Spectra (1A)

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Single-Sided Spectrum

$$x(t) = a_0 + \sum_{k=1}^{\infty} \left(a_k \cos(k\omega_0 t) + b_k \sin(k\omega_0 t) \right)$$

$$x(t) = g_0 + \sum_{k=1}^{\infty} g_k \cos(k\omega_0 t + \phi_k)$$

$$a_0 = \frac{1}{T} \int_0^T x(t)\, dt$$

$$a_k = \frac{2}{T} \int_0^T x(t) \cos(k\omega_0 t)\, dt$$

$$b_k = \frac{2}{T} \int_0^T x(t) \sin(k\omega_0 t)\, dt$$

$$k = +1, +2, \dots$$

$$g_0 = a_0$$

$$g_k = \sqrt{a_k^2 + b_k^2}$$

$$\phi_k = \tan^{-1}\left(-\frac{b_k}{a_k} \right)$$

$$k = +1, +2, \dots$$

$$\cos(\alpha + \beta) = \underline{\cos(\alpha)\cos(\beta)} - \underline{\sin(\alpha)\sin(\beta)}$$

$$g_k \cos(k\omega_0 t + \phi_k) = \underline{g_k \cos(\phi_k)} \cos(k\omega_0 t) - \underline{g_k \sin(\phi_k)} \sin(k\omega_0 t)$$

$$\underline{} a_k \cos(k\omega_0 t) + \underline{} b_k \sin(k\omega_0 t)$$

$$a_k = g_k \cos(\phi_k)$$
$$-b_k = g_k \sin(\phi_k)$$

$$a_k^2 + b_k^2 = g_k^2$$

$$-\frac{b_k}{a_k} = \tan(\phi_k)$$

Two-Sided Spectrum

$$x(t) = \sum_{k=-\infty}^{+\infty} C_k \, e^{+jk\omega_0 t}$$

$$x(t) = \sum_{k=-\infty}^{+\infty} C_k \, e^{+jk\omega_0 t}$$

$$C_k = \frac{1}{T} \int_0^T x(t) \, e^{-jk\omega_0 t} \, dt$$

$$k = \dots, -2, -1, 0, +1, +2, \dots$$

$$C_k = \frac{1}{T} \int_0^T x(t) \, e^{-jk\omega_0 t} \, dt$$

$$k = \dots, -2, -1, 0, +1, +2, \dots$$

$$C_k = \begin{cases} a_0 & (k = 0) \\ \frac{1}{2}(a_k - jb_k) & (k > 0) \\ \frac{1}{2}(a_k + jb_k) & (k < 0) \end{cases}$$

$$C_k = \begin{cases} a_0 & (k = 0) \\ \frac{1}{2} g_k \, e^{+j\phi_k} & (k > 0) \\ \frac{1}{2} g_k \, e^{-j\phi_k} & (k < 0) \end{cases}$$

$$|C_k| = \begin{cases} a_0 & (k = 0) \\ \frac{1}{2}\sqrt{a_k^2 + b_k^2} & (k \ne 0) \end{cases}$$

$$Arg(C_k) = \begin{cases} \tan^{-1}(-b_k / a_k) & (k > 0) \\ \tan^{-1}(+b_k / a_k) & (k < 0) \end{cases}$$

$$|C_k| = \begin{cases} a_0 & (k = 0) \\ \frac{1}{2}|g_k| & (k \ne 0) \end{cases}$$

$$Arg(C_k) = \begin{cases} +\phi_k & (k > 0) \\ -\phi_k & (k < 0) \end{cases}$$

Power Spectrum   *Two-Sided*

$$|C_k|^2 + |C_{-k}|^2 = \frac{1}{2} g_k^2 = \frac{1}{2}(a_k^2 + b_k^2)$$

Periodogram   *One-Sided*

$$2 \cdot |C_k| = g_k = \sqrt{a_k^2 + b_k^2}$$

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Power Spectrum (1)

$$x(t) = a_0 + \sum_{n=1}^{\infty} \left(a_k \cos(k\omega_0 t) + b_k \sin(k\omega_0 t) \right)$$

$$x(t) = \sum_{k=-\infty}^{+\infty} C_k e^{+jk\omega_0 t}$$

$$a_0 = \frac{1}{T} \int_0^T x(t)\, dt$$

$$a_k = \frac{2}{T} \int_0^T x(t) \cos(k\omega_0 t)\, dt$$

$$b_k = \frac{2}{T} \int_0^T x(t) \sin(k\omega_0 t)\, dt$$

$$k = 1, 2, \ldots$$

$$C_k = \frac{1}{T} \int_0^T x(t)\, e^{-jk\omega_0 t}\, dt$$

$$k = -2, -1, 0, +1, +2, \ldots$$

$$C_k = \begin{cases} a_0 & (k = 0) \\ \frac{1}{2}(a_k - jb_k) & (k > 0) \\ \frac{1}{2}(a_k + jb_k) & (k < 0) \end{cases}$$

