix} a_{21} & a_{22} & \cdots & a_{2n} \end{bmatrix}
\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{bmatrix} b_2 \end{bmatrix}
$$

$$\sum_{j=1}^{n} a_{2j} \cdot x_j = b_2$$

$$
\begin{bmatrix} a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}
\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = b_m
$$

$$\sum_{j=1}^{n} a_{mj} \cdot x_j = b_m$$

Example

$$
\begin{aligned}
a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\
a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\
\vdots \qquad \vdots \qquad\qquad \vdots \quad &\quad \vdots \\
a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m
\end{aligned}
$$

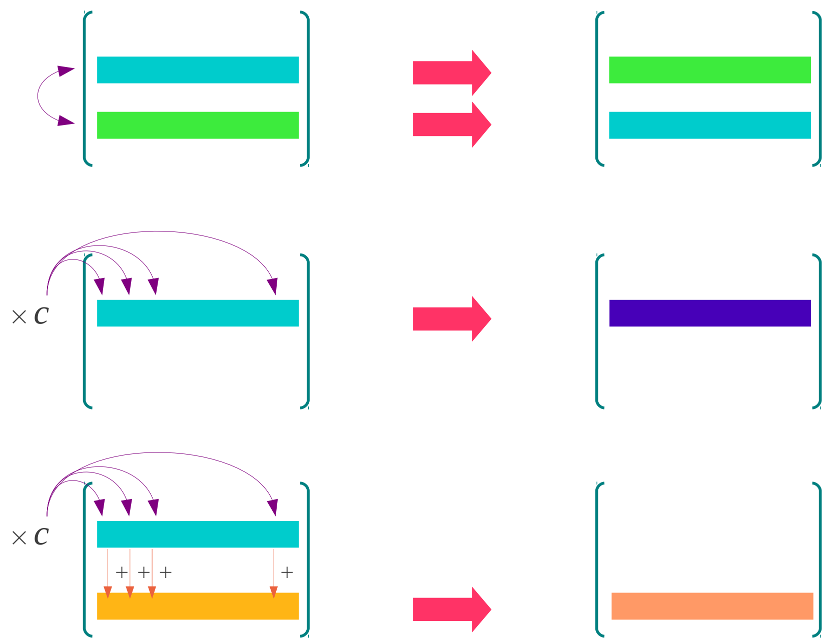
$$
\begin{aligned}
2x_1 + 1x_2 - 1x_3 &= +8 \\
-3x_1 - 1x_2 + 2x_3 &= -11 \\
-2x_1 + 1x_2 + 2x_3 &= -3
\end{aligned}
$$

$$
\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}
\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}
=
\begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix}
$$

$$
\begin{pmatrix} +2 & +1 & -1 \\ -3 & -1 & +2 \\ -2 & +1 & +2 \end{pmatrix}
\begin{pmatrix} x_1 \\ x_2 \\ x_n \end{pmatrix}
=
\begin{pmatrix} +8 \\ -11 \\ -3 \end{pmatrix}
$$

Gauss-Jordan Elimination

$$\begin{pmatrix} +2 & +1 & -1 \\ -3 & -1 & +2 \\ -2 & +1 & +2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} +8 \\ -11 \\ -3 \end{pmatrix} \Rightarrow \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} * \\ * \\ * \end{pmatrix}$$



Gauss-Jordan Elimination – Step 1

$$+2x_1 + x_2 - x_3 = 8 \qquad (L_1)$$
$$-3x_1 - x_2 + 2x_3 = -11 \qquad (L_2)$$
$$-2x_1 + x_2 + 2x_3 = -3 \qquad (L_3)$$

$$\left[\begin{array}{ccc|c} +2 & +1 & -1 & +8 \\ -3 & -1 & +2 & -11 \\ -2 & +1 & +2 & -3 \end{array}\right]$$

$$+1x_1 + \tfrac{1}{2}x_2 - \tfrac{1}{2}x_3 = 4 \qquad (\tfrac{1}{2} \times L_1)$$

+2/2 +1/2 –1/2 +8/2

$$+1x_1 + \tfrac{1}{2}x_2 - \tfrac{1}{2}x_3 = 4 \qquad (\tfrac{1}{2} \times L_1)$$
$$-3x_1 - x_2 + 2x_3 = -11 \qquad (L_2)$$
$$-2x_1 + x_2 + 2x_3 = -3 \qquad (L_3)$$

$$\left[\begin{array}{ccc|c} +1 & +1/2 & -1/2 & +4 \\ -3 & -1 & +2 & -11 \\ -2 & +1 & +2 & -3 \end{array}\right]$$

Gauss-Jordan Elimination – Step 2

$$+1x_1 + \tfrac{1}{2}x_2 - \tfrac{1}{2}x_3 = +4 \quad (L_1)$$
$$-3x_1 - x_2 + 2x_3 = -11 \quad (L_2)$$
$$-2x_1 + x_2 + 2x_3 = -3 \quad (L_3)$$

$$\left(\begin{array}{ccc|c} +1 & +1/2 & -1/2 & +4 \\ \boxed{-3} & -1 & +2 & -11 \\ \boxed{-2} & +1 & +2 & -3 \end{array}\right)$$

$$+3x_1 + \tfrac{3}{2}x_2 - \tfrac{3}{2}x_3 = +12 \quad (\boxed{3 \times L_1})$$
$$-3x_1 - x_2 + 2x_3 = -11 \quad (L_2)$$

+3	+3/2	–3/2	+12
–3	–1	+2	–11

$$+2x_1 + \tfrac{2}{2}x_2 - \tfrac{2}{2}x_3 = +8 \quad (\boxed{2 \times L_1})$$
$$-2x_1 + x_2 + 2x_3 = -3 \quad (L_3)$$

+2	+2/2	–2/2	+8
–2	+1	+2	–3

$$+1x_1 + \tfrac{1}{2}x_2 - \tfrac{1}{2}x_3 = +4 \quad (L_1)$$
$$0x_1 + \tfrac{1}{2}x_2 + \tfrac{1}{2}x_3 = +1 \quad (\boxed{3 \times L_1} + L_2)$$
$$0x_1 + 2x_2 + 1x_3 = +5 \quad (\boxed{2 \times L_1} + L_3)$$

$$\left(\begin{array}{ccc|c} +1 & +1/2 & -1/2 & +4 \\ \boxed{0} & +1/2 & +1/2 & +1 \\ \boxed{0} & +2 & +1 & +5 \end{array}\right)$$

