Derivatives (2A)

- Partial Derivative
- Directional Derivative
- Tangent Planes and Normal Lines

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Partial Derivatives

Function of one variable $y = f(x)$

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

Function of two variable $z = f(x, y)$

$$\frac{\partial z}{\partial x} = \lim_{\Delta x \to 0} \frac{f(x+\Delta x, y) - f(x, y)}{\Delta x}$$

treating y as a constant

$$\frac{\partial z}{\partial y} = \lim_{\Delta x \to 0} \frac{f(x, y+\Delta y) - f(x, y)}{\Delta y}$$

treating x as a constant

Partial Derivatives Notations

Function of one variable $y = f(x)$

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

Function of two variables $z = f(x, y)$

$$\frac{\partial z}{\partial x} = \frac{\partial f}{\partial x} = z_x = f_x$$

$$\frac{\partial z}{\partial x} = \lim_{\Delta x \to 0} \frac{f(x+\Delta x, y) - f(x, y)}{\Delta x}$$

treating y as a constant

$$\frac{\partial z}{\partial y} = \frac{\partial f}{\partial y} = z_y = f_y$$

$$\frac{\partial z}{\partial y} = \lim_{\Delta x \to 0} \frac{f(x, y+\Delta y) - f(x, y)}{\Delta y}$$

treating x as a constant

Higher-Order & Mixed Partial Derivatives

Second-order Partial Derivatives

$$\frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) \qquad \frac{\partial^2 z}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right)$$

Third-order Partial Derivatives

$$\frac{\partial^3 z}{\partial x^3} = \frac{\partial}{\partial x}\left(\frac{\partial^2 z}{\partial x^2}\right) \qquad \frac{\partial^3 z}{\partial y^3} = \frac{\partial}{\partial y}\left(\frac{\partial^2 z}{\partial y^2}\right)$$

Mixed Partial Derivatives

$$\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial^2 z}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right)$$

Chain Rule (1)

Function of two variable

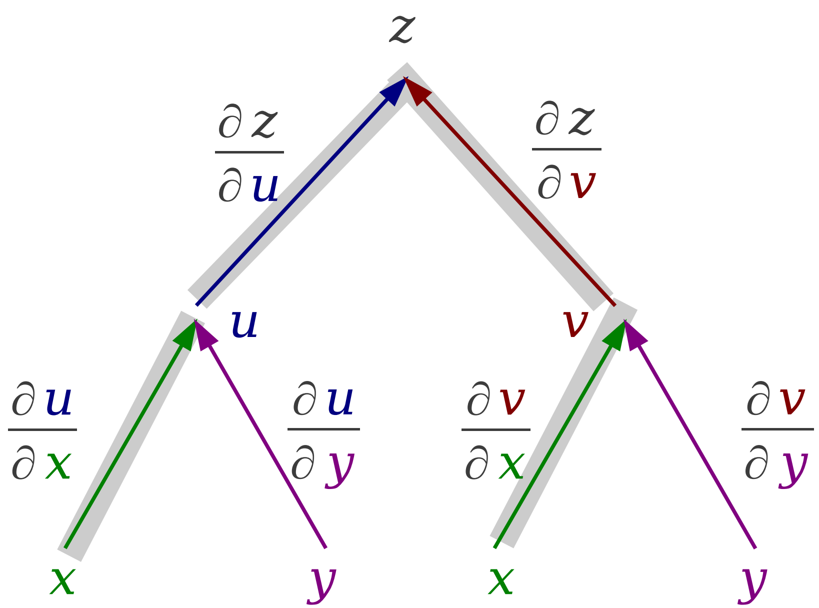
$$z = f(u, v)$$

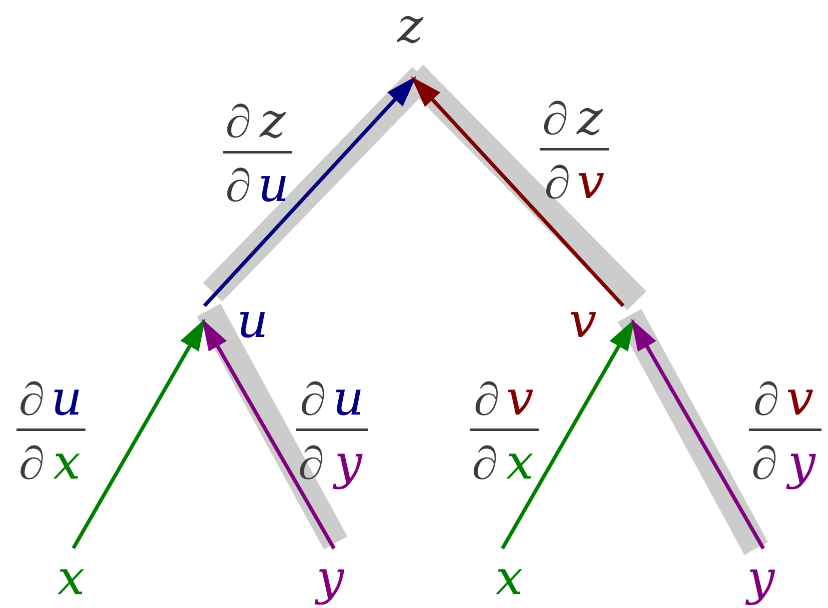
$$u = g(x, y) \qquad v = h(x, y)$$

$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial x} \qquad \frac{\partial z}{\partial y} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial y}$$

Chain Rule

Function of two variables $z = f(u, v)$





$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial x}$$

$$\frac{\partial z}{\partial y} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial y}$$

Directional Derivatives

Function of two variables $z = f(x, y)$

$$\frac{\partial z}{\partial x} = \lim_{\Delta x \to 0} \frac{f(x+\Delta x, y) - f(x, y)}{\Delta x}$$

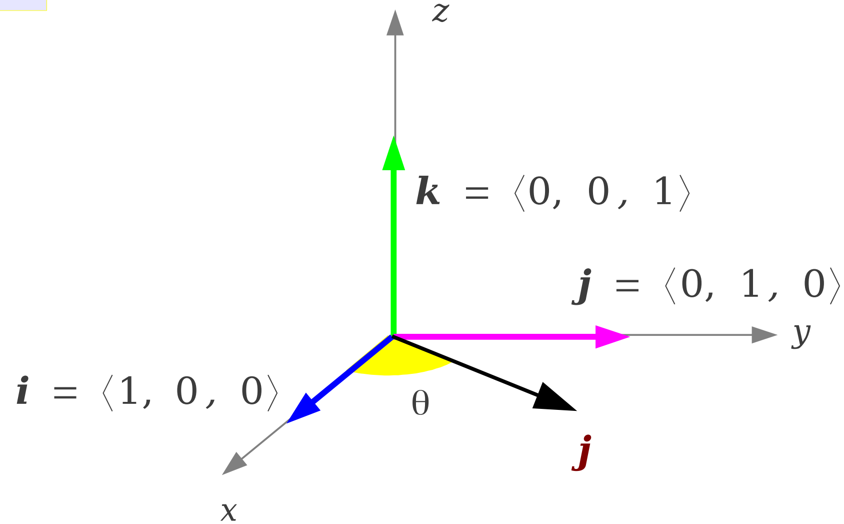
→ value

Rate of change of **f** in the **x** direction

$$\frac{\partial z}{\partial y} = \lim_{\Delta x \to 0} \frac{f(x, y+\Delta y) - f(x, y)}{\Delta y}$$