Average Power $\quad P = \dfrac{1}{T} \int_0^T |x(t)|^2\, dt$

$$P = a_0^2 + \frac{1}{2} \sum_{k=1}^{+\infty} \left(a_k^2 + b_k^2 \right)$$

Power Spectrum

$$P_k = \begin{cases} a_0^2 & (k = 0) \\ \frac{1}{2}\left(a_k^2 + b_k^2 \right) & (k > 0) \end{cases}$$

Average Power $\quad P = \dfrac{1}{T} \int_0^T |x(t)|^2\, dt$

$$P = |C_0|^2 + \sum_{k=1}^{+\infty} \left(|C_k|^2 + |C_{-k}|^2 \right)$$

Power Spectrum

$$P_k = \begin{cases} |C_0|^2 & (k = 0) \\ \left(|C_k|^2 + |C_{-k}|^2 \right) & (k > 0) \end{cases}$$

Power Spectrum (2)

$$x(t) = a_0 + \sum_{n=1}^{\infty} \left(a_n \cos(n\omega_0 t) + b_n \sin(n\omega_0 t) \right)$$

$$x(t) = \sum_{k=-\infty}^{+\infty} C_k \, e^{+jk\omega_0 t}$$

Average Power $P = \dfrac{1}{T} \int_0^T |x(t)|^2 \, dt$

Average Power $P = \dfrac{1}{T} \int_0^T |x(t)|^2 \, dt$

$$P = a_0^2 + \frac{1}{2} \sum_{k=1}^{+\infty} \left(a_k^2 + b_k^2 \right)$$

$$P = |C_0|^2 + \sum_{k=1}^{+\infty} \left(|C_k|^2 + |C_{-k}|^2 \right)$$

$$|C_k|^2 + |C_{-k}|^2 = \tfrac{1}{2} \left(a_k^2 + b_k^2 \right)$$

$$|C_k|^2 = \tfrac{1}{4} \left(a_k^2 + b_k^2 \right)$$
$$|C_{-k}|^2 = \tfrac{1}{4} \left(a_k^2 + b_k^2 \right)$$

$$C_k = \begin{cases} a_0 & (k = 0) \\ \tfrac{1}{2}(a_k - jb_k) & (k > 0) \\ \tfrac{1}{2}(a_k + jb_k) & (k < 0) \end{cases}$$

Power Spectrum

$$P_k = \begin{cases} a_0^2 & (k = 0) \\ \dfrac{1}{2}\left(a_k^2 + b_k^2 \right) & (k > 0) \end{cases}$$

Power Spectrum

$$P_k = \begin{cases} |C_0|^2 & (k = 0) \\ \left(|C_k|^2 + |C_{-k}|^2 \right) & (k > 0) \end{cases}$$

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Periodogram

$$x(t) = a_0 + \sum_{n=1}^{\infty} \left(a_n \cos(n\omega_0 t) + b_n \sin(n\omega_0 t) \right)$$