Gauss-Jordan Elimination – Step 3

$$+1x_1 + \tfrac{1}{2}x_2 - \tfrac{1}{2}x_3 = +4 \qquad (L_1)$$
$$0x_1 + \tfrac{1}{2}x_2 + \tfrac{1}{2}x_3 = +1 \qquad (L_2)$$
$$0x_1 + 2x_2 + 1x_3 = +5 \qquad (L_3)$$

$$\left[\begin{array}{ccc|c} +1 & +1/2 & -1/2 & +4 \\ 0 & +1/2 & +1/2 & +1 \\ 0 & +2 & +1 & +5 \end{array}\right]$$

$$0x_1 + 1x_2 + 1x_3 = +2 \qquad (2 \times L_2)$$

0	+1	+1	+2

$$+1x_1 + \tfrac{1}{2}x_2 - \tfrac{1}{2}x_3 = +4 \qquad (L_1)$$
$$0x_1 + 1x_2 + 1x_3 = +2 \qquad (2 \times L_2)$$
$$0x_1 + 2x_2 + 1x_3 = +5 \qquad (L_3)$$

$$\left[\begin{array}{ccc|c} +1 & +1/2 & -1/2 & +4 \\ 0 & +1 & +1 & +2 \\ 0 & +2 & +1 & +5 \end{array}\right]$$

Gauss-Jordan Elimination – Step 4

$$+1x_1 + \tfrac{1}{2}x_2 - \tfrac{1}{2}x_3 = +4 \qquad (L_1)$$
$$0x_1 + 1x_2 + 1x_3 = +2 \qquad (L_2)$$
$$0x_1 + 2x_2 + 1x_3 = +5 \qquad (L_3)$$

$$\left(\begin{array}{ccc|c} +1 & +1/2 & -1/2 & +4 \\ 0 & +1 & +1 & +2 \\ 0 & \boxed{+2} & +1 & +5 \end{array}\right)$$

$$0x_1 - 2x_2 - 2x_3 = -4 \qquad \boxed{(-2 \times L_2)}$$
$$0x_1 + 2x_2 + 1x_3 = +5 \qquad (L_3)$$

0	–2	–2	–4
0	+2	+1	+5

$$+1x_1 + \tfrac{1}{2}x_2 - \tfrac{1}{2}x_3 = +4 \qquad (L_1)$$
$$0x_1 + 1x_2 + 1x_3 = +2 \qquad (L_2)$$
$$0x_1 + 0x_2 - 1x_3 = +1 \qquad \boxed{(-2 \times L_2)} + L_3)$$

$$\left(\begin{array}{ccc|c} +1 & +1/2 & -1/2 & +4 \\ 0 & +1 & +1 & +2 \\ 0 & \boxed{0} & -1 & +1 \end{array}\right)$$

Gauss-Jordan Elimination – Step 5

$$+1\,x_1 + \tfrac{1}{2}x_2 - \tfrac{1}{2}x_3 = +4 \qquad (L_1)$$
$$0\,x_1 + 1\,x_2 + 1\,x_3 = +2 \qquad (L_2)$$
$$0\,x_1 + 0\,x_2 - 1\,x_3 = +1 \qquad (L_3)$$

$$\left(\begin{array}{ccc|c} +1 & +1/2 & -1/2 & +4 \\ 0 & +1 & +1 & +2 \\ 0 & 0 & -1 & +1 \end{array}\right)$$

$$0\,x_1 - 0\,x_2 + 1\,x_3 = -1 \qquad (-1 \times L_3)$$

0	0	+1	–1

$$+1\,x_1 + \tfrac{1}{2}x_2 - \tfrac{1}{2}x_3 = +4 \qquad (L_1)$$
$$0\,x_1 + 1\,x_2 + 1\,x_3 = +2 \qquad (L_2)$$
$$0\,x_1 + 0\,x_2 + 1\,x_3 = -1 \qquad (-1 \times L_3)$$

$$\left(\begin{array}{ccc|c} +1 & +1/2 & -1/2 & +4 \\ 0 & +1 & +1 & +2 \\ 0 & 0 & +1 & -1 \end{array}\right)$$

Forward Phase

$$\left(\begin{array}{ccc|c} +2 & +1 & -1 & +8 \\ -3 & -1 & +2 & -11 \\ -2 & +1 & +2 & -3 \end{array}\right) \Rightarrow \left(\begin{array}{ccc|c} +1 & +1/2 & -1/2 & +4 \\ -3 & -1 & +2 & -11 \\ -2 & +1 & +2 & -3 \end{array}\right) \Rightarrow \left(\begin{array}{ccc|c} +1 & +1/2 & -1/2 & +4 \\ 0 & +1/2 & +1/2 & +1 \\ 0 & +2 & +1 & +5 \end{array}\right) \Rightarrow$$

$$\left(\begin{array}{ccc|c} +1 & +1/2 & -1/2 & +4 \\ 0 & +1 & +1 & +2 \\ 0 & +2 & +1 & +5 \end{array}\right) \Rightarrow \left(\begin{array}{ccc|c} +1 & +1/2 & -1/2 & +4 \\ 0 & +1 & +1 & +2 \\ 0 & 0 & -1 & +1 \end{array}\right) \Rightarrow \left(\begin{array}{ccc|c} +1 & +1/2 & -1/2 & +4 \\ 0 & +1 & +1 & +2 \\ 0 & 0 & +1 & -1 \end{array}\right) \Rightarrow$$

Forward Phase - Gaussian Elimination

$$+1x_1 + \tfrac{1}{2}x_2 - \tfrac{1}{2}x_3 = +4 \quad (L_1)$$
$$0x_1 + 1x_2 + 1x_3 = +2 \quad (L_2)$$
$$0x_1 + 0x_2 + 1x_3 = -1 \quad (L_3)$$

+1	+1/2	–1/2	+4
0	+1	+1	+2
0	0	+1	–1

$$0x_1 + 0x_2 + \tfrac{1}{2}x_3 = -\tfrac{1}{2} \quad (+\tfrac{1}{2} \times L_3)$$
$$+1x_1 + \tfrac{1}{2}x_2 - \tfrac{1}{2}x_3 = +4 \quad (L_1)$$

0	0	+1/2	–1/2
+1	+1/2	–1/2	+4

$$0x_1 + 0x_2 - 1x_3 = +1 \quad (-1 \times L_3)$$
$$0x_1 + 1x_2 + 1x_3 = +2 \quad (L_2)$$

0	0	–1	+1
0	+1	+1	+2

$$+1x_1 + \tfrac{1}{2}x_2 + 0x_3 = +\tfrac{7}{2} \quad (+\tfrac{1}{2} \times L_3 + L_1)$$
$$0x_1 + 1x_2 + 0x_3 = +3 \quad (-1 \times L_3 + L_2)$$
$$0x_1 + 0x_2 + 1x_3 = -1 \quad (L_3)$$

+1	+1/2	0	+7/2
0	+1	0	+3
0	0	+1	–1

Gauss-Jordan Elimination – Step 7

$$+1\,x_1 + \tfrac{1}{2}x_2 + 0\,x_3 = +\tfrac{7}{2} \qquad (L_1)$$
$$0\,x_1 + 1\,x_2 + 0\,x_3 = +3 \qquad (L_2)$$
$$0\,x_1 + 0\,x_2 + 1\,x_3 = -1 \qquad (L_3)$$

$$\left(\begin{array}{ccc|c} +1 & \boxed{+1/2} & 0 & +7/2 \\ 0 & +1 & 0 & +3 \\ 0 & 0 & +1 & -1 \end{array}\right)$$

$$0\,x_1 - \tfrac{1}{2}x_2 + 0\,x_3 = -\tfrac{3}{2} \qquad \boxed{(-\tfrac{1}{2} \times L_2)}$$
$$+1\,x_1 + 0\,x_2 - 0\,x_3 = +2 \qquad (L_1)$$

