Reconstructor Spectra (9B)

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Please send corrections (or suggestions) to youngwlim@hotmail.com.

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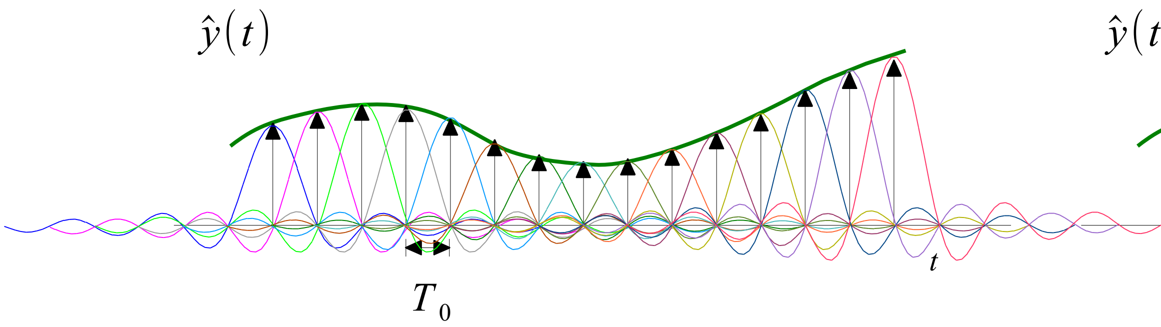
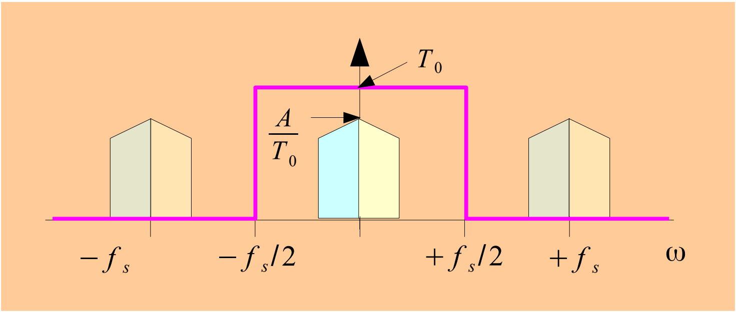
Reconstructors in Frequency Domain