An estimate of the spectral density

$$a_0 = \frac{1}{T} \int_0^T x(t)\, dt$$

$$a_n = \frac{2}{T} \int_0^T x(t) \cos(n\omega_0 t)\, dt$$

$$b_n = \frac{2}{T} \int_0^T x(t) \sin(n\omega_0 t)\, dt$$

$$n = 1, 2, \ldots$$

$$\frac{T}{2} a_n = \int_{t_1}^{t_1+T} x(t) \cos(kt)\, dt$$

$$\frac{T}{2} b_n = \int_{t_1}^{t_1+T} x(t) \sin(kt)\, dt$$

T: integer multiple $\dfrac{2\pi}{k}$

$$T = \frac{2\pi}{k} \cdot n$$

$$k = \frac{2\pi}{T} \cdot n = n\omega_0$$

Positive Frequency Only

$$f = \frac{n}{T} \qquad n = 1, 2, \ldots$$

Single-sided Spectrum

$$\sqrt{a_n^2 + b_n^2} \qquad sometimes \ \left(a_n^2 + b_n^2 \right)$$

abscissa $\dfrac{2\pi}{k}$

ordinates $r = \sqrt{a_n^2 + b_n^2}$

Periodogram and Power Spectrum

Single-Sided
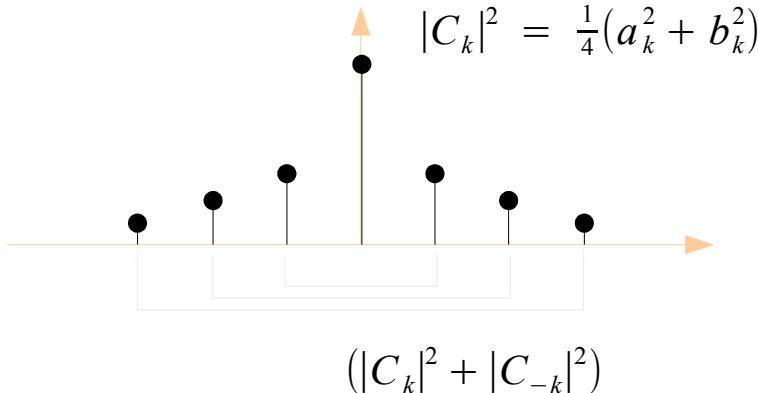
$$|C_k|^2 = \tfrac{1}{4}(a_k^2 + b_k^2)$$

$$(|C_k|^2 + |C_{-k}|^2)$$

Power Spectrum   Two-Sided

$$P_{avg} = (V_{rms})^2$$

$$\tfrac{1}{2}(a_k^2 + b_k^2)$$

$$|C_k|^2 + |C_{-k}|^2 = \tfrac{1}{2}|g_k|^2 = \tfrac{1}{2}(a_k^2 + b_k^2)$$

$$|C_k| = \tfrac{1}{2}\sqrt{a_k^2 + b_k^2}$$

$$(|C_k| + |C_{-k}|)$$

Periodogram

$$V_{peak}$$

$$\sqrt{a_k^2 + b_k^2}$$

$$|C_k| + |C_{-k}| = |g_k|^2 = \sqrt{a_k^2 + b_k^2}$$

Root Mean Square

$|g[k]|^2 \Delta$

$N \Delta$

Δ
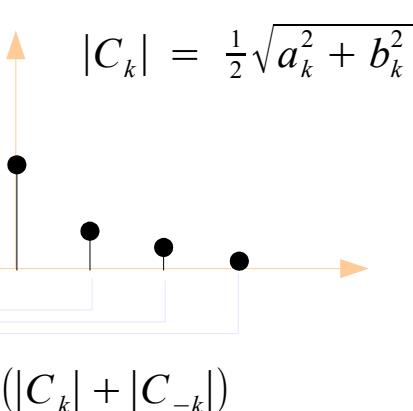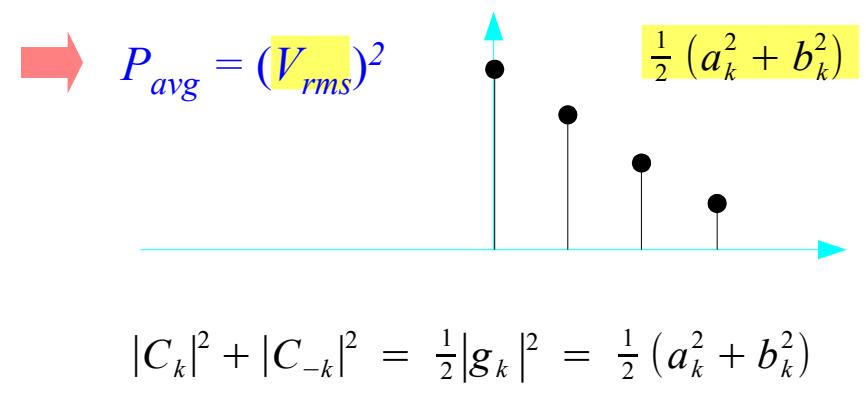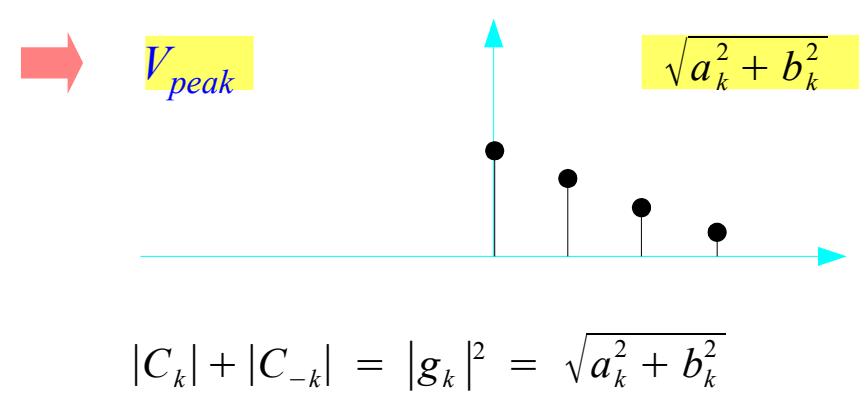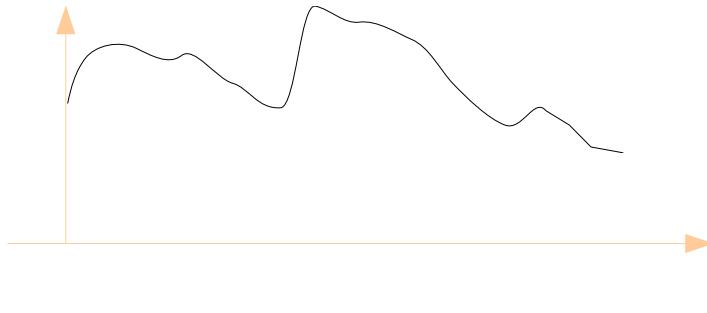
mean squared amplitude