0	–1/2	0	–3/2
+1	+1/2	0	+7/2

$$+1\,x_1 + 0\,x_2 - 0\,x_3 = +2 \qquad (+1 \times L_3 + L_1)$$
$$0\,x_1 + 1\,x_2 + 0\,x_3 = +3 \qquad (L_2)$$
$$0\,x_1 + 0\,x_2 + 1\,x_3 = -1 \qquad (L_3)$$

$$\left(\begin{array}{ccc|c} +1 & \boxed{0} & 0 & +2 \\ 0 & +1 & 0 & +3 \\ 0 & 0 & +1 & -1 \end{array}\right)$$

Backward Phase

$$\left(\begin{array}{ccc|c} +1 & +1/2 & -1/2 & +4 \\ 0 & +1 & +1 & +2 \\ 0 & 0 & +1 & -1 \end{array}\right) \Rightarrow \left(\begin{array}{ccc|c} +1 & +1/2 & 0 & +7/2 \\ 0 & +1 & 0 & +3 \\ 0 & 0 & +1 & -1 \end{array}\right) \Rightarrow \left(\begin{array}{ccc|c} +1 & 0 & 0 & +2 \\ 0 & +1 & 0 & +3 \\ 0 & 0 & +1 & -1 \end{array}\right) \Rightarrow$$

Gauss-Jordan Elimination

Forward Phase - Gaussian Elimination

$$\left[\begin{array}{ccc|c} +2 & +1 & -1 & +8 \\ -3 & -1 & +2 & -11 \\ -2 & +1 & +2 & -3 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} +1 & +1/2 & -1/2 & +4 \\ -3 & -1 & +2 & -11 \\ -2 & +1 & +2 & -3 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} +1 & +1/2 & -1/2 & +4 \\ 0 & +1/2 & +1/2 & +1 \\ 0 & +2 & +1 & +5 \end{array}\right] \rightarrow$$

$$\left[\begin{array}{ccc|c} +1 & +1/2 & -1/2 & +4 \\ 0 & +1 & +1 & +2 \\ 0 & +2 & +1 & +5 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} +1 & +1/2 & -1/2 & +4 \\ 0 & +1 & +1 & +2 \\ 0 & 0 & -1 & +1 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} +1 & +1/2 & -1/2 & +4 \\ 0 & +1 & +1 & +2 \\ 0 & 0 & +1 & -1 \end{array}\right] \rightarrow$$

Backward Phase

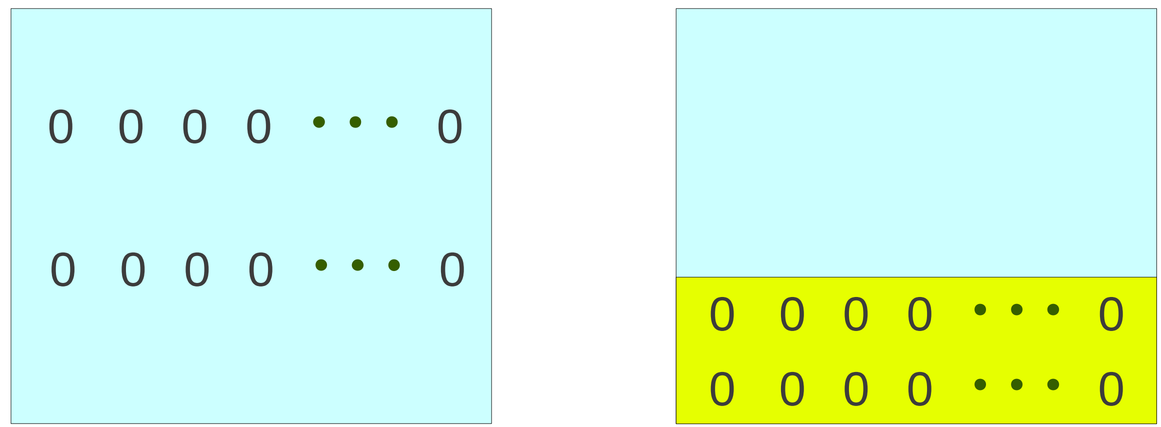
$$\left[\begin{array}{ccc|c} +1 & +1/2 & -1/2 & +4 \\ 0 & +1 & +1 & +2 \\ 0 & 0 & +1 & -1 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} +1 & +1/2 & 0 & +7/2 \\ 0 & +1 & 0 & +3 \\ 0 & 0 & +1 & -1 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} +1 & 0 & 0 & +2 \\ 0 & +1 & 0 & +3 \\ 0 & 0 & +1 & -1 \end{array}\right] \rightarrow$$

REF: Row Echelon Forms (1)

zero rows	➡	Should be grouped at the bottom
non-zero row	➡	A leading **1** The 1st non-zero element should be one
Any successive non-zero rows	➡	The leading **1** of the lower row should be farther to the **right** than the leading **1** of the higher row

REF: Row Echelon Forms (2)

zero rows → Should be grouped at the bottom



REF: Row Echelon Forms (3)