→ value

Rate of change of **f** in the **y** direction



Rate of change of **f** in the **u** direction

→ value **?**

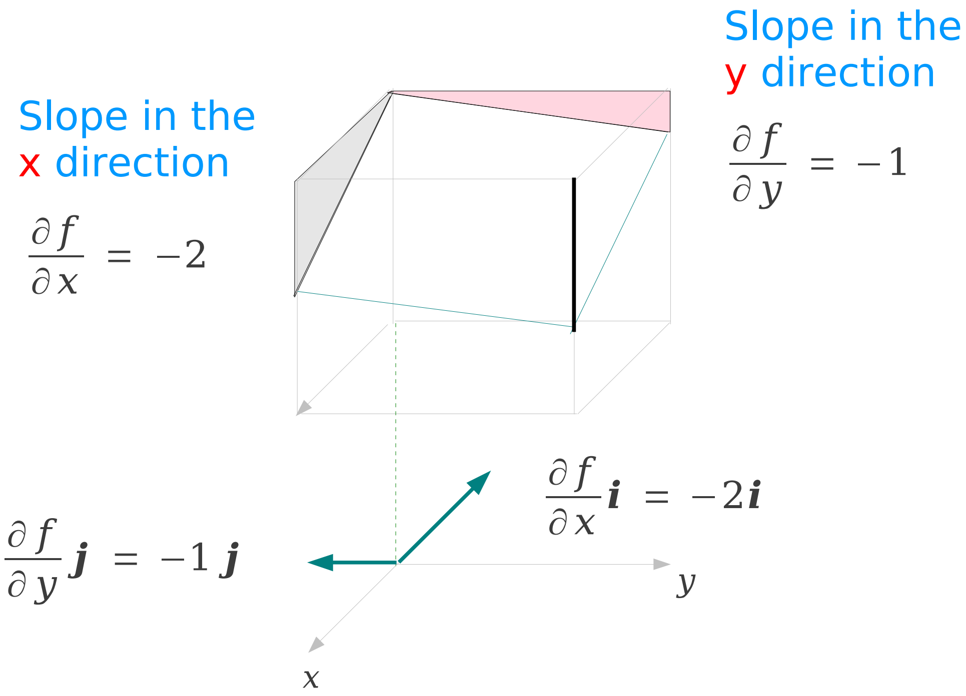
Gradient of a 2 Variable Function

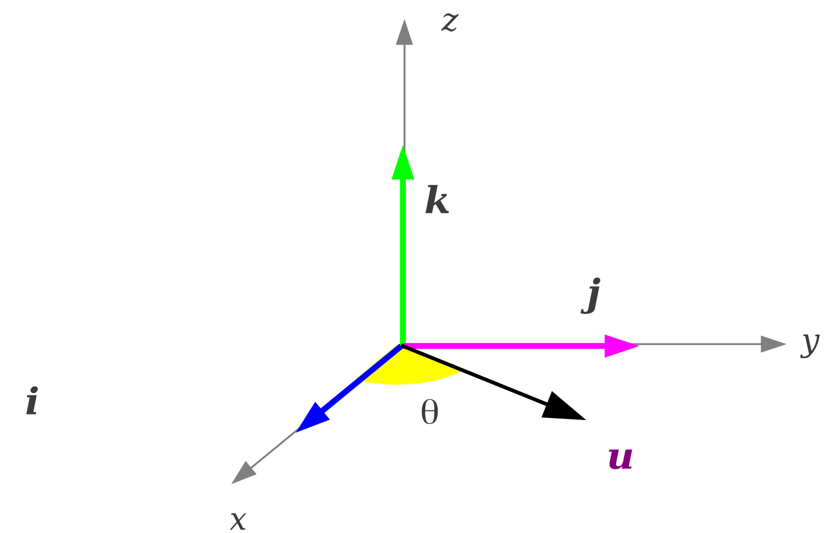
Function of two variables $f(x, y)$

$$\nabla f(x,y) = \frac{\partial f}{\partial x}\,\boldsymbol{i} + \frac{\partial f}{\partial y}\,\boldsymbol{j}$$

vector

Rate of change of **f** in the **x** direction
Rate of change of **f** in the **y** direction





Gradient of a 3 Variable Function

Function of two variables $f(x, y)$

$$\nabla f(x,y) = \frac{\partial f}{\partial x}\,\boldsymbol{i} + \frac{\partial f}{\partial y}\,\boldsymbol{j}$$

⇨ vector

Rate of change of **f** in the **x** direction
Rate of change of **f** in the **y** direction

Function of three variables $F(x, y, z)$

$$\nabla F(x,y,z) = \frac{\partial F}{\partial x}\,\boldsymbol{i} + \frac{\partial F}{\partial y}\,\boldsymbol{j} + \frac{\partial F}{\partial z}\,\boldsymbol{k}$$

⇨ vector

Rate of change of **f** in the **x** direction
Rate of change of **f** in the **y** direction
Rate of change of **f** in the **z** direction

General Partial Differentiation

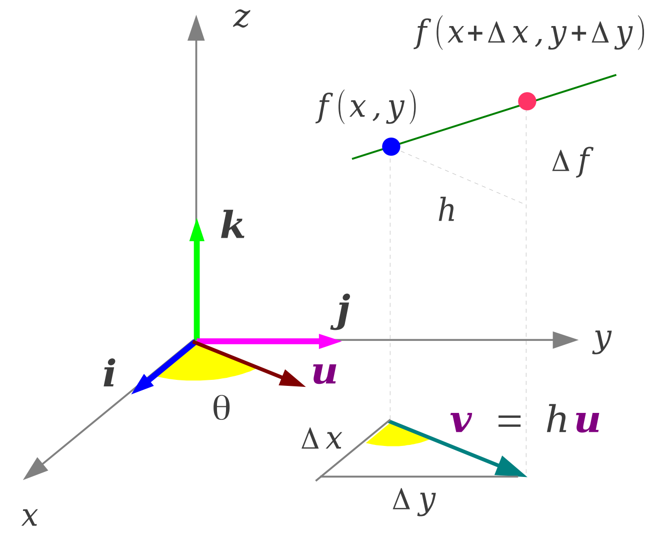
Function of two variables $f(x, y)$

$$\boldsymbol{u} = \cos\theta\, \boldsymbol{i} + \sin\theta\, \boldsymbol{j}$$

$$(x,\ y,\ 0) \xrightarrow{\boldsymbol{v} = h\boldsymbol{u}} (x+\Delta x,\ y+\Delta y,\ 0)$$