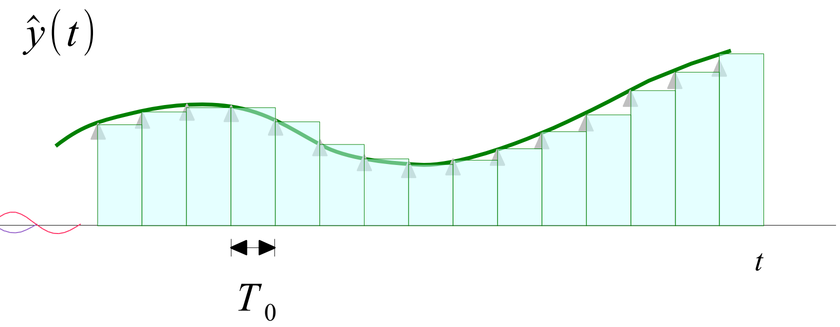
Ideal Reconstructor



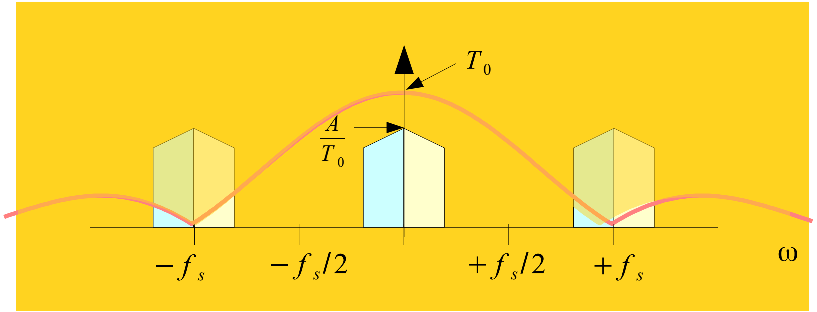
CTFT



Practical Reconstructor

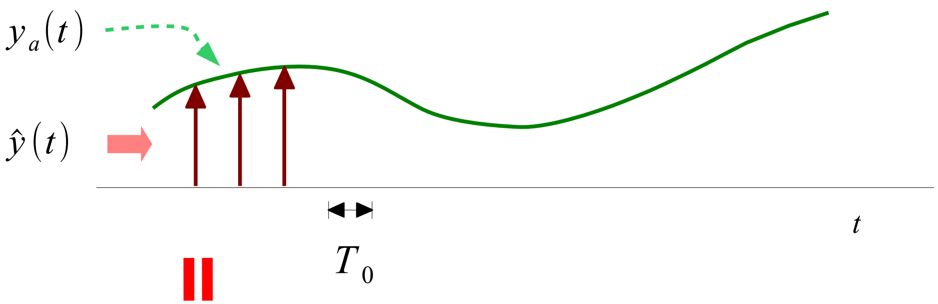


CTFT

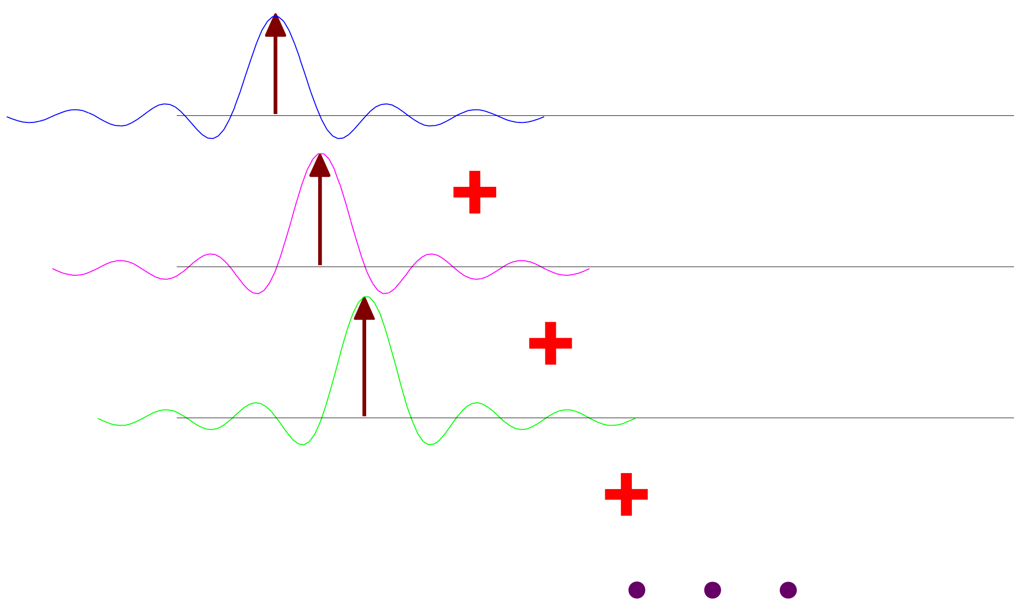


Reconstructors in Time Domain

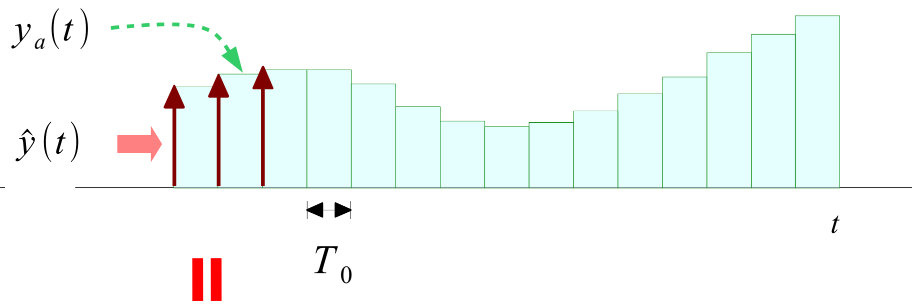
Ideal Reconstructor



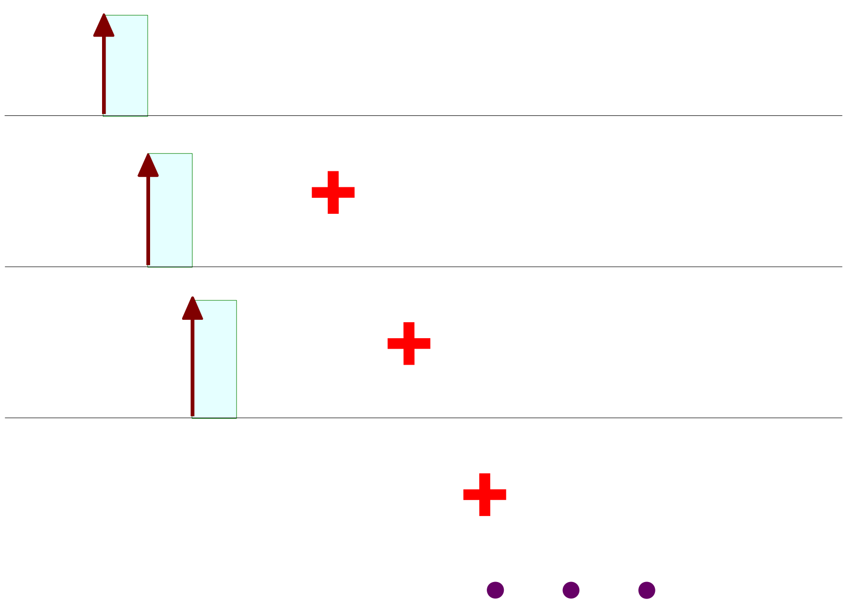
$$y_a(t) = \sum_{n=-\infty}^{\infty} y(nT_0)\, h(t-nT_0)$$



Practical Reconstructor



$$y_a(t) = \sum_{n=-\infty}^{\infty} y(nT_0)\, h(t-nT_0)$$



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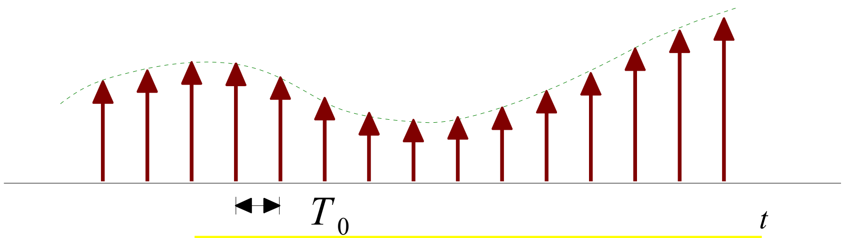
Reconstruct via Convolution