non-zero row → A leading one

The 1st non-zero element should be one

0 9 * * • • • *

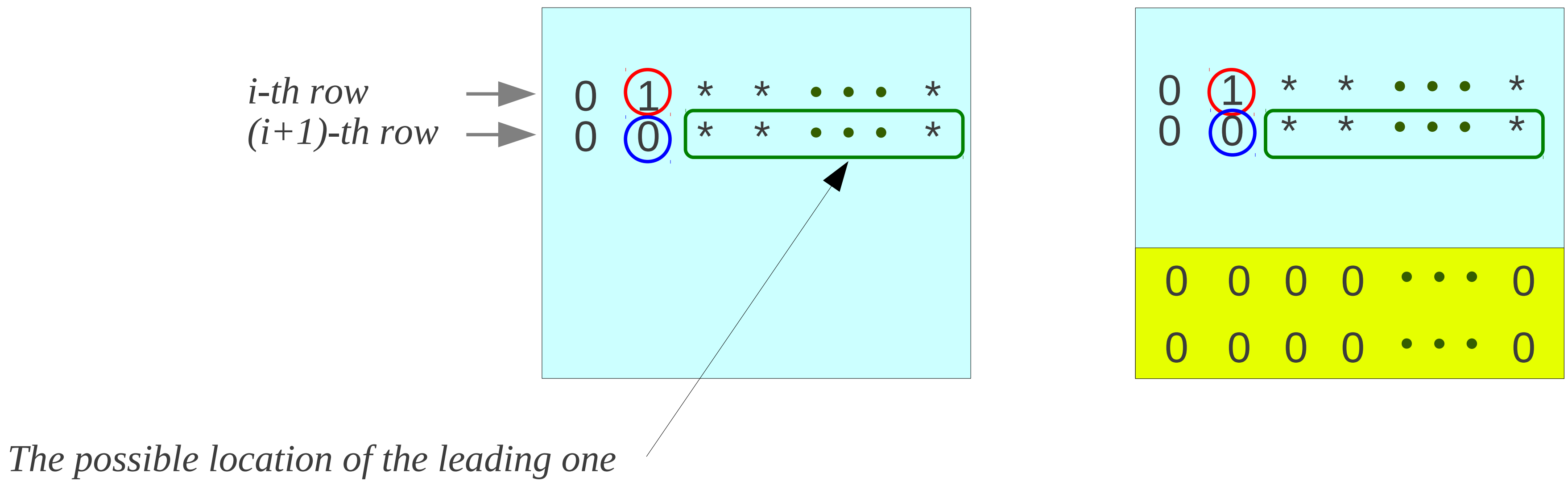
0 1 * * • • • *

0 0 0 0 • • • 0

0 0 0 0 • • • 0

REF: Row Echelon Forms (4)

Any successive non-zero rows → The leading **1** of the lower row should be farther to the **right** than the leading **1** of the higher row



Could be like this 0 0 1 * • • • *

Or like this 0 0 0 1 • • • *

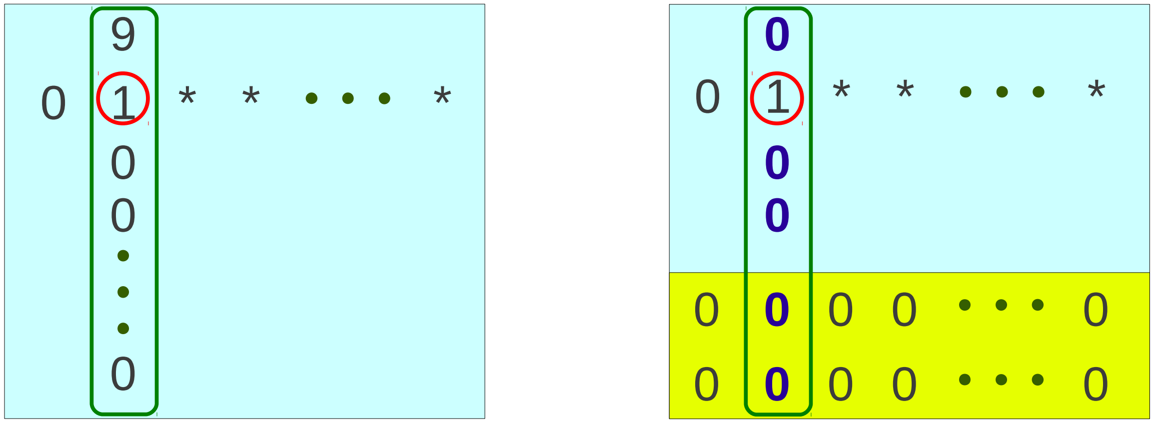
Or like this 0 0 0 0 • • • 1

RREF: Reduced Row Echelon Forms (1)

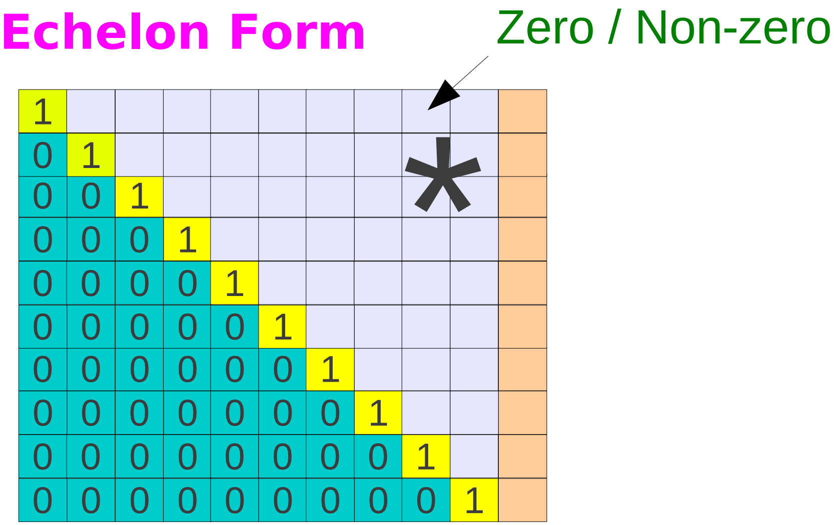
zero rows	→	Should be grouped at the bottom
non-zero row	→	A leading **1** The 1st non-zero element should be one
Any successive non-zero rows	→	The leading **1** of the lower row should be farther to the **right** than the leading **1** of the higher row
Any column that contains a leading **1**	→	All other elements except the leading one are **all zeros**

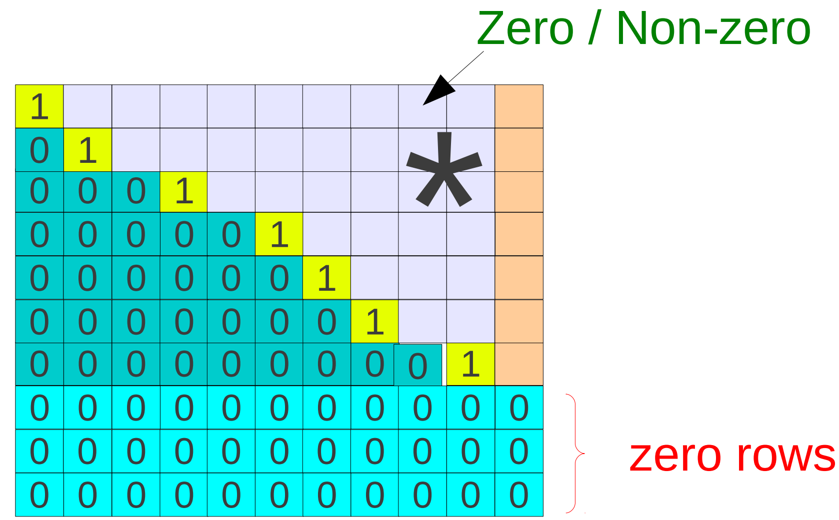
RREF: Reduced Row Echelon Forms (2)

Any column that contains a leading one → All other elements except the leading one are **all zeros**