$$\Delta x = h\cos\theta$$
$$\Delta y = h\sin\theta$$

$$h = \sqrt{(\Delta x)^2 + (\Delta y)^2}$$



$$\frac{f(x+\Delta x,\ y+\Delta y) - f(x,\ y)}{h}$$

$$= \frac{f(x + h\cos\theta,\ y + h\sin\theta) - f(x,\ y)}{h}$$

Rate of change of **f** in the **u** direction → value

Directional Derivative

Function of two variable $f(x, y)$

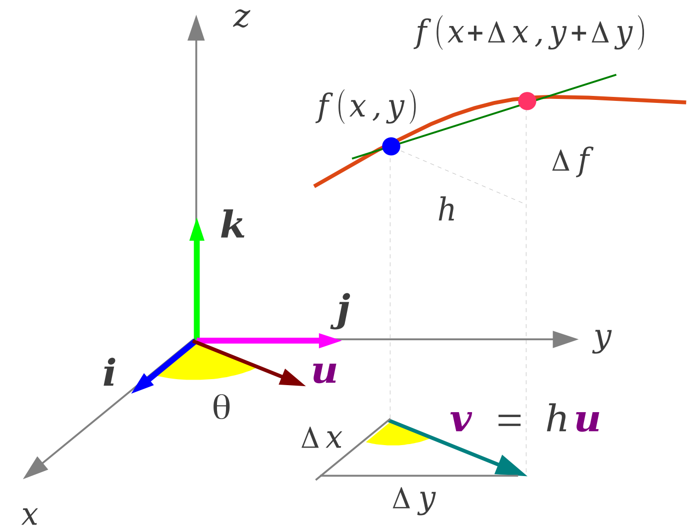
$$\boldsymbol{u} = \cos\theta\, \boldsymbol{i} + \sin\theta\, \boldsymbol{j}$$

$$(x, y, 0) \xrightarrow{\boldsymbol{v} = h\boldsymbol{u}} (x+\Delta x, y+\Delta y, 0)$$

$\Delta x = h\cos\theta$

$\Delta y = h\sin\theta$

$h = \sqrt{(\Delta x)^2 + (\Delta y)^2}$



Rate of change of **f** in the **u** direction

$$D_{\boldsymbol{u}} f(x,y) = \lim_{h\to 0} \frac{f(x + h\cos\theta, y + h\sin\theta) - f(x, y)}{h}$$

→ value

Computing Directional Derivative (1)

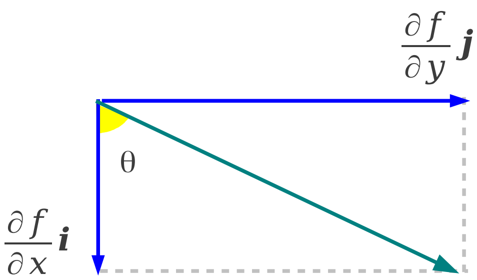
Function of two variables $f(x, y)$

$$\boldsymbol{u} = \cos\theta\, \boldsymbol{i} + \sin\theta\, \boldsymbol{j}$$

$$D_{\boldsymbol{u}} f(x,y) = \lim_{h \to 0} \frac{f(x + h\cos\theta,\, y + h\sin\theta) - f(x, y)}{h}$$ → value

$\theta = 0^\circ$ ⇒ $\boldsymbol{i} = \cos 0\, \boldsymbol{i} + \sin 0\, \boldsymbol{j}$

⇒ $$D_{\boldsymbol{i}} f(x,y) = \lim_{h \to 0} \frac{f(x + h,\, y) - f(x, y)}{h} = \frac{\partial f}{\partial x}$$



$\theta = 90^\circ$ ⇒ $\boldsymbol{j} = \cos 90^\circ \boldsymbol{i} + \sin 90^\circ \boldsymbol{j}$

$$\nabla f(x,y) = \frac{\partial f}{\partial x}\boldsymbol{i} + \frac{\partial f}{\partial y}\boldsymbol{j}$$

⇒ $$D_{\boldsymbol{j}} f(x,y) = \lim_{h \to 0} \frac{f(x,\, y + h) - f(x, y)}{h} = \frac{\partial f}{\partial y}$$

Computing Directional Derivative (2)

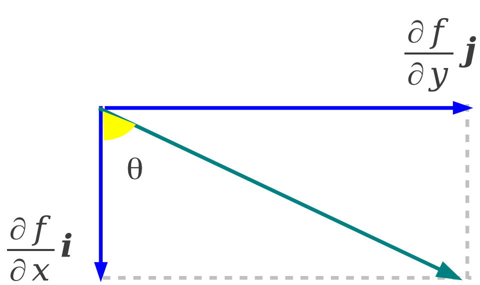
Function of two variables $f(x, y)$

$$\boldsymbol{u} = \cos\theta\, \boldsymbol{i} + \sin\theta\, \boldsymbol{j}$$

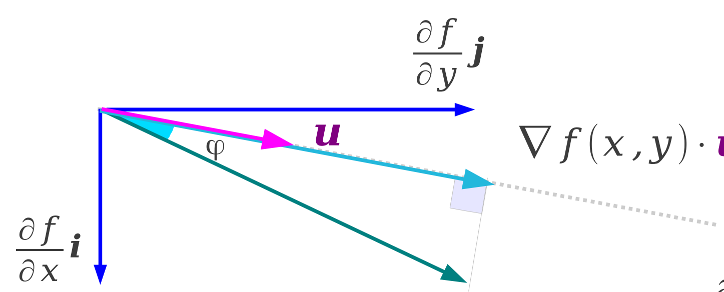
$$D_{\boldsymbol{u}} f(x,y) = \lim_{h \to 0} \frac{f(x + h\cos\theta,\, y + h\sin\theta) - f(x,y)}{h}$$

→ value

$$\nabla f(x,y) = \frac{\partial f}{\partial x}\boldsymbol{i} + \frac{\partial f}{\partial y}\boldsymbol{j} \qquad \boldsymbol{u} = \cos\theta\, \boldsymbol{i} + \sin\theta\, \boldsymbol{j}$$



$$\nabla f(x,y) = \frac{\partial f}{\partial x}\boldsymbol{i} + \frac{\partial f}{\partial y}\boldsymbol{j}$$



$$\nabla f(x,y) \cdot \boldsymbol{u} = D_{\boldsymbol{u}} f(x,y)$$

$$\nabla f(x,y) = \frac{\partial f}{\partial x}\boldsymbol{i} + \frac{\partial f}{\partial y}\boldsymbol{j}$$

Computing Directional Derivative (3)