$$\frac{1}{T} \int_0^T g^2(t) \, dt$$

mean squared amplitude

$$\frac{1}{N\Delta} \sum_{k=0}^{N-1} |g[k]|^2 \Delta = \frac{1}{N} \sum_{k=0}^{N-1} |g[k]|^2$$

root mean squared

$$g_{rms} = \sqrt{\frac{1}{T} \int_0^T g^2(t) \, dt}$$

root mean squared

$$g_{rms} = \frac{1}{N} \sum_{k=0}^{N-1} |g[k]|^2$$

CTFS and CTFT

**Continuous Time Fourier <u>Series</u>**

$$C_k = \frac{1}{T} \int_0^T x(t)\, e^{-jk\omega_0 t}\, dt \qquad \Longleftrightarrow \qquad x(t) = \sum_{k=0}^{\infty} C_k\, e^{+jk\omega_0 t}$$

Continuous Time Fourier <u>Transform</u>

$$X(j\omega) = \int_{-\infty}^{+\infty} x(t)\, e^{-j\omega t}\, dt \qquad \Longleftrightarrow \qquad x(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} X(j\omega)\, e^{+j\omega t}\, d\omega$$

$$X(f) = \int_{-\infty}^{+\infty} x(t)\, e^{-j2\pi f t}\, dt \qquad \Longleftrightarrow \qquad x(t) = \int_{-\infty}^{+\infty} X(f)\, e^{+j2\pi f t}\, df$$

Continuous Periodic Signal
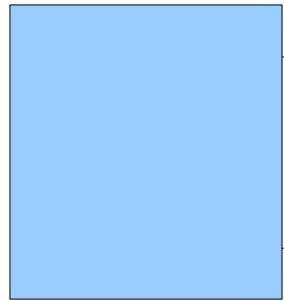
instantaneous power $\quad x^2(t)$

$1\,\Omega \qquad v(t) = x(t)$

average power $\qquad \dfrac{1}{T}\displaystyle\int_0^T x^2(t)\,dt$

$T\,:\,period$

$v(t) = x(t)$ Continuous Periodic

➡   CTFS (Fourier Series)   Parseval's Theorem

$$x(t) = \sum_{k=-\infty}^{+\infty} C_k\, e^{+jk\omega_0 t}$$

$$C_k = \frac{1}{T}\int_0^T x(t)\, e^{-jk\omega_0 t}\, dt$$

$$k = \ldots,\ -2,\ -1,\ 0,\ +1,\ +2,\ \ldots$$

$$\frac{1}{T}\int_0^T x^2(t)\,dt \;=\; \sum_{n=-\infty}^{+\infty} |C_n|^2$$

average power   sum of

power spectrum $\;C_n$

Continuous Aperiodic Signal

instantaneous power $\quad x^2(t)$

$1\,\Omega \quad v(t) = x(t)$

$+$

$-$
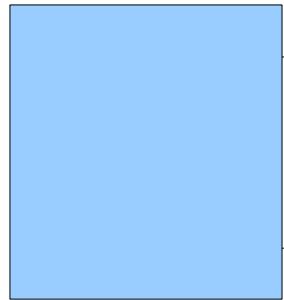
total energy $\quad \displaystyle\int_{-\infty}^{+\infty} x^2(t)\, dt$