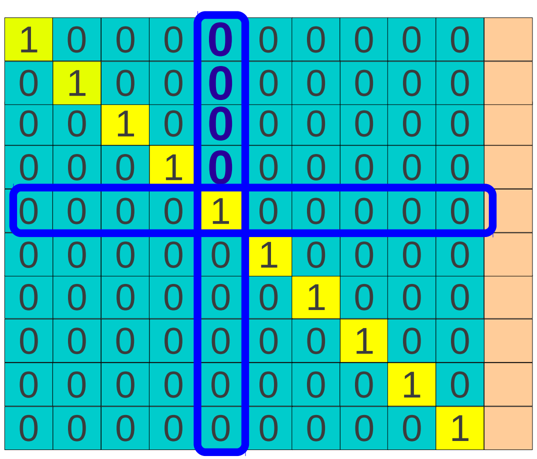


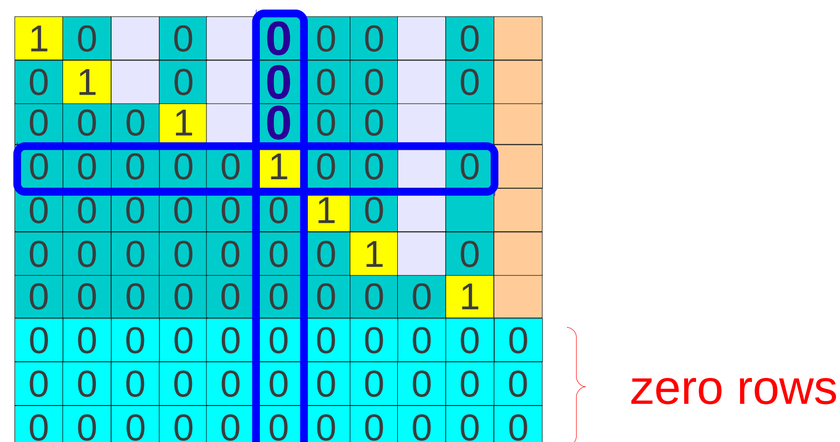
Row Echelon Form





Reduced Row Echelon Form



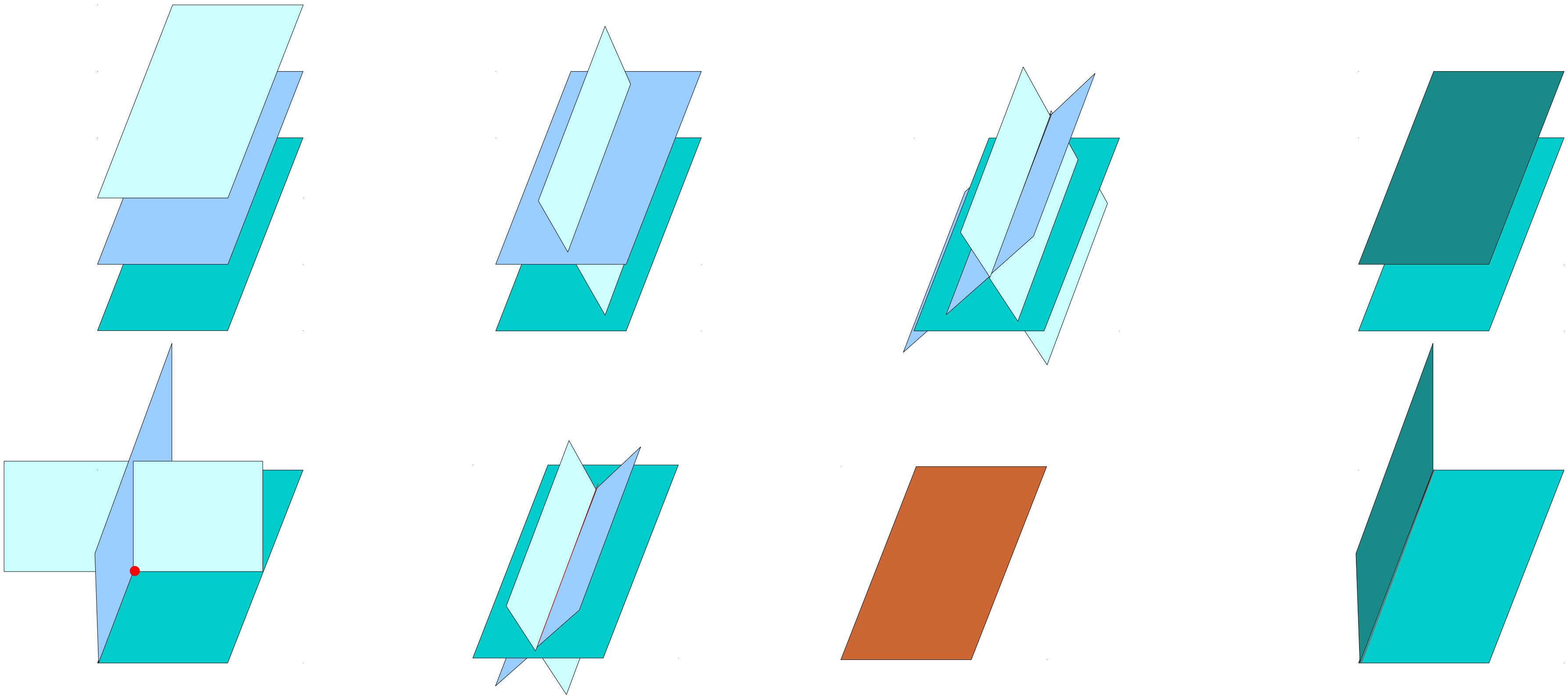


Linear Systems of 3 Unknowns

(Eq 1) ➡ $a_{11}\ x_1 + a_{12}\ x_2 + a_{13}\ x_3 = b_1$

(Eq 2) ➡ $a_{21}\ x_1 + a_{22}\ x_2 + a_{23}\ x_3 = b_2$

(Eq 3) ➡ $a_{31}\ x_1 + a_{32}\ x_2 + a_{33}\ x_3 = b_3$



Leading and Free Variables

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right) \quad \left(\begin{array}{ccc|c} 1 & 0 & 3 & -1 \\ 0 & 1 & -4 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right) \quad \left(\begin{array}{ccc|c} 1 & -5 & 1 & 4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$$0 \cdot x_1 + 0 \cdot x_2 + 0 \cdot x_3 = 1$$

$$0 = 1$$

$$1 \cdot x_1 + 3 \cdot x_3 = -1$$

$$1 \cdot x_2 - 4 \cdot x_3 = 2$$

$$1 \cdot x_1 - 5 \cdot x_2 + 1 \cdot x_3 = 4$$

with a leading 1

leading variables

Other remaining varaible

free variables

Free Variables as Parameters (1)

1	0	0	0
0	1	2	0
0	0	0	1

1	0	3	-1
0	1	-4	2
0	0	0	0

1	-5	1	4
0	0	0	0
0	0	0	0

$$0\cdot x_1 + 0\cdot x_2 + 0\cdot x_3 = 1$$

$$0 = 1$$

$$1\cdot x_1 \qquad + 3\cdot x_3 = -1$$
$$1\cdot x_2 - 4\cdot x_3 = 2$$

$$1\cdot x_1 - 5\cdot x_2 + 1\cdot x_3 = 4$$

Solve for a leading variable

$$x_1 = -1 - 3\cdot x_3$$
$$x_2 = 2 + 4\cdot x_3$$

$$x_1 = 4 + 5\cdot x_2 - 1\cdot x_3$$

Treat a free variable as a parameter

$$x_3 = t$$

$$x_2 = s \qquad x_3 = t$$

$$\begin{cases} x_1 = -1 - 3t \\ x_2 = 2 + 4t \\ x_3 = t \end{cases}$$

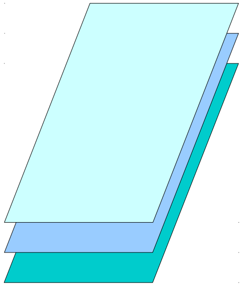
$$\begin{cases} x_1 = 4 + 5\cdot x_2 - 1\cdot x_3 \\ x_2 = s \\ x_3 = t \end{cases}$$