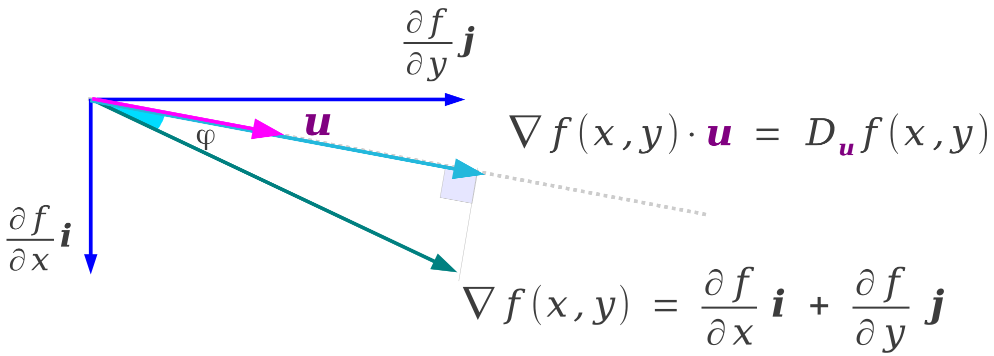
Function of two variables $f(x, y)$

$$\boldsymbol{u} = \cos\theta\, \boldsymbol{i} + \sin\theta\, \boldsymbol{j}$$

$$D_{\boldsymbol{u}} f(x,y) = \lim_{h \to 0} \frac{f(x + h\cos\theta, y + h\sin\theta) - f(x, y)}{h}$$ → value

$$\nabla f(x,y) = \frac{\partial f}{\partial x} \boldsymbol{i} + \frac{\partial f}{\partial y} \boldsymbol{j}$$

$$\boldsymbol{u} = \cos\theta\, \boldsymbol{i} + \sin\theta\, \boldsymbol{j}$$

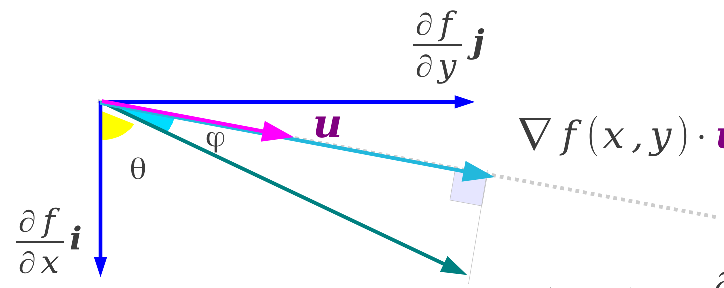


$$D_{\boldsymbol{u}} f(x,y) = \nabla f(x,y) \cdot \boldsymbol{u}$$ → value

Rate of change of **f** in the **u** direction

Computing Directional Derivative (4)

$$\nabla f(x,y) = \frac{\partial f}{\partial x}\boldsymbol{i} + \frac{\partial f}{\partial y}\boldsymbol{j}$$



$$\nabla f(x,y)\cdot\boldsymbol{u} = D_{\boldsymbol{u}} f(x,y)$$

$$\nabla f(x,y) = \frac{\partial f}{\partial x}\boldsymbol{i} + \frac{\partial f}{\partial y}\boldsymbol{j}$$

$$\boldsymbol{u} = \cos\theta\,\boldsymbol{i} + \sin\theta\,\boldsymbol{j}$$

$$D_{\boldsymbol{u}} f(x,y) = \nabla f(x,y)\cdot\boldsymbol{u}$$ → value

Rate of change of **f** in the **u** direction

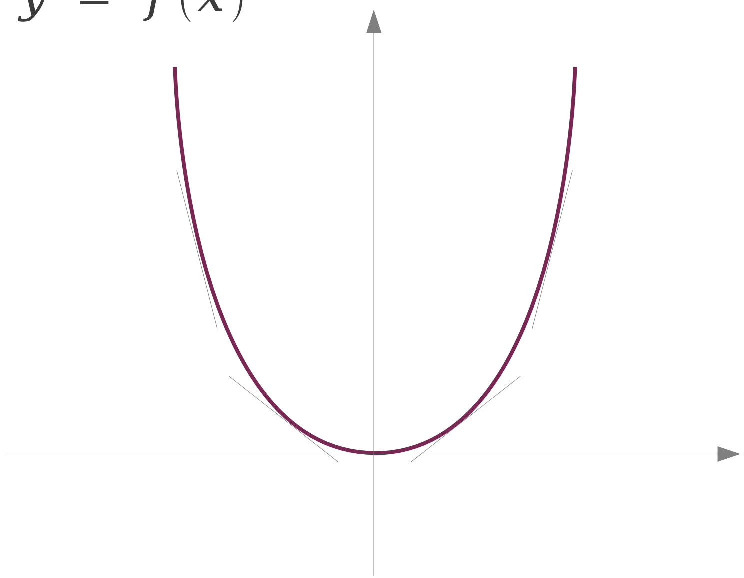
$$D_{\boldsymbol{u}} f(x,y) = \|\nabla f(x,y)\|\,\|\boldsymbol{u}\|\cos\varphi = \|\nabla f(x,y)\|\cos\varphi$$

min $-\|\nabla f(x,y)\| \le D_{\boldsymbol{u}} f(x,y) \le \|\nabla f(x,y)\|$ max

the direction of $\nabla f(x,y)$ → $f(x,y)$ Increases most rapidly

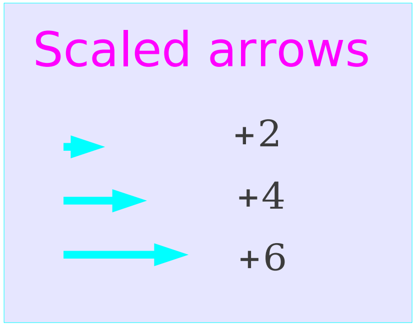
Gradient Example (1)