$v(t) = x(t)$ Continuous Aperiodic

CTFT (Fourier Integral)

Parseval's Theorem

$$x(t) = \int_{-\infty}^{+\infty} X(f)\, e^{+j2\pi f t}\, df$$

$$X(f) = \int_{-\infty}^{+\infty} x(t)\, e^{-j2\pi f t}\, dt$$

$$\int_{-\infty}^{+\infty} x^2(t)\, dt = \int_{-\infty}^{\infty} |X(f)|^2\, df$$

total energy

integral of
energy spectral
density $\quad |X(f)|^2$

Parseval's Theorem – CTFS

Average Power

$$E[x^2(t)] = \frac{1}{T}\int_{-\infty}^{+\infty} x^2(t)\, dt$$

CTFS

$$x(t) = \sum_{k=-\infty}^{+\infty} C_k\, e^{+jk\omega t}$$

$$x^2(t) = x(t)\cdot x^*(t) = \left(\sum_{k=-\infty}^{+\infty} C_k\, e^{+jk\omega t}\right)\left(\sum_{l=-\infty}^{+\infty} C_l^*\, e^{-jl\omega t}\right)$$

$$x^2(t) = \sum_{k=-\infty}^{+\infty}\sum_{l=-\infty}^{+\infty} C_k C_l^*\, e^{+j(k-l)\omega t}$$

$$\frac{1}{T}\int_{-\infty}^{+\infty} x^2(t)\, dt = \sum_{k=-\infty}^{+\infty}\sum_{l=-\infty}^{+\infty} C_k C_l^* \left(\frac{1}{T}\int_{-\infty}^{+\infty} e^{+j(k-l)\omega t}\, dt\right)$$

Average Power

$$\frac{1}{T}\int_{-\infty}^{+\infty} x^2(t)\, dt = \sum_{k=-\infty}^{+\infty} |C_k|^2$$

Power Spectrum

$$|C_k|^2$$

power associated with individual frequency components

Parseval's Theorem − CTFT

Total Energy

$$\int_{-\infty}^{+\infty} x^2(t)\, dt$$

CTFT

$$x(t) = \int_{-\infty}^{+\infty} X(f)\, e^{+j2\pi ft}\, df$$

$$x^2(t) = x(t) \cdot x^*(t) = \left(\int_{-\infty}^{+\infty} X(f) e^{+j2\pi ft}\, df \right) \left(\int_{-\infty}^{+\infty} X^*(\nu) e^{-j2\pi\nu t}\, d\nu \right)$$

$$x^2(t) = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} |X(f)|^2\, e^{+j2\pi(f-\nu)t}\, df\, d\nu$$

$$\int_{-\infty}^{+\infty} x^2(t)\, dt = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} |X(f)|^2 \int_{-\infty}^{+\infty} e^{+j2\pi(f-\nu)t}\, dt\, df\, d\nu$$

Total Energy

$$\int_{-\infty}^{+\infty} x^2(t)\, dt = \int_{-\infty}^{+\infty} |X(f)|^2\, df \qquad = \frac{1}{2\pi} \int_{-\infty}^{+\infty} |X(\omega)|^2\, d\omega$$

Energy Spectral Density

$$|X(f)|^2$$

energy associated with individual frequency components

Parseval's Theorem – DTFT

Total Energy

$$\int_{-\infty}^{+\infty} x^2(t) \, dt$$

Total energy over all discrete time n
Total energy in fundamental period of frequency