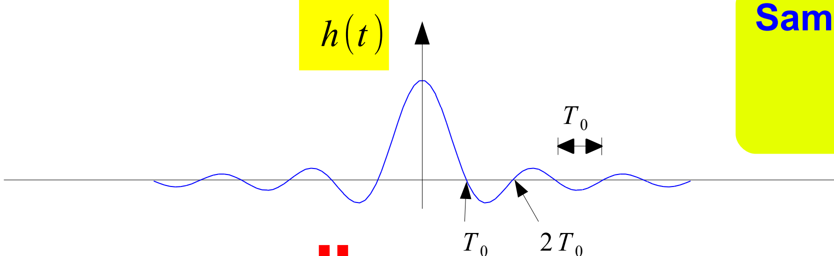
Ideal Reconstructor



$$\hat{y}(t) = \sum_{n=-\infty}^{\infty} y(nT_0)\delta(t - nT_0)$$

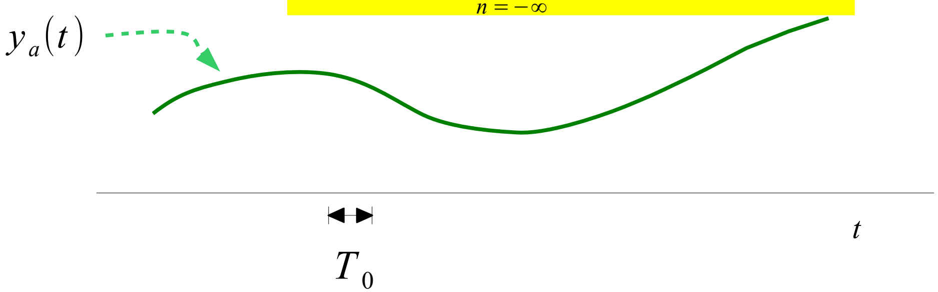
*****

$h(t)$

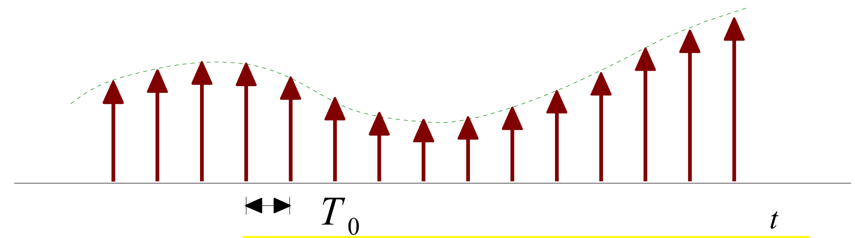


=

$$y_a(t) = \sum_{n=-\infty}^{\infty} y(nT_0)\, h(t-nT_0)$$



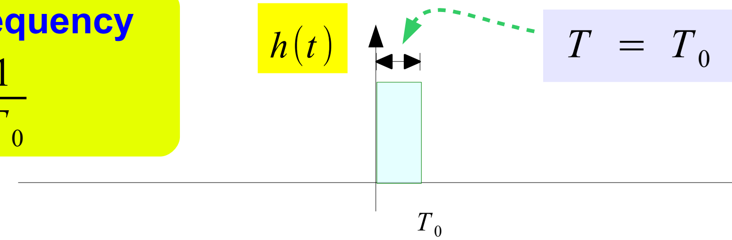
Practical Reconstructor



$$\hat{y}(t) = \sum_{n=-\infty}^{\infty} y(nT_0)\delta(t - nT_0)$$

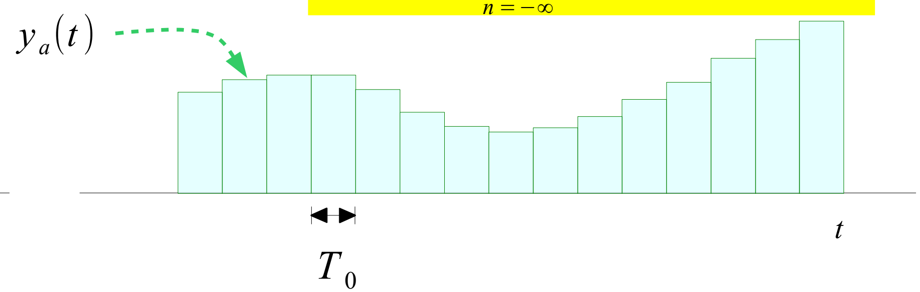
*****

$h(t)$



=

$$y_a(t) = \sum_{n=-\infty}^{\infty} y(nT_0)\, h(t-nT_0)$$

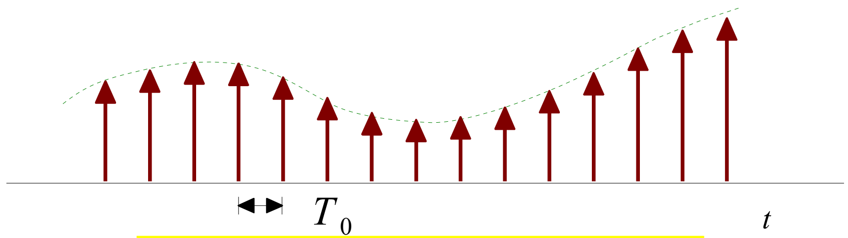


Sampling frequency

$$f_s = \frac{1}{T_0}$$

Reconstructor Frequency Response

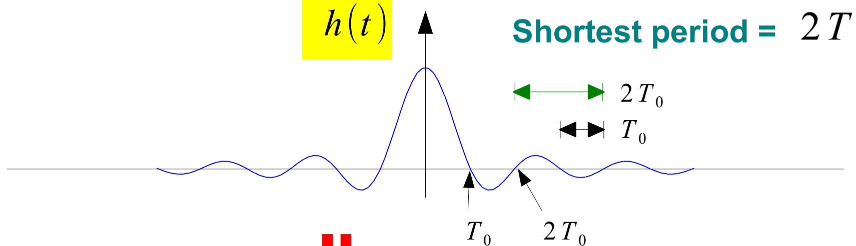
Ideal Reconstructor



$$\hat{y}(t) = \sum_{n=-\infty}^{\infty} y(nT_0)\delta(t - nT_0)$$

*

$h(t)$



Shortest period = $2T_0$

=

Highest frequency = $\frac{1}{2T_0}$

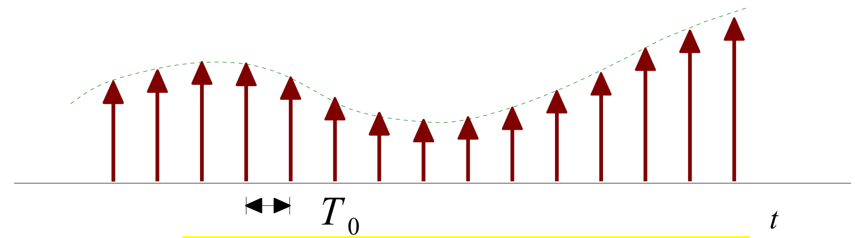
$$\frac{1}{2T_0} \leq \frac{f_s}{2}$$

Nyquist frequency = $f_s = \frac{1}{T_0}$

Sampling frequency

$$f_s = \frac{1}{T_0}$$

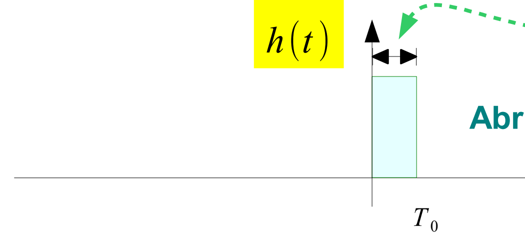
Practical Reconstructor



$$\hat{y}(t) = \sum_{n=-\infty}^{\infty} y(nT_0)\delta(t - nT_0)$$

*

$h(t)$



$T = T_0$

Abrupt Jump at $t = 0, \ T_0$

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Infinity frequency span