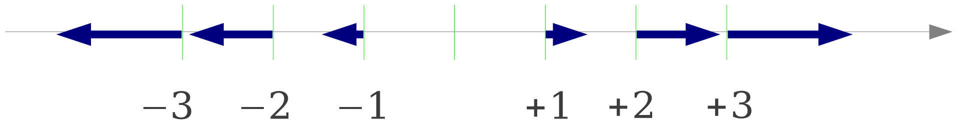
$y = f(x)$



$f(x) = x^2$ $\qquad \frac{df}{dx} = 2x$

x	-3	-2	-1	0	$+1$	$+2$	$+3$
$\frac{df}{dx}$	-6	-4	-2	0	$+2$	$+4$	$+6$

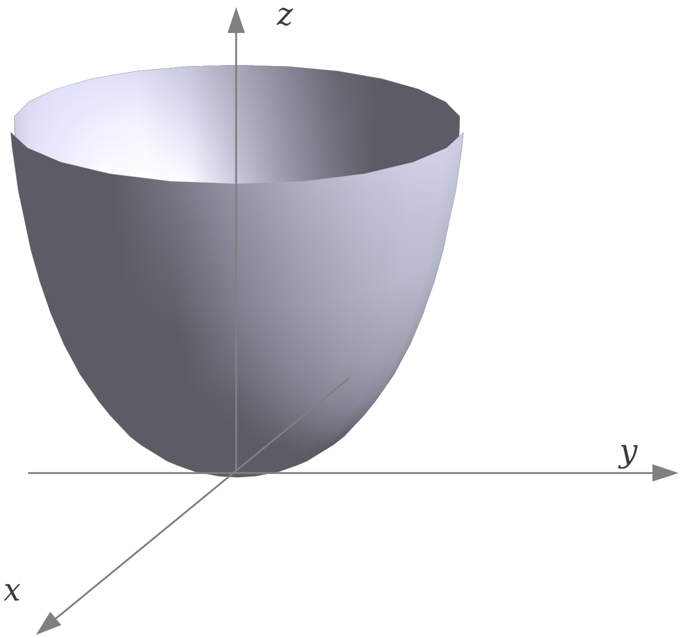


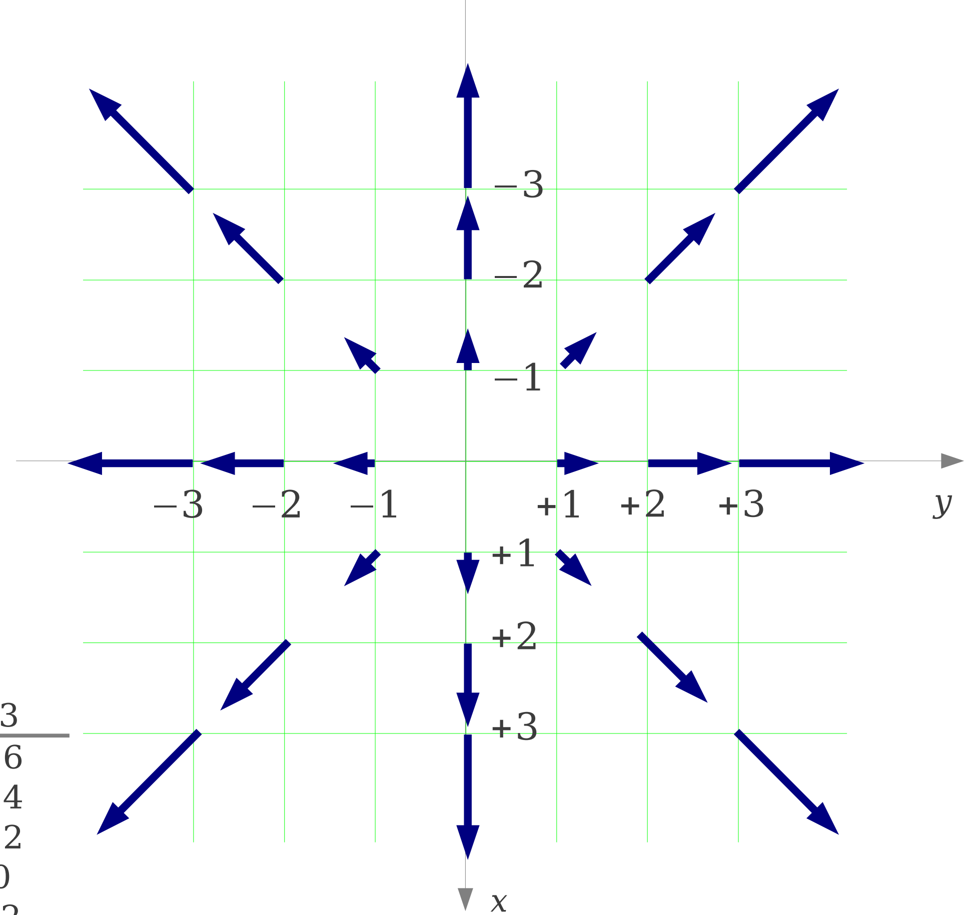


Gradient Example (2)

$$y = f(x, y)$$

$$f(x,y) = x^2 + y^2 \qquad \frac{\partial f}{\partial x} = 2x \qquad \frac{\partial f}{\partial y} = 2y$$

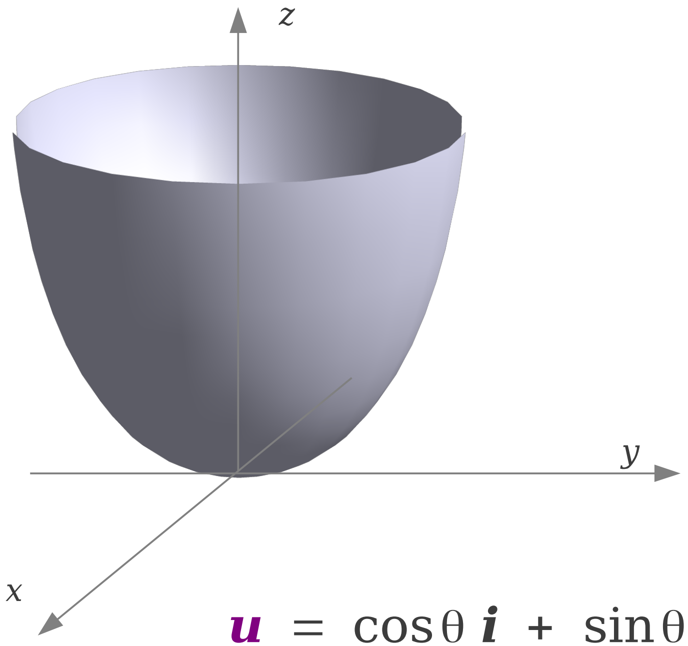




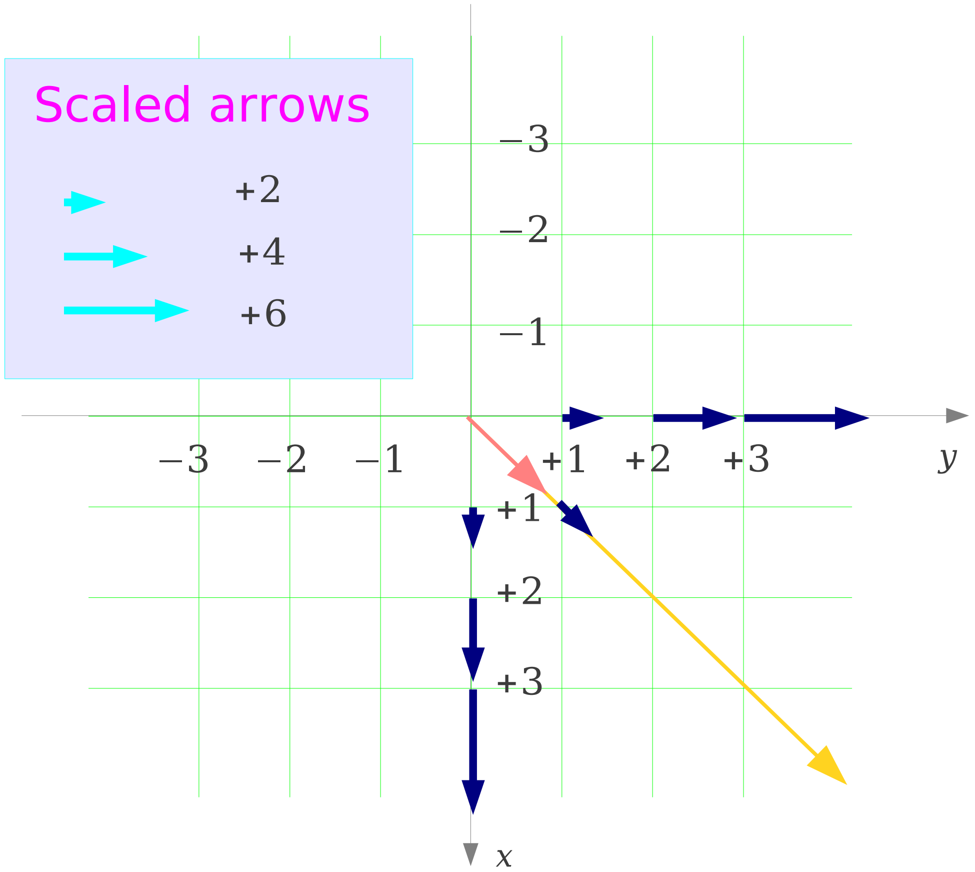
	$x=-3$	$x=-2$	$x=-1$	$x=0$	$x=+1$	$x=+2$	$x=+3$
$y=-3$	−6,−6	−4,−6	−2,−6	0,−6	+2,−6	+4,−6	+6,−6
$y=-2$	−6,−4	−4,−4	−2,−4	0,−4	+2,−4	+4,−4	+6,−4
$y=-1$	−6,−2	−4,−2	−2,−2	0,−2	+2,−2	+4,−2	+6,−2
$y=0$	−6,0	−4,0	−2,0	0,0	+2,0	+4,0	+6,0
$y=+1$	−6,+2	−4,+2	−2,+2	0,+2	+2,+2	+4,+2	+6,+2
$y=+2$	−6,+4	−4,+4	−2,+4	0,+4	+2,+4	+4,+4	+6,+4
$y=+3$	−6,+6	−4,+6	−2,+6	0,+6	+2,+6	+4,+6	+6,+6