DTFT

$$x[n] = \int_{-0.5}^{+0.5} X(e^{j\hat{f}}) \, e^{+j2\pi\hat{f}n} \, d\hat{f}$$

$$x^2[n] = x[n] \cdot x^*[n] = \left(\int_{-0.5}^{+0.5} X(e^{j\hat{f}}) \, e^{+j2\pi\hat{f}n} \, d\hat{f} \right) \left(\int_{-0.5}^{+0.5} X^*(e^{j\hat{v}}) \, e^{+j2\pi\hat{v}n} \, d\hat{v} \right)$$

$$x^2[n] = \int_{-0.5}^{+0.5} \int_{-0.5}^{+0.5} |X(e^{j\hat{f}})|^2 \, e^{+j2\pi(\hat{f}-\hat{v})n} \, df \, dv$$

$$\sum_{n=-\infty}^{+\infty} x^2[n] \, dt = \int_{-0.5}^{+0.5} \int_{-0.5}^{+0.5} |X(e^{j\hat{f}})|^2 \sum_{n=-\infty}^{+\infty} e^{+j2\pi(\hat{f}-\hat{v})n} \, df \, dv$$

Total Energy

$$\sum_{n=-\infty}^{+\infty} |x[n]|^2 = \int_{-0.5}^{+0.5} |X(\hat{f})|^2 \, d\hat{f} \qquad = \frac{1}{2\pi} \int_{-\pi}^{+\pi} |X(\hat{\omega})|^2 \, d\hat{\omega}$$

Energy Spectral Density

$$|X(\hat{f})|^2$$

energy associated with individual frequency components

Parseval's Theorem – DFT

Average Power

$$\frac{1}{N}\sum_{n=0}^{N-1} x^2[n]$$

DFT

$$x[n] = \frac{1}{N}\sum_{k=0}^{N-1} X[k]\,e^{+j\left(\frac{2\pi}{N}\right)kn}$$

$$x^2[n] = x[n]\cdot x^*[n] = \frac{1}{N^2}\left(\sum_{k=0}^{N-1} X[k]\,e^{+j\left(\frac{2\pi}{N}\right)kn}\right)\left(\sum_{l=0}^{N-1} X^*[l]\,e^{-j\left(\frac{2\pi}{N}\right)ln}\right)$$

$$x^2[n] = \frac{1}{N^2}\sum_{k=0}^{N-1}\sum_{l=0}^{N-1} X[k]X^*[l]\,e^{+j\left(\frac{2\pi}{N}\right)(k-l)n}$$

$$\frac{1}{N}\sum_{n=0}^{N-1} x^2[n]\,dt = \frac{1}{N^3}\sum_{k=0}^{N-1}\sum_{l=0}^{N-1} X[k]X^*[l]\sum_{n=0}^{N-1} e^{+j\left(\frac{2\pi}{N}\right)(k-l)n}$$

Average Power

$$\frac{1}{N}\sum_{n=0}^{N-1}|x[n]|^2 = \frac{1}{N^2}\sum_{k=0}^{N-1}|X[k]|^2 \qquad \sum_{n=0}^{N-1}|x[n]|^2 = \frac{1}{N}\sum_{k=0}^{N-1}|X[k]|^2$$

Periodogram

$$\frac{1}{N}|X[k]|^2$$

Average Power of Random Signals

A truncated sample function

$$x_T(t) = x(t) \quad -\frac{T}{2} < t < +\frac{T}{2}$$

$$= 0 \quad otherwise$$

Fourier Transform

$$X_T(\omega) = \int_{-\infty}^{+\infty} x_T(t)\, e^{-j\omega t}\, dt \quad \Longleftrightarrow \quad x_T(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} X_T(\omega)\, e^{+j\omega t}\, d\omega$$

$$X_T(f) = \int_{-\infty}^{+\infty} x_T(t)\, e^{-j2\pi f t}\, dt \quad \Longleftrightarrow \quad x_T(t) = \int_{-\infty}^{+\infty} X_T(f)\, e^{+j2\pi f t}\, df$$

Parseval's Therorem

$$\int_{-\infty}^{+\infty} x_T^2(t)\, dt = \int_{-\infty}^{+\infty} |X_T(f)|^2\, df \qquad \text{total energy}$$

Average Power

$$\frac{1}{T}\int_{-\infty}^{+\infty} x_T^2(t)\, dt = \int_{-\infty}^{+\infty} \frac{|X_T(f)|^2}{T}\, dt \qquad \text{total energy / T}$$

Power Spectral Density of Random Signals

Average Power

$$\frac{1}{T}\int_{-\infty}^{+\infty} x_T^2(t)\, dt \quad = \quad \int_{-\infty}^{+\infty} \frac{|X_T(f)|^2}{T}\, df \quad \Rightarrow \quad \int_{-\infty}^{+\infty} \hat{S}_{xx}(f)\, df$$