Free Variables as Parameters (2)

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right)$$

$$0 \cdot x_1 + 0 \cdot x_2 + 0 \cdot x_3 = 1$$

$$0 = 1$$

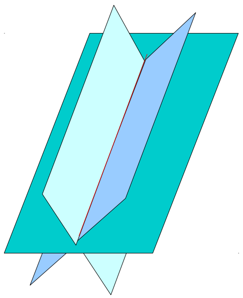


$$\left(\begin{array}{ccc|c} 1 & 0 & 3 & -1 \\ 0 & 1 & -4 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$$\begin{aligned} 1 \cdot x_1 \quad\quad + 3 \cdot x_3 &= -1 \\ 1 \cdot x_2 - 4 \cdot x_3 &= 2 \end{aligned}$$

$$\begin{cases} x_1 = -1 - 3t \\ x_2 = 2 + 4t \\ x_3 = t \end{cases}$$

$x_3 = t$ ⟸ free variable



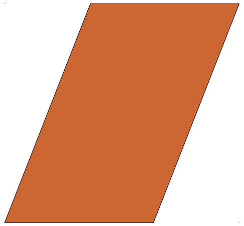
$$\left(\begin{array}{ccc|c} 1 & -5 & 1 & 4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$$1 \cdot x_1 - 5 \cdot x_2 + 1 \cdot x_3 = 4$$

$$\begin{cases} x_1 = 4 + 5 \cdot x_2 - 1 \cdot x_3 \\ x_2 = s \\ x_3 = t \end{cases}$$

$x_2 = s$ ⟸ free variable

$x_3 = t$ ⟸ free variable



Free Variables as Parameters (3)

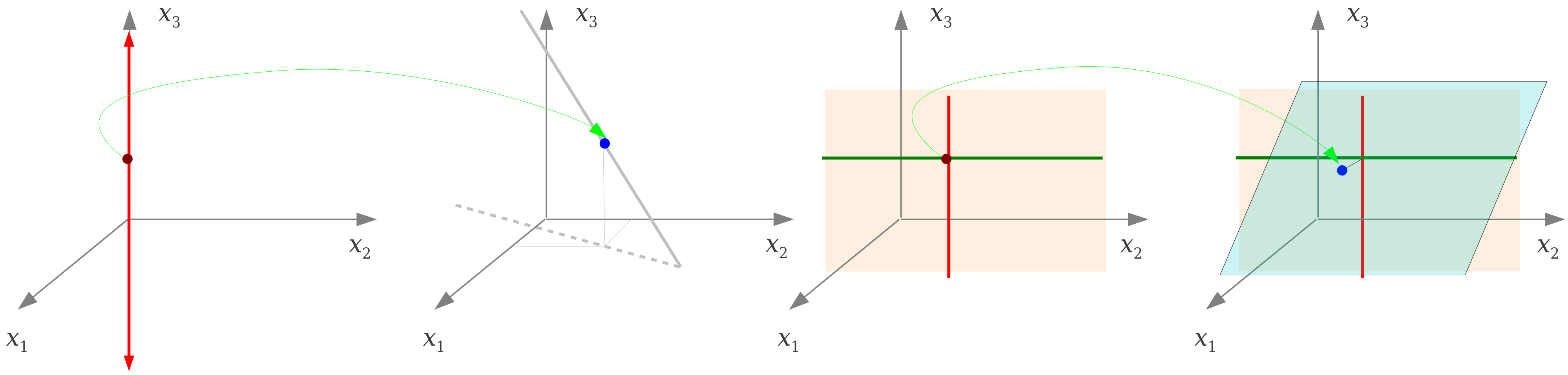
$$\begin{cases} x_1 = -1 - 3t \\ x_2 = 2 + 4t \\ x_3 = t \end{cases}$$

$x_3 = t$ ⇐ free variable

$$\begin{cases} x_1 = 4 + 5 \cdot x_2 - 1 \cdot x_3 \\ x_2 = s \\ x_3 = t \end{cases}$$

$x_2 = s$ ⇐ free variable

$x_3 = t$ ⇐ free variable



infinitely many solutions

infinitely many solutions

Consistent Linear System

A linear system with at least one solution

➡ A Consistent Linear System

A linear system with no solutions

➡ A Inconsistent Linear System

General Solution

A linear system with infinitely many solutions

Solve for a leading variable

Treat a free variable as a parameter

⇒ A set of parametric equations

All solutions can be obtained
by assigning numerical values to those parmeters

⇒ Called a general solution

Homogeneous System

$$\begin{aligned} a_{11} x_1 + a_{12} x_2 + \cdots + a_{1n} x_n &= 0 \\ a_{21} x_1 + a_{22} x_2 + \cdots + a_{2n} x_n &= 0 \\ \vdots \qquad \vdots \qquad\qquad \vdots \quad & \quad \vdots \\ a_{m1} x_1 + a_{m2} x_2 + \cdots + a_{mn} x_n &= 0 \end{aligned}$$

All constant terms are zero

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

All constant terms are zero

Solutions of a Homogeneous System

All homogeneous system passes through the origin

The homogeneous system has

* only the trivial solution

* many solutions in addition to the trivial solution

$$
\begin{aligned}
a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= 0 \\
a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= 0 \\
\vdots \qquad \vdots \qquad\qquad \vdots \quad &\quad \vdots \\
a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= 0
\end{aligned}
$$

$$
\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}
\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}
=
\begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}
$$

Trivial Solution

$$\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

satisfies all homogeneous equation

$$\begin{matrix} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = 0 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = 0 \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = 0 \end{matrix}$$

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

All homogeneous system passes through the origin

Augmented Matrix

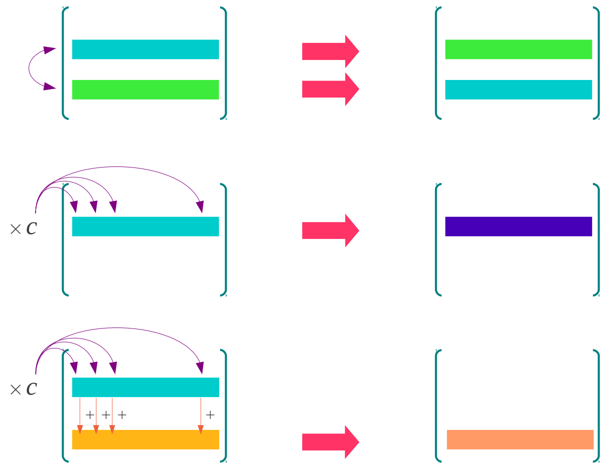
$$\begin{array}{ccccccccc} a_{11} x_1 & + & a_{12} x_2 & + & \cdots & + & a_{1n} x_n & = & 0 \\ a_{21} x_1 & + & a_{22} x_2 & + & \cdots & + & a_{2n} x_n & = & 0 \\ \vdots & & \vdots & & & & \vdots & & \vdots \\ a_{m1} x_1 & + & a_{m2} x_2 & + & \cdots & + & a_{mn} x_n & = & 0 \end{array}$$