Distortion

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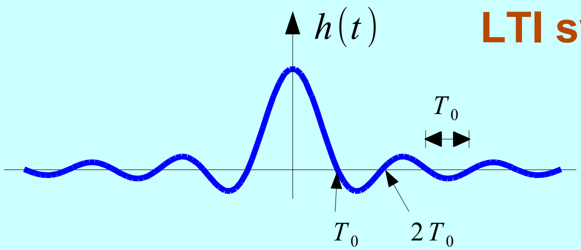
Reconstructor As a LTI System

Ideal Reconstructor

$$\hat{y}(t) = \sum_{n=-\infty}^{\infty} y(nT_0)\,\delta(t - nT_0)$$

input

LTI system



output

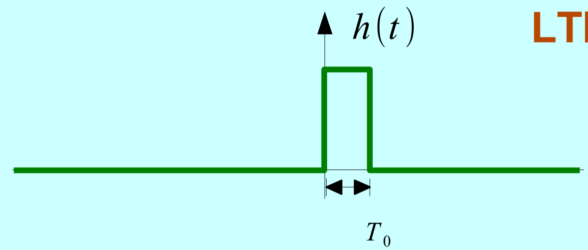
$$y_a(t) = \sum_{n=-\infty}^{\infty} y(nT_0)\,\boldsymbol{h}(t - nT_0)$$

Practical Reconstructor

$$\hat{y}(t) = \sum_{n=-\infty}^{\infty} y(nT_0)\,\delta(t - nT_0)$$

input

LTI system

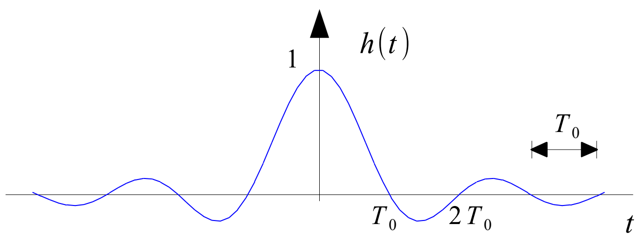


output

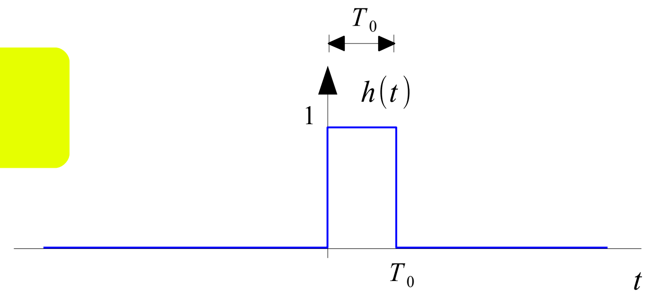
$$y_a(t) = \sum_{n=-\infty}^{\infty} y(nT_0)\,\boldsymbol{h}(t - nT_0)$$

CTFT of Reconstructors (1)

$t = \pm T_0,\ \pm 2T_0,\ \pm 3T_0,\ \cdots \Rightarrow h(t) = 0$



$$\frac{1}{T_0} \equiv f_s$$

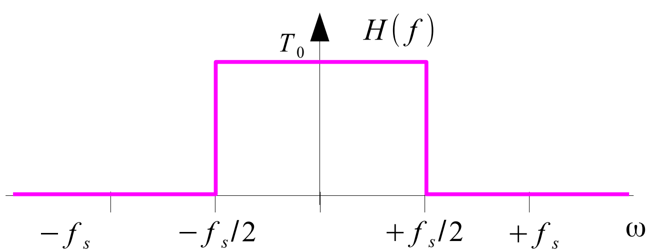


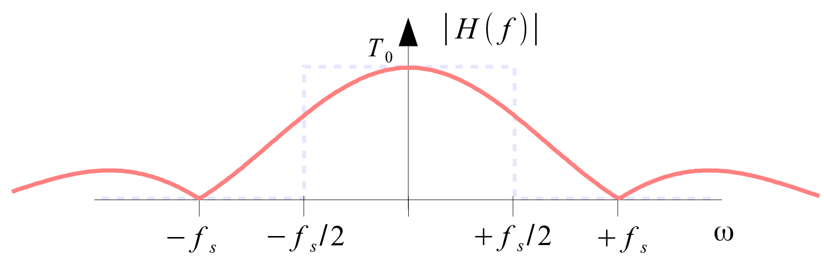
$$h(t) = \frac{\sin(\pi t/T_0)}{\pi t/T_0} = \frac{\sin(\pi f_s t)}{\pi f_s t}$$

$$h(t) = u(t) - u(t-T_0) = \begin{cases} 1, & 0 \le t \le T_0 \\ 0, & otherwise \end{cases}$$

CTFT

CTFT





$$H(f) = \begin{cases} T_0, & |f| \le f_s/2 \\ 0, & otherwise \end{cases}$$

$$H(f) = T_0 \cdot \frac{\sin(\pi f T_0)}{\pi f T_0} e^{-j\pi f T_0}$$

$t = \pm\frac{1}{T_0},\ \pm\frac{2}{T_0},\ \pm\frac{3}{T_0},\ \cdots \Rightarrow H(f) = 0$