Raw Power Spectral Density

$$\frac{|X_T(f)|^2}{T} \;=\; \hat{S}_{xx}(f)$$

⟷

Power Spectral Density

$$\lim_{T\to\infty} \frac{\boldsymbol{E}\big[\,|X_T(f)|^2\,\big]}{T} \;=\; S_{xx}(f)$$

$$\lim_{T\to\infty} \frac{1}{T}\int_{-\infty}^{+\infty} x_T^2(t)\, dt \quad = \quad \lim_{T\to\infty} \int_{-\infty}^{+\infty} \frac{|X_T(f)|^2}{T}\, df \qquad \Longrightarrow \quad \textit{not converge}$$

$$E\!\left[\lim_{T\to\infty} \frac{1}{T}\int_{-\infty}^{+\infty} x_T^2(t)\, dt\right] \quad = \quad E\!\left[\lim_{T\to\infty} \int_{-\infty}^{+\infty} \frac{|X_T(f)|^2}{T}\, df\right] \quad \Longleftarrow \quad \textit{random signal}$$

$$Var\big(x(t)\big) \;=\; \sigma_x^2 \quad = \quad \int_{-\infty}^{+\infty} \lim_{T\to\infty} \frac{\boldsymbol{E}\big[\,|X_T(f)|^2\,\big]}{T}\, df \;\Rightarrow\; \int_{-\infty}^{+\infty} S_{xx}(f)\, df$$

Power and Power Density Spectra (1)

Average Power

$$E[x^2(t)] = \frac{\displaystyle\lim_{T \to \infty} \frac{1}{T}}{} \int_{-T/2}^{T/2} x^2(t)\, dt$$

$|x[n]|^2 T$

T

NT

Average Power of N sample of x(t)

$$\frac{1}{NT} \int_0^{NT} x^2(t)\, dt \;\approx\; \frac{1}{N} \sum_{n=0}^{N-1} x^2[n] \;=\; \frac{1}{N^2} \sum_{k=0}^{N-1} |X_k|^2$$

$$\sum_{n=0}^{N-1} x^2[n] \;=\; \frac{1}{N} \sum_{k=0}^{N-1} |X[k]|^2$$

$$P_{xx}(k) \;=\; \frac{1}{N} |X[k]|^2$$
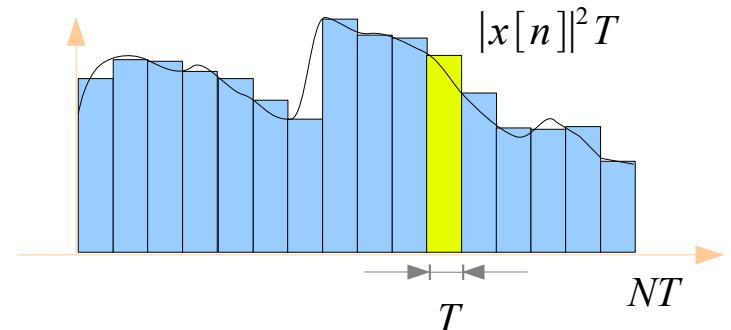
Periodogram:

$$k \;=\; 0, 1, \cdots, N-1$$

Average Power

$$\sum_{n=0}^{N-1} x^2[n] \;=\; \frac{1}{N} \sum_{k=0}^{N-1} P_{xx}(k)$$

Total Energy

$$T \sum_{n=0}^{N-1} x^2[n] \;=\; T \sum_{k=0}^{N-1} P_{xx}(k) \;\approx\; \int_0^{NT} x^2(t)\, dt$$

Power and Power Density Spectra (2)

Periodogram:

$$P_{xx}(k) = \frac{1}{N}|X[k]|^2$$

$$k = 0, 1, \cdots, N-1$$

$|x[n]|^2 T$

NT

T
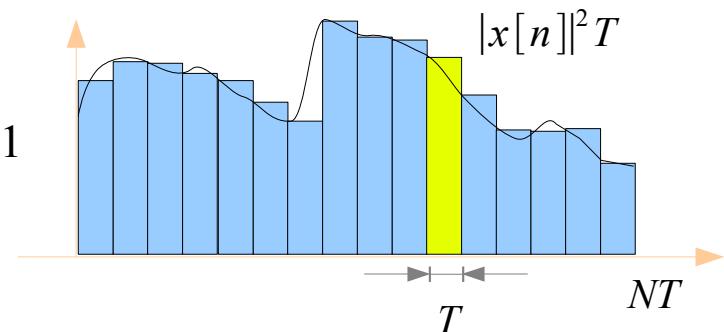
Average Power

$$\sum_{n=0}^{N-1} x^2[n] = \frac{1}{N}\sum_{k=0}^{N-1} P_{xx}(k)$$

$$if \quad P_{xx}(k) \quad is \quad constant \quad P$$

$$\sum_{n=0}^{N-1} x^2[n] = \int_{-0.5}^{+0.5} P_{xx}(\hat{f})df$$

$$P_{xx}(\hat{f}) = P$$

$$\sum_{n=0}^{N-1} x^2[n] = \int_{-1/2T}^{+1/2T} P_{xx}(f)df$$

$$P_{xx}(f) = TP$$

$$\sum_{n=0}^{N-1} x^2[n] = \int_{-\pi/T}^{+\pi/T} P_{xx}(\omega)d\omega$$

$$P_{xx}(\omega) = TP/2\pi$$

Correlation and Power Spectrum (1)

Periodogram:

$$P_{xx}(k) = \frac{1}{N}|X[k]|^2$$

$$k = 0, 1, \cdots, N-1$$

Autocorrelation Function

$$R_{xx}[m] = \frac{1}{N}\sum_{l=0}^{N-1} x[l]x[l+m]$$

$$m = 0, 1, \cdots, N-1$$

Autocorrelation → DFT → Periodogram

Average Power

$$\frac{1}{N}\sum_{n=0}^{N-1} x^2[n] = R_{xx}[0]$$

$$\sum_{m=0}^{N-1}\left(\frac{1}{N}\sum_{l=0}^{N-1} x[l]x[l+m]\right)e^{-j\left(\frac{2\pi}{N}\right)mk}$$

$$= \sum_{m=0}^{N-1}\left(\frac{1}{N}\sum_{l=0}^{N-1}\left(\frac{1}{N}\sum_{\alpha=0}^{N-1} X_\alpha e^{+j\left(\frac{2\pi}{N}\right)l\alpha}\right)\left(\frac{1}{N}\sum_{\beta=0}^{N-1} X_\beta^* e^{-j\left(\frac{2\pi}{N}\right)(l+m)\beta}\right)\right)e^{-j\left(\frac{2\pi}{N}\right)mk}$$

$$= \frac{1}{N^3}\sum_{\alpha=0}^{N-1}\sum_{\beta=0}^{N-1} X_\alpha X_\beta^* \sum_{m=0}^{N-1}\sum_{l=0}^{N-1}\left(e^{+j\left(\frac{2\pi}{N}\right)l\alpha}\right)\left(e^{-j\left(\frac{2\pi}{N}\right)(l+m)\beta}\right)e^{-j\left(\frac{2\pi}{N}\right)mk}$$

$$= \frac{1}{N^3}\sum_{\alpha=0}^{N-1}\sum_{\beta=0}^{N-1} X_\alpha X_\beta^* \sum_{m=0}^{N-1}\sum_{l=0}^{N-1}\left(e^{+j\left(\frac{2\pi}{N}\right)(l(\alpha-\beta) - m(\beta+k))}\right) \qquad = \frac{1}{N}\sum_{\alpha=0}^{N-1}\sum_{\beta=0}^{N-1} X_\alpha X_\beta^*$$

	Periodic Signals	Aperiodic Signals	Random Signals	
Frequency Spacing	$\Delta f = \frac{1}{N \Delta t}$	$\Delta f = \frac{1}{N \Delta t}$	$\sum S \Delta f = \frac{1}{N \Delta t} \sum S$	$\frac{1}{N \Delta t} \sum x^2 \Delta t$
Two Sided	$\frac{1}{N} X(k)$	$\frac{\Delta t}{N} X(k)$	$S(k) = \frac{\Delta t}{N} \lvert X(k) \rvert^2$	$P = \sum_{k=0}^{N-1} S(k) \Delta f$
One Sided				
$k=0,\ \frac{N}{2}$	$\frac{1}{N} X(k)$	$\frac{\Delta t}{N} X(k)$	$S_1(k) = 2S(k)$	$P = \sum_{k=0}^{N/2} S_1(k) \Delta f$
$k=1, \cdots, \frac{N}{2}-1$	$\frac{2}{N} X(k)$	$\frac{2\Delta t}{N} X(k)$	$S_1(k) = S(k)$	
Frequency Scale	$k \Delta f$	$k \Delta f$	$k \Delta f$	

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