Gradient Example (2)

$y = f(x, y)$



$f(x,y) = x^2 + y^2 \qquad \frac{\partial f}{\partial x} = 2x \qquad \frac{\partial f}{\partial y} = 2y$

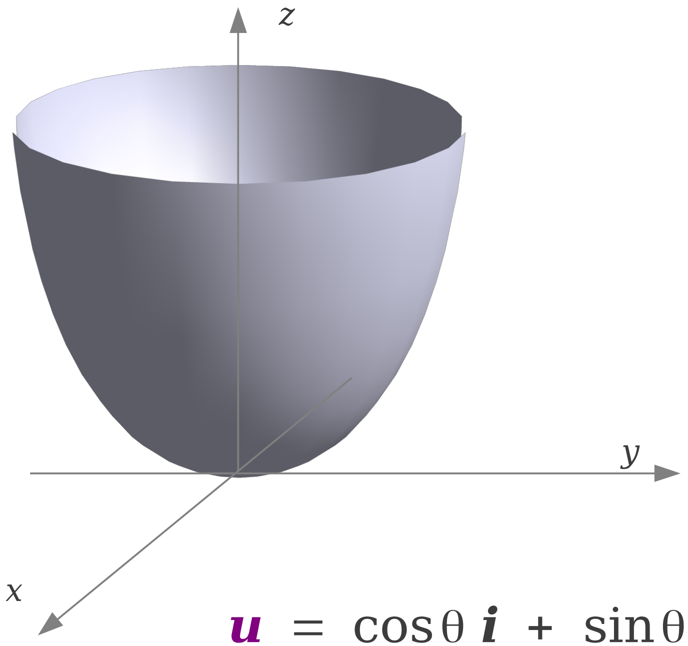


$$\boldsymbol{u} = \cos\theta\, \boldsymbol{i} + \sin\theta\, \boldsymbol{j} = \frac{1}{\sqrt{2}}\boldsymbol{i} + \frac{1}{\sqrt{2}}\boldsymbol{j}$$

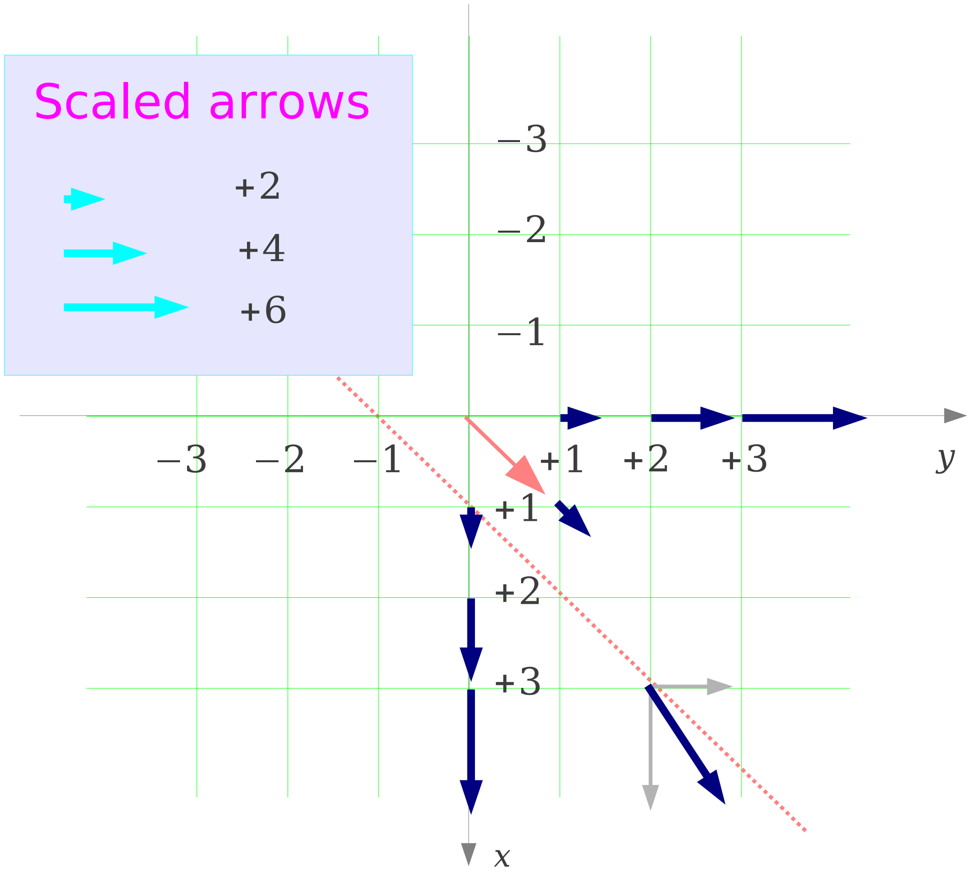
$$D_{\boldsymbol{u}} f(1,1) = \nabla f(1,1) \cdot \boldsymbol{u} = (2,2)\cdot\left(\tfrac{1}{\sqrt{2}}, \tfrac{1}{\sqrt{2}}\right) = 2\sqrt{2}$$

Gradient Example (3)

$$y = f(x, y)$$



$$f(x, y) = x^2 + y^2 \qquad \frac{\partial f}{\partial x} = 2x \qquad \frac{\partial f}{\partial y} = 2y$$