CTFT of Reconstructors (2)

$$h(t) = \frac{\sin(\pi t/T_0)}{\pi t/T_0} = \frac{\sin(\pi f_s t)}{\pi f_s t}$$

inverse ↑ **CTFT**

$$h(t) = \frac{1}{2\pi}\int_{-\pi f_s}^{+\pi f_s} T_0 \cdot e^{+j\omega t}\, d\omega$$

$$= \frac{T_0}{2\pi}\left[\frac{1}{jt} e^{+j\omega t}\right]_{-\pi f_s}^{+\pi f_s} = \frac{T_0}{2\pi}\frac{e^{+j\pi f_s t} - e^{-j\pi f_s t}}{jt}$$

$$= \frac{e^{+j\pi f_s t} - e^{-j\pi f_s t}}{2jt\pi/T_0} \quad \pi/T_0 \rightarrow \pi f_s$$

$$= \frac{\sin(\pi f_s t)}{\pi f_s t}$$

$f_s = 1/T_0$ $\quad$ $\pi f = \pi/T_0$

$$H(f) = \begin{cases} T_0, & |f| \le f_s/2 \\ 0, & otherwise \end{cases}$$

$$h(t) = u(t) - u(t - T_0) = \begin{cases} 1, & 0 \le t \le T_0 \\ 0, & otherwise \end{cases}$$

↓ **CTFT**

$$H(j\omega) = \int_0^{T_0} 1 \cdot e^{-j\omega t}\, dt$$

$$= \left[\frac{-1}{j\omega} e^{-j\omega t}\right]_0^{T_0} = -\frac{e^{-j\omega T_0} - 1}{j\omega}$$

$$= e^{-j\omega T_0/2} \cdot \left(\frac{e^{+j\omega T_0/2} - e^{-j\omega T_0/2}}{j\omega}\right) \quad j\omega \rightarrow 2j\omega/2$$

$$= \frac{\sin(\omega T_0/2)}{\omega/2} \cdot e^{-j\omega T_0/2} \qquad \times \frac{T_0}{T_0}$$

$\omega = 2\pi f$ $\quad$ $\omega/2 = \pi f$

$$H(f) = T_0 \cdot \frac{\sin(\pi f T_0)}{\pi f T_0} e^{-j\pi f T_0}$$

CTFT of Reconstructors (3)

$$h(t) = \frac{\sin(\pi t/T_0)}{\pi t/T_0} = \frac{\sin(\pi f_s t)}{\pi f_s t}$$

inverse CTFT $f_s = 1/T_0$

$$H(f) = \begin{cases} T_0, & |f| \le f_s/2 \\ 0, & otherwise \end{cases}$$

$$h(t) = u(t) - u(t-T_0) = \begin{cases} 1, & 0 \le t \le T_0 \\ 0, & otherwise \end{cases}$$

CTFT $\omega = 2\pi f$

$$H(f) = T_0 \cdot \frac{\sin(\pi f T_0)}{\pi f T_0} e^{-j\pi f T_0}$$

$$h(t) = \frac{\sin(\pi f_s t)}{\pi f_s t}$$

$$\lim_{t \to 0} \frac{\sin(\pi f_s t)}{\pi f_s t} = \lim_{t \to 0} \frac{\pi f_s \cos(\pi f_s t)}{\pi f_s}$$

$$h(0) = 1$$

$$H(f) = T_0 \cdot \frac{\sin(\pi f T_0)}{\pi f T_0} e^{-j\pi f T_0}$$

$$\lim_{f \to 0} \frac{\sin(\pi f T_0)}{\pi f T_0} = \lim_{f \to 0} \frac{\pi T_0 \sin(\pi f T_0)}{\pi T_0}$$

$$H(0) = T_0$$

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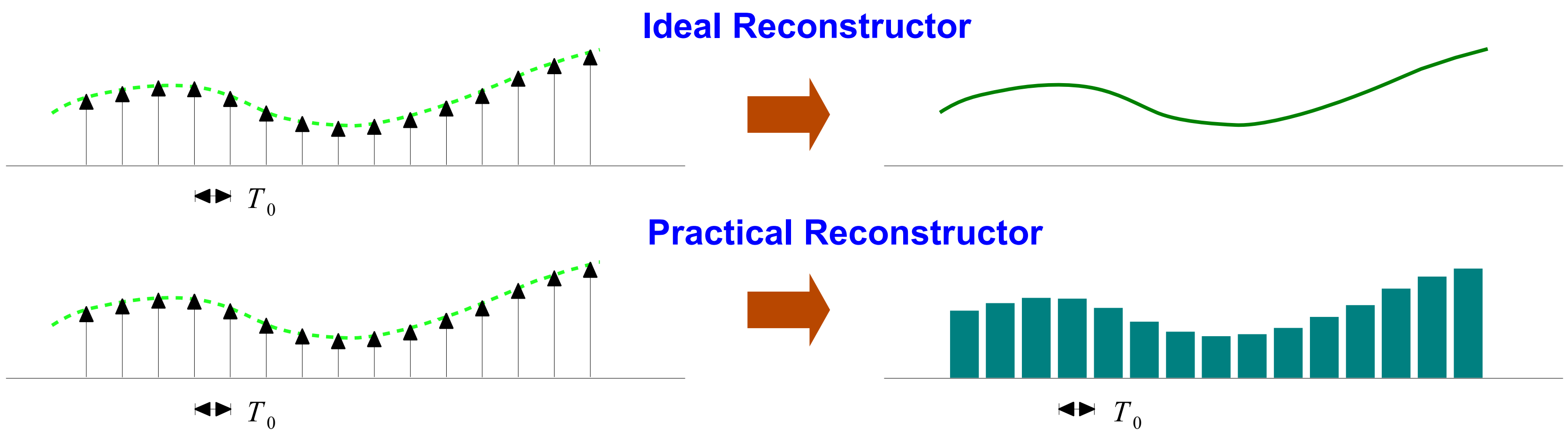
Analog Reconstructor (1)