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \begin{matrix} = \\ = \\ \\ = \end{matrix} \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

Augmented matrix of a homogeneous system

$$\left(\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & 0 \\ a_{21} & a_{22} & \cdots & a_{2n} & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & 0 \end{array} \right)$$

Reduced Row Echelon Form



Elementary row operations do not alter the zero column of a matrix

homogeneous system

The augmented zero column is preserved in the reduced row echelon form

Reduced Echelon Form

m leading variables

1	0	0	0	0	0	0	0	0	0	0
0	1	0	0	0	0	0	0	0	0	0
0	0	1	0	0	0	0	0	0	0	0
0	0	0	1	0	0	0	0	0	0	0
0	0	0	0	1	0	0	0	0	0	0
		0	0	0	1	0	0	0	0	0
		0	0	0	0	1	0	0	0	0
		0	0	0	0	0	1	0	0	0
0	0	0	0	0	0	0	0	1	0	0
0	0	0	0	0	0	0	0	0	1	0

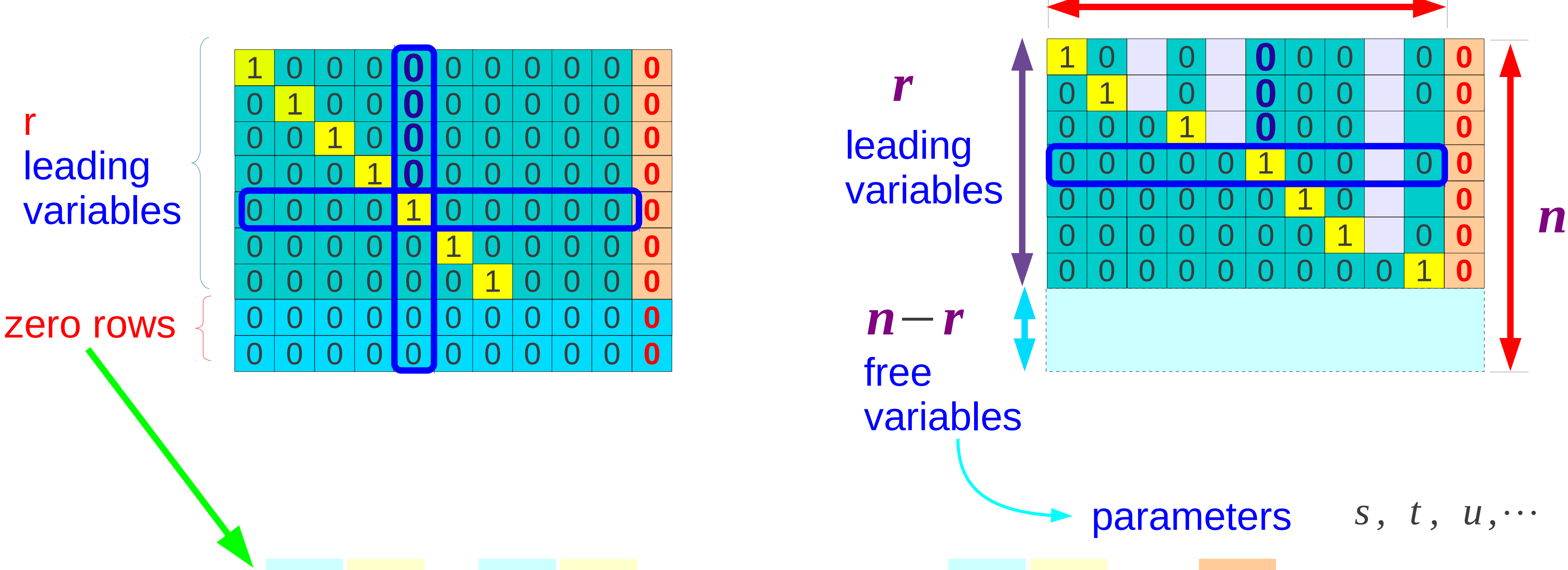
r leading variables

1	0		0		0	0	0		0	0
0	1		0		0	0	0		0	0
0	0	0	1		0	0	0			0
0	0	0	0	0	1	0	0		0	0
0	0	0	0	0	0	1	0			0
0	0	0	0	0	0	0	1		0	0
0	0	0	0	0	0	0	0	0	1	0
0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0

zero rows

Free Variable Theorem

Reduced Echelon Form



$$0\ x_1 + 0\ x_2 + \cdots + 0\ x_n = 0$$

A homogeneous linear system with *n* unknowns

If the reduced row echelon form of its augmented matrix has

r non-zero rows ➡ ***n – r*** free variables ➡ *infinitely many solutions*

Free Variable Theorem Example

Reduced Echelon Form

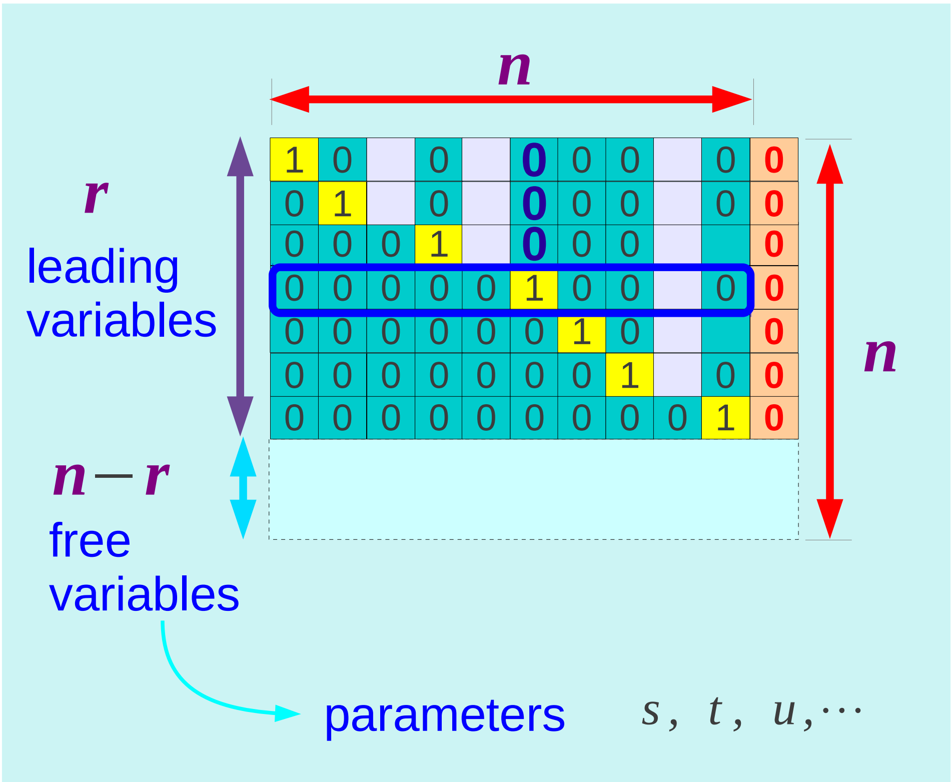
$$\left[\begin{array}{ccc|c} 1 & 0 & 3 & -1 \\ 0 & 1 & -4 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right] \qquad \left[\begin{array}{ccc|c} 1 & -5 & 1 & 4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$1 \cdot x_1 + 3 \cdot x_3 = -1$$
$$1 \cdot x_2 - 4 \cdot x_3 = 2$$