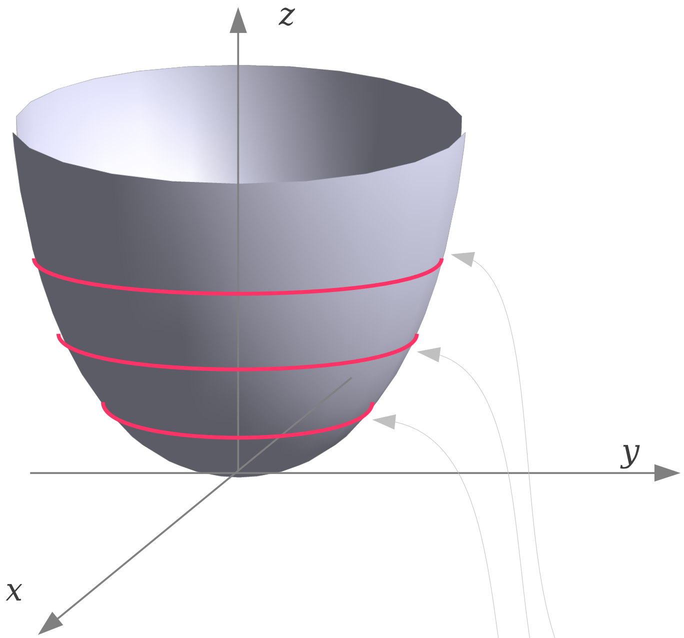


$$\boldsymbol{u} = \cos\theta\, \boldsymbol{i} + \sin\theta\, \boldsymbol{j}$$
$$= \frac{1}{\sqrt{2}} \boldsymbol{i} + \frac{1}{\sqrt{2}} \boldsymbol{j}$$

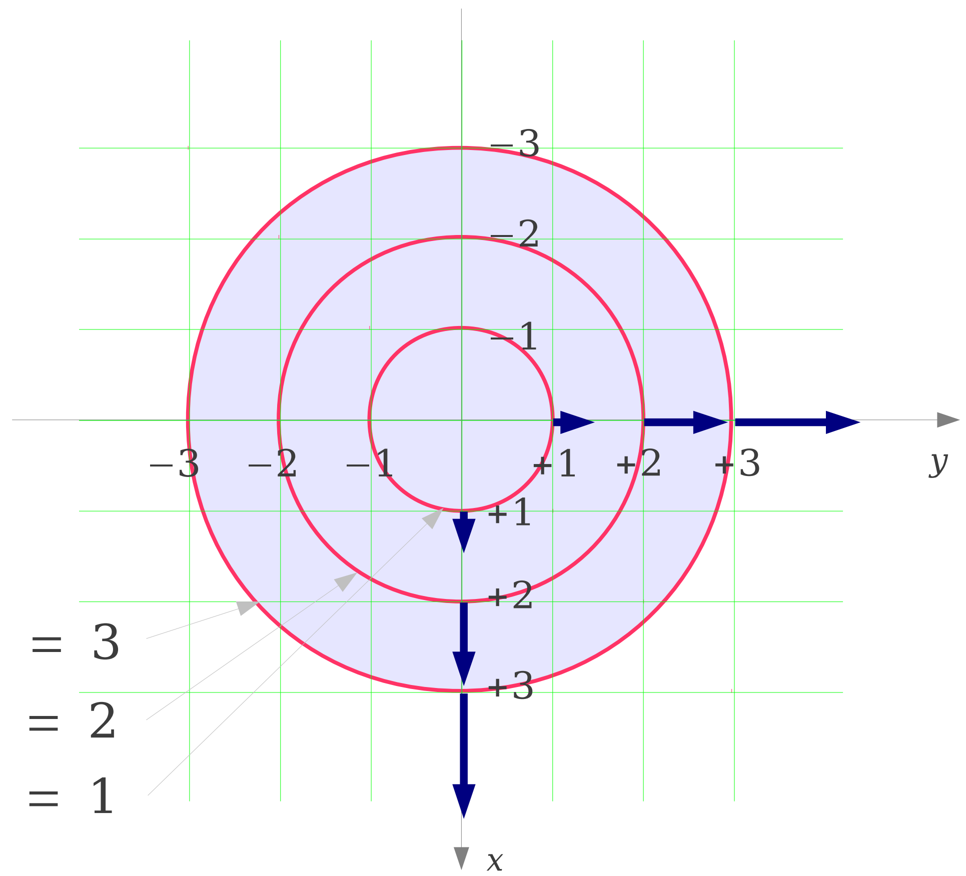
$$D_{\boldsymbol{u}} f(3,2) = \nabla f(3,2) \cdot \boldsymbol{u}$$
$$= (6,4) \cdot \left(\tfrac{1}{\sqrt{2}}, \tfrac{1}{\sqrt{2}}\right)$$
$$= 3\sqrt{2} + 2\sqrt{2} = 5\sqrt{2}$$

Level Curve (1)

$$y = f(x, y)$$



$$f(x,y) = x^2 + y^2 \qquad \frac{\partial f}{\partial x} = 2x \qquad \frac{\partial f}{\partial y} = 2y$$



$$f(x,y) = 3$$

$$f(x,y) = 2$$

$$f(x,y) = 1$$

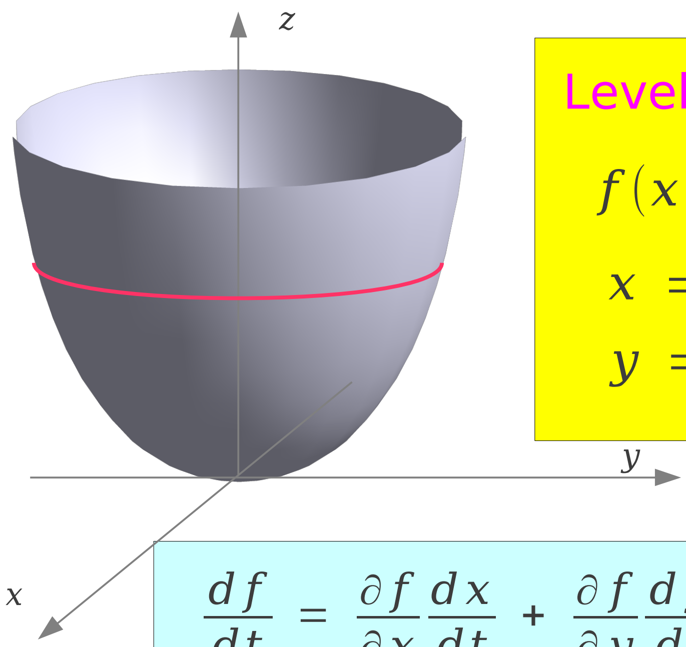
Level Curve

$$f(x,y) = c$$

Level Curve (2)

$$y = f(x, y)$$

$$f(x,y) = x^2 + y^2 \qquad \frac{\partial f}{\partial x} = 2x \qquad \frac{\partial f}{\partial y} = 2y$$