$$y_a(t) = \int_{-\infty}^{+\infty} h(t-t')\hat{y}(t')\,dt'$$

$$\hat{y}(t) = \sum_{n=-\infty}^{+\infty} y(nT_0)\,\delta(t-nT_0)$$

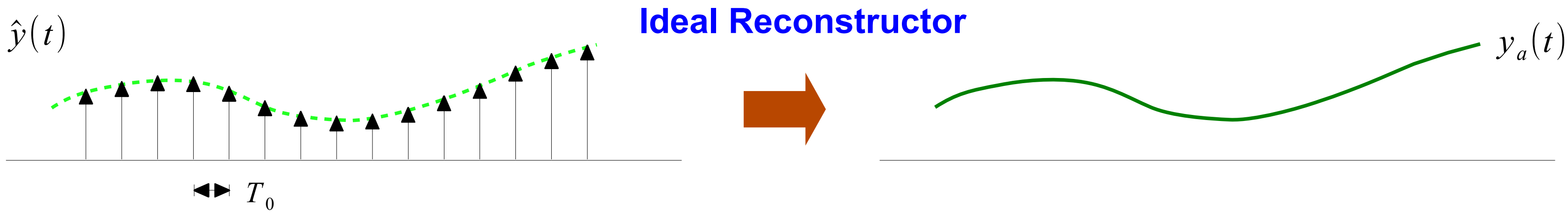
Analog Reconstructor
h(t)

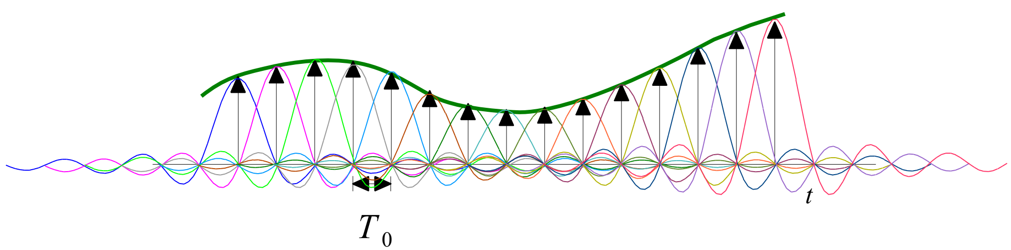
$$y_a(t) = \sum_{n=-\infty}^{+\infty} y(nT_0)h(t-nT_0)$$

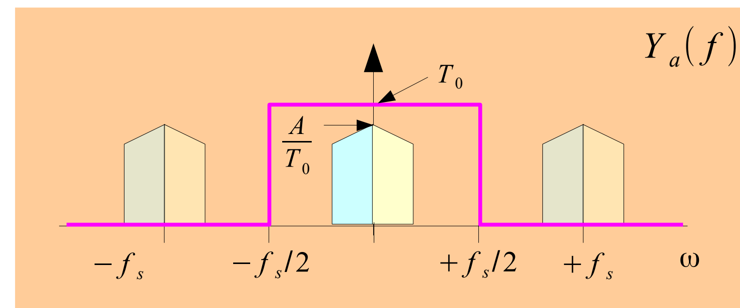
$$\hat{Y}(f) = \frac{1}{T}\sum_{m=-\infty}^{+\infty} Y(f-mf_s)$$

$$Y_a(f) = H(f)\hat{Y}(f)$$

Analog Reconstructor (2)







$$h(t) = \frac{\sin(\pi t / T_0)}{\pi t / T_0} = \frac{\sin(\pi f_s t)}{\pi f_s t}$$

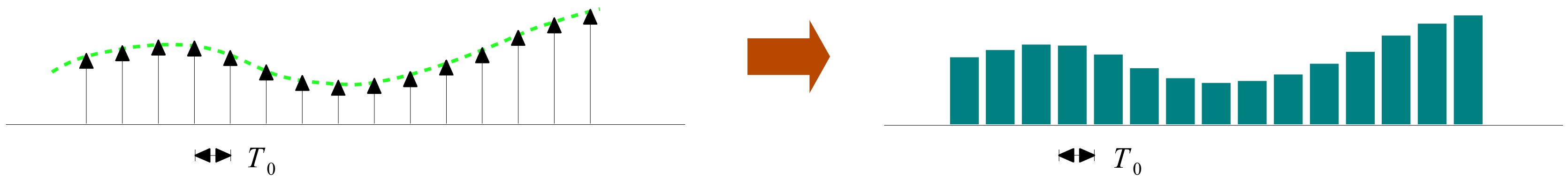
$$\hat{Y}(f) = \frac{1}{T_0} Y(f) \qquad -\frac{f_s}{2} \le f \le +\frac{f_s}{2}$$

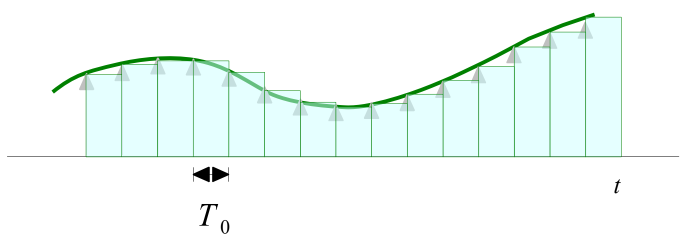
$$Y_a(f) = H(f)\,\hat{Y}(f)$$

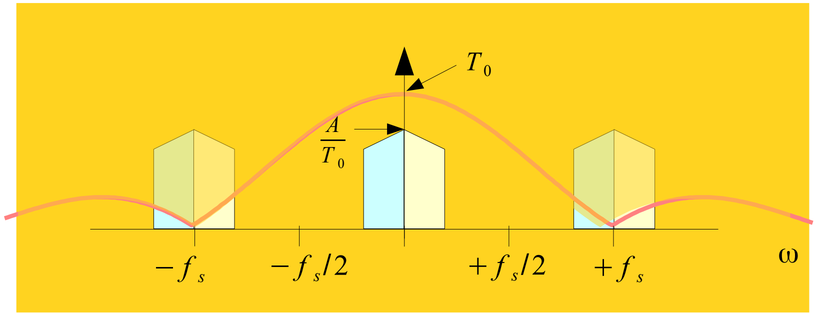
$$= T_0 \cdot \frac{Y(f)}{T_0} = Y(f)$$

Analog Reconstructor (3)