$$1 \cdot x_1 - 5 \cdot x_2 + 1 \cdot x_3 = 4$$

$$\begin{cases} x_1 = -1 - 3t \\ x_2 = 2 + 4t \\ x_3 = t \end{cases}$$ free variable (x_3)

$$\begin{cases} x_1 = 4 + 5 \cdot x_2 - 1 \cdot x_3 \\ x_2 = s \\ x_3 = t \end{cases}$$ free variable (x_2), free variable (x_3)



A homogeneous linear system with ***n*** unknowns

If the reduced row echelon form of its augmented matrix has

r non-zero rows ➡ ***n – r*** free variables ➡ *infinitely many solutions*

Linear System Ax = B

$$A \; x = \mathbf{0}$$

Always consistent

$rank(A) = n$

unique solution $x = \mathbf{0}$

$rank(A) < n$

Infinitely many solution

$n - r$ parameters

$$A = [a_{ij}]_{m \times n}$$

m equations

n unknowns

$$A \; x = b$$

$rank(A) = rank(A|b)$

Consistent

$rank(A) = n$

unique solution $x = \mathbf{0}$

$rank(A) < n$

Infinitely many solution

$n - r$ parameters

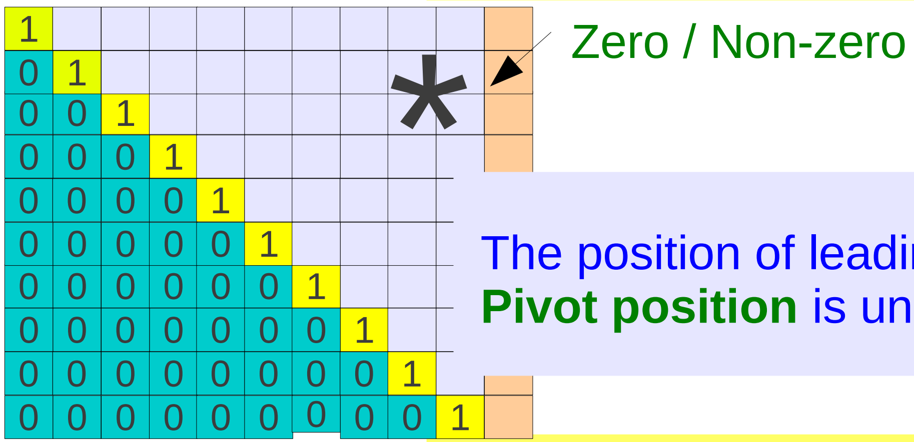
$rank(A) < rank(A|b)$

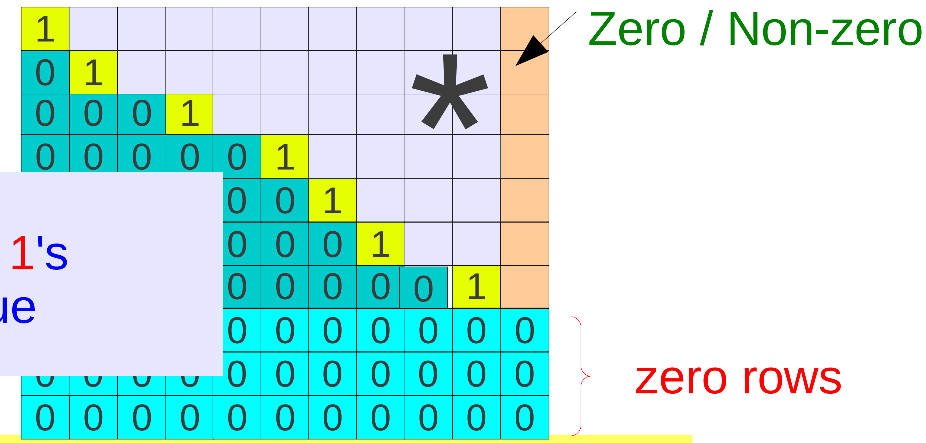
Inconsistent

Pivot Positions

Row Echelon Form → **Not unique**

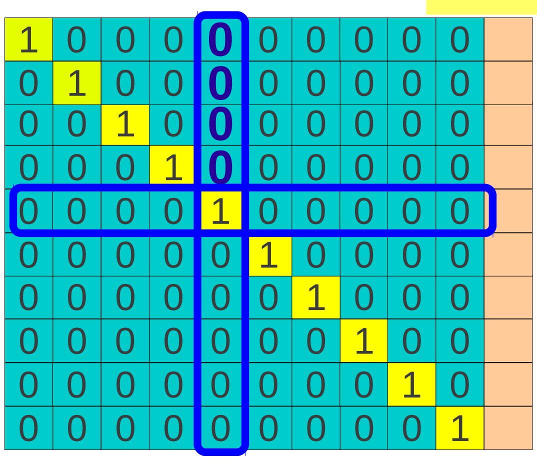
Depend on the sequence of elementary row operations

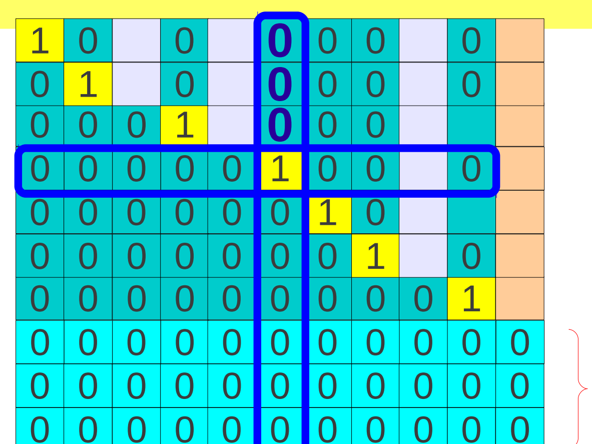




The position of leading 1's
Pivot position is unique

Reduced Row Echelon Form → **Unique**





Pulse

References

[1] http://en.wikipedia.org/
[2] Anton & Busby, “Contemporary Linear Algebra”
[3] Anton & Rorres, “Elementary Linear Algebra”