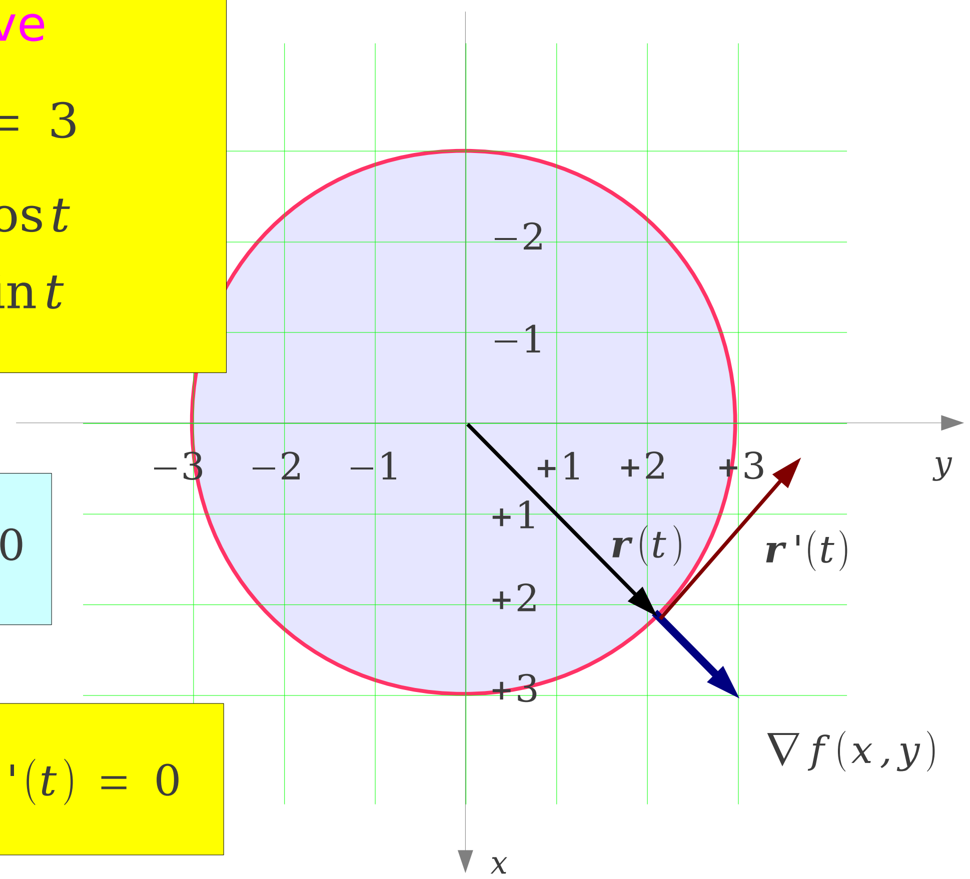


Level Curve

$$f(x,y) = 3$$

$$x = 3\cos t$$

$$y = 3\sin t$$



$$\frac{df}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} = 0$$

$$\begin{cases} \nabla f(x,y) = \dfrac{\partial f}{\partial x}\mathbf{i} + \dfrac{\partial f}{\partial y}\mathbf{j} \\ \mathbf{r}'(t) = \dfrac{dx}{dt}\mathbf{i} + \dfrac{dy}{dt}\mathbf{j} \end{cases}$$

$$\nabla f(x,y) \cdot \mathbf{r}'(t) = 0$$

Level Surface

Function of two variable $f(x, y)$

Level Curve $f(x, y) = c$

$$\begin{cases} x = g(t) \\ y = h(t) \end{cases} \Rightarrow \frac{df}{dt} = \frac{\partial g}{\partial x}\frac{dx}{dt} + \frac{\partial h}{\partial y}\frac{dy}{dt} = 0 \Rightarrow \begin{cases} \nabla f(x,y) = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} \\ \mathbf{r}'(t) = \frac{dx}{dt}\mathbf{i} + \frac{dy}{dt}\mathbf{j} \end{cases}$$

∇f orthogonal to the level curve at P

$$\nabla f(x,y) \cdot \mathbf{r}'(t) = 0$$

Function of three variable $F(x, y, z)$

Level Surface $F(x, y, z) = c$

$$\begin{cases} x = f(t) \\ y = g(t) \\ z = h(t) \end{cases} \Rightarrow \frac{dF}{dt} = 0 \Rightarrow \begin{cases} \nabla F(x,y,z) = \frac{\partial F}{\partial x}\mathbf{i} + \frac{\partial F}{\partial y}\mathbf{j} + \frac{\partial F}{\partial z}\mathbf{k} \\ \mathbf{r}'(t) = \frac{dx}{dt}\mathbf{i} + \frac{dy}{dt}\mathbf{j} + \frac{dz}{dt}\mathbf{k} \end{cases}$$

$$\frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial g}{\partial y}\frac{dy}{dt} + \frac{\partial h}{\partial z}\frac{dz}{dt} = 0$$

∇F normal to the level surface at P

$$\nabla F(x,y,z) \cdot \mathbf{r}'(t) = 0$$

Tangent Plane

Function of three variable $F(x, y, z)$

Level Surface $F(x, y, z) = c$

P0 $F(x_0, y_0, z_0) = c$

$$\nabla F(x_0, y_0, z_0) \cdot (\boldsymbol{r} - \boldsymbol{r}_0) = 0$$

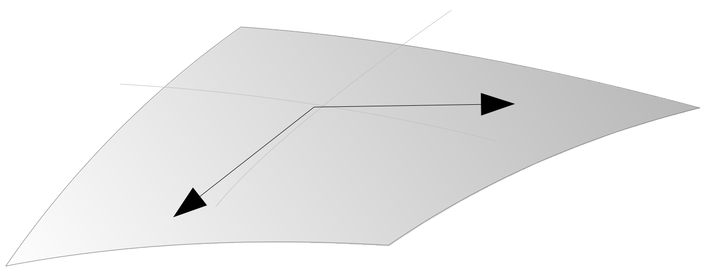
$$\frac{\partial F}{\partial x}(x_0, y_0, z_0)(x - x_0) + \frac{\partial F}{\partial y}(x_0, y_0, z_0)(y - y_0) + \frac{\partial F}{\partial z}(x_0, y_0, z_0)(z - z_0) = 0$$

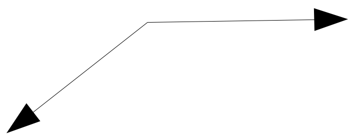
Chain Rule

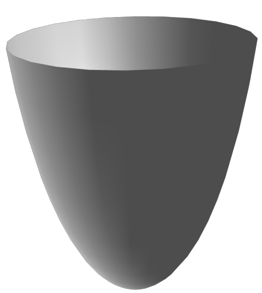
Function of two variable

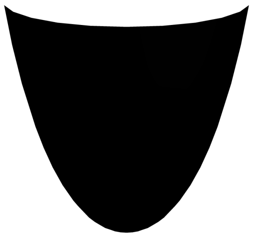
$$y = f(u, v)$$

$$u = g(x, y) \qquad v = h(x, y)$$









References

[1] http://en.wikipedia.org/
[2] http://planetmath.org/
[3] M.L. Boas, “Mathematical Methods in the Physical Sciences”
[4] D.G. Zill, “Advanced Engineering Mathematics”