Practical Reconstructor





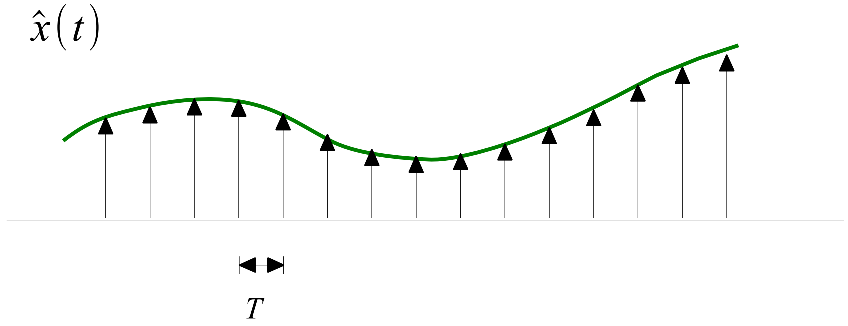


$$\hat{Y}(f) = \frac{1}{T_0} Y(f) \qquad -\frac{f_s}{2} \le f \le +\frac{f_s}{2}$$

$$Y_a(f) = H(f)\,\hat{Y}(f)$$

$$= T_0 \cdot \frac{Y(f)}{T_0} \left| \frac{\sin(\pi f T_0)}{\pi f T_0} \right|$$

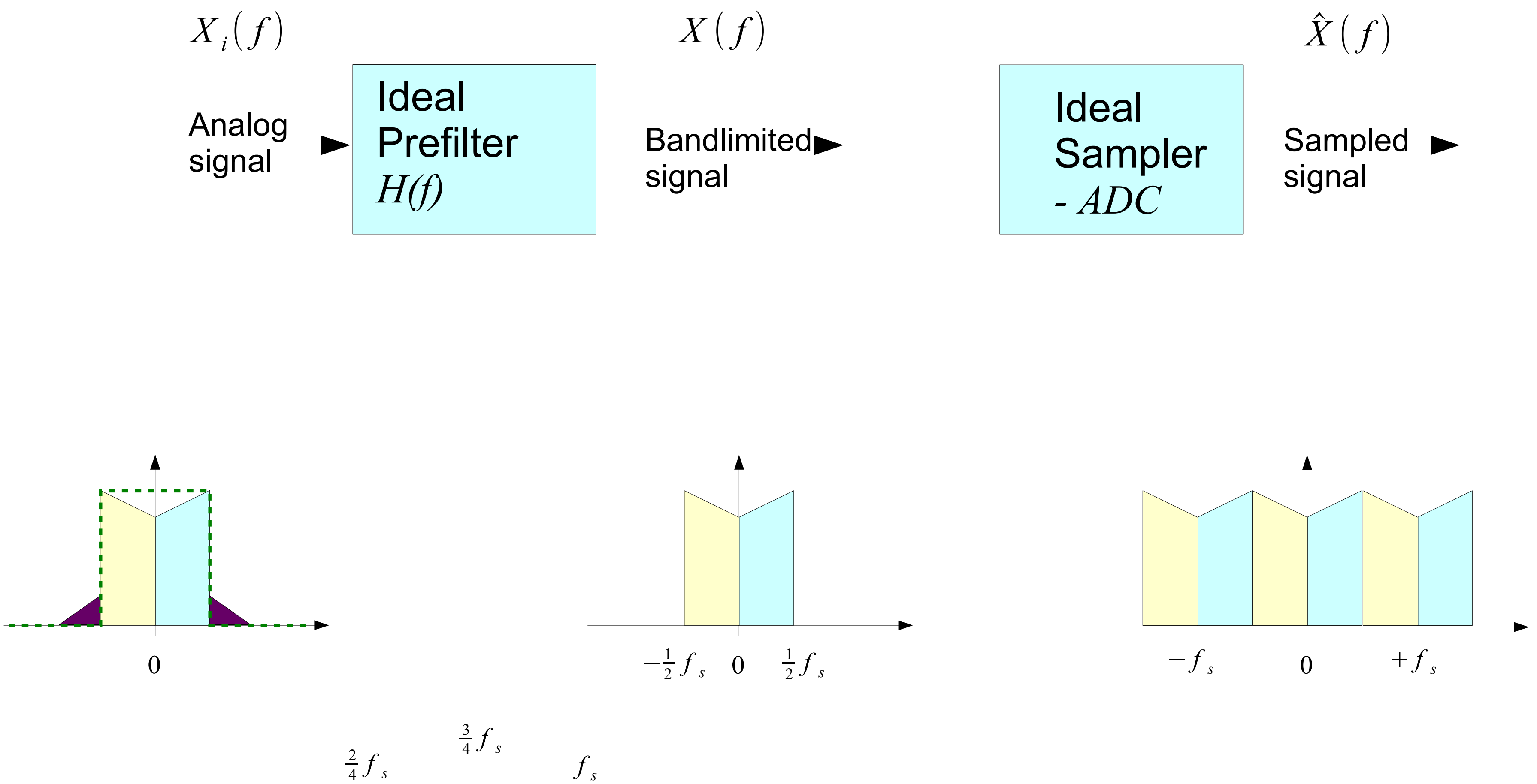
Impulse Response of Ideal Reconstructor



$$\hat{Y}(f) = \frac{1}{T} Y(f) \qquad -\frac{f_s}{2} \le f \le +\frac{f_s}{2}$$

$$y(t) = \sum_{n=-\infty}^{+\infty} y(nT) h(t-nT)$$

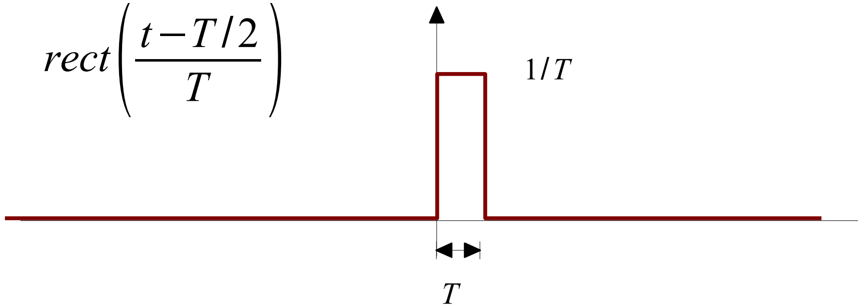
$$h(t) = \frac{\sin(\pi t/T)}{\pi t/T} = \frac{\sin(\pi f_s t)}{\pi f_s t}$$

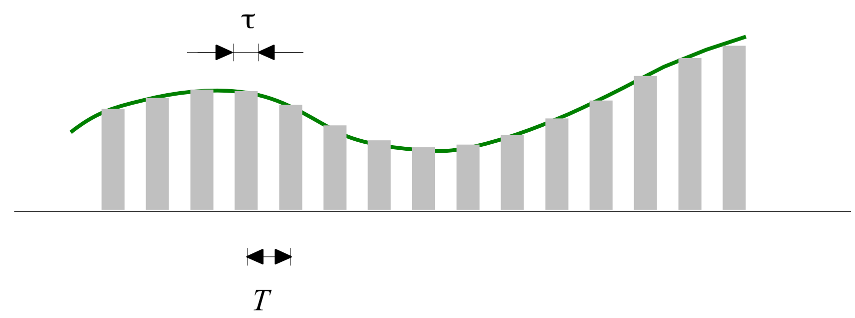


$X_i(f)$
Analog signal
Ideal Prefilter $H(f)$
$X(f)$
Bandlimited signal
Ideal Sampler - ADC
$\hat{X}(f)$
Sampled signal
0
$-\frac{1}{2}f_s$
0
$\frac{1}{2}f_s$
$-f_s$
0
$+f_s$
$\frac{2}{4}f_s$
$\frac{3}{4}f_s$
f_s

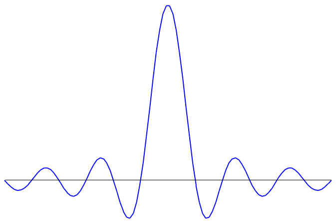
Zero Order Hold (ZOH)

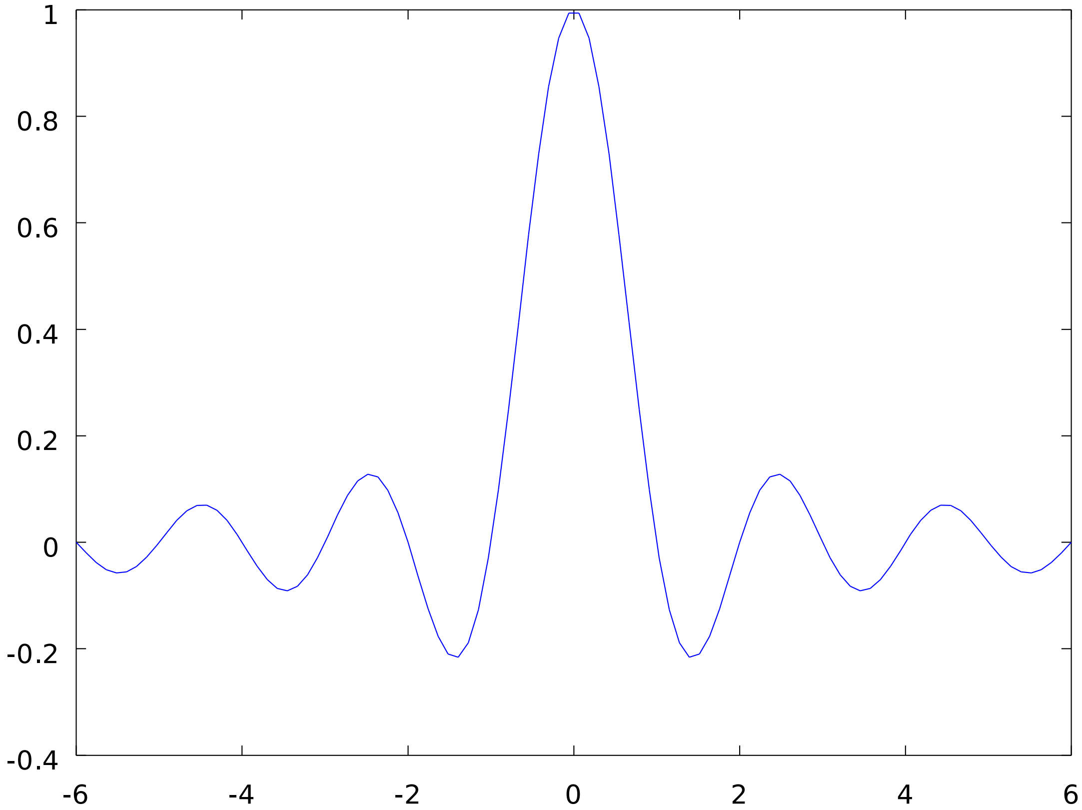
$$rect\left(\frac{t-T/2}{T}\right)$$





$$x_{ZOH}(t) = \sum_{n=-\infty}^{+\infty} x[n] \cdot rect\left(\frac{t-T/2-nT}{T}\right)$$




1
0.8
0.6
0.4
0.2
0
-0.2
-0.4
-6
-4
-2
0
2
4
6

References

[1] http://en.wikipedia.org/
[2] J.H. McClellan, et al., Signal Processing First, Pearson Prentice Hall, 2003
[3] A "graphical interpretation" of the DFT and FFT, by Steve Mann
[4] R. G. Lyons, Understanding Digital Signal Processing, 1997
[5] AVR121: Enhancing ADC resolution by oversampling
[6] S.J. Orfanidis, Introduction to Signal Processing
www.ece.rutgers.edu/~orfanidi/intro2sp