

Complex Phase Factors (DFT.A1)

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DFT

Discrete Fourier Transform

Inverse Discrete Fourier Transform

$$X[k] = \sum_{n=0}^{N-1} x[n] e^{-j(2\pi/N)kn} \iff x[n] = \frac{1}{N} \sum_{k=0}^{N-1} X[k] e^{+j(2\pi/N)kn}$$

$$W_N \triangleq e^{-j(2\pi/N)}$$

$$W_N^{nk} \triangleq e^{-j(2\pi/N)nk}$$

$$X[k] = \sum_{n=0}^{N-1} x[n] W_N^{kn} \iff x[n] = \frac{1}{N} \sum_{k=0}^{N-1} X[k] W_N^{-kn}$$

Complex Phase Factor

$$W_N^{+kN} = 1$$

$$W_N^{-kN} = 1$$

$$W_N^{\pm N} = e^{-j\left(\frac{2\pi}{N}\right)(\pm N)}$$

$$W_N^{k \pm N} = W_N^k$$

$$W_N^{-k \pm N} = W_N^{-k}$$

$$W_N^{\pm k \pm N} = W_N^{\pm k} \cdot W_N^{\pm N}$$

Modular N System

$$W_N^{nk} \triangleq e^{-j(2\pi/N)nk}$$

DFT Index (1)

$$X[k] = \sum_{n=0}^{N-1} x[n] e^{-j(2\pi/N)kn}$$

$$X[k] = \sum_{n=0}^{N-1} x[n] W_N^{kn}$$

	n=0	n=1	n=2	n=N-1
k=0	0·0	0·1	0·2	•	•	•	0·(N-1)
k=1	1·0	1·1	1·2	•	•	•	1·(N-1)
k=2	2·0	2·1	2·2	•	•	•	2·(N-1)
•	•	•	•				•
•	•	•	•				•
•	•	•	•				•
k=N-1	(N-1)·0	(N-1)·1	(N-1)·2	•	•	•	(N-1)·(N-1)

*A Multiplication Table
in Modular N System*

DFT Index (2)

$$X[k] = \sum_{n=0}^{N-1} x[n] e^{-j(2\pi/N)kn}$$

$$X[k] = \sum_{n=0}^{N-1} x[n] W_N^{kn}$$

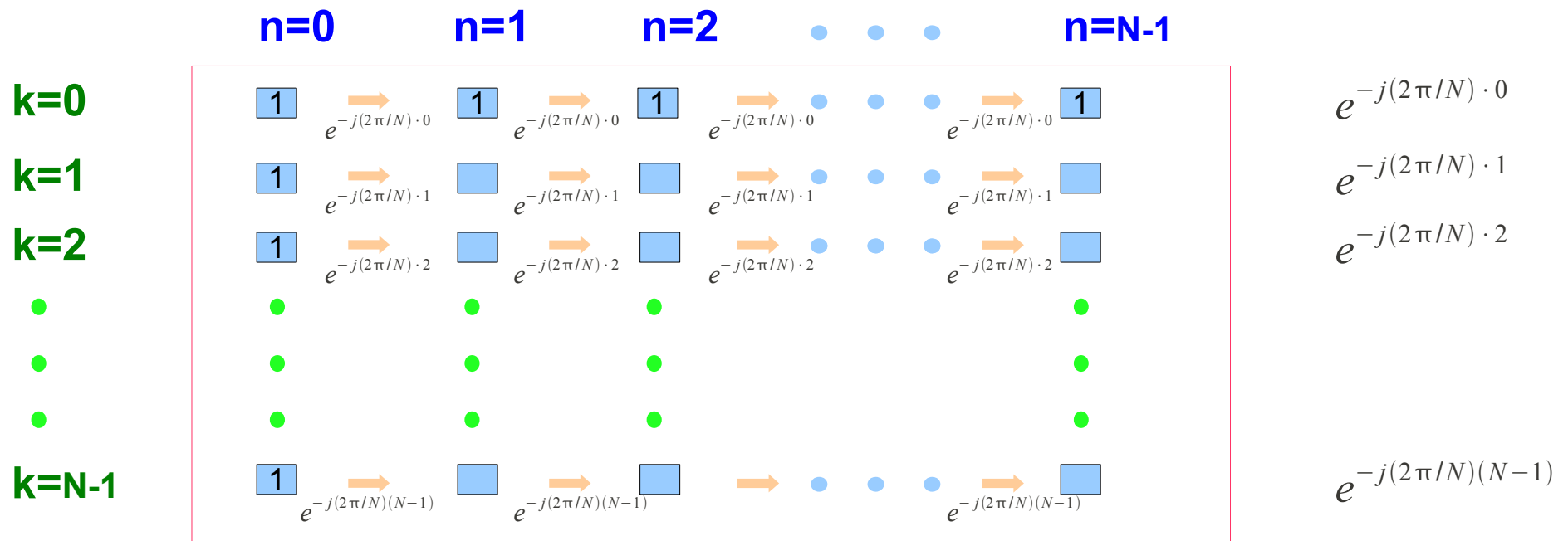
	n=0	n=1	n=2	...	n=N-1	
k=0	0·0 → +0	0·1 → +0	0·2 → +0	...	0·(N-1) → +0	+ 0 (mod N)
k=1	1·0 → +1	1·1 → +1	1·2 → +1	...	1·(N-1) → +1	+ 1 (mod N)
k=2	2·0 → +2	2·1 → +2	2·2 → +2	...	2·(N-1) → +2	+ 2 (mod N)
•	•	•	•	•	•	
•	•	•	•	•	•	
•	•	•	•	•	•	
k=N-1	(N-1)·0 → +(N-1)	(N-1)·1 → +(N-1)	(N-1)·2 → +(N-1)	...	(N-1)·(N-1) → +(N-1)	+ N-1 (mod N)

*A Multiplication Table
in Modular N System*

DFT Matrix

$$X[k] = \sum_{n=0}^{N-1} x[n] e^{-j(2\pi/N)kn}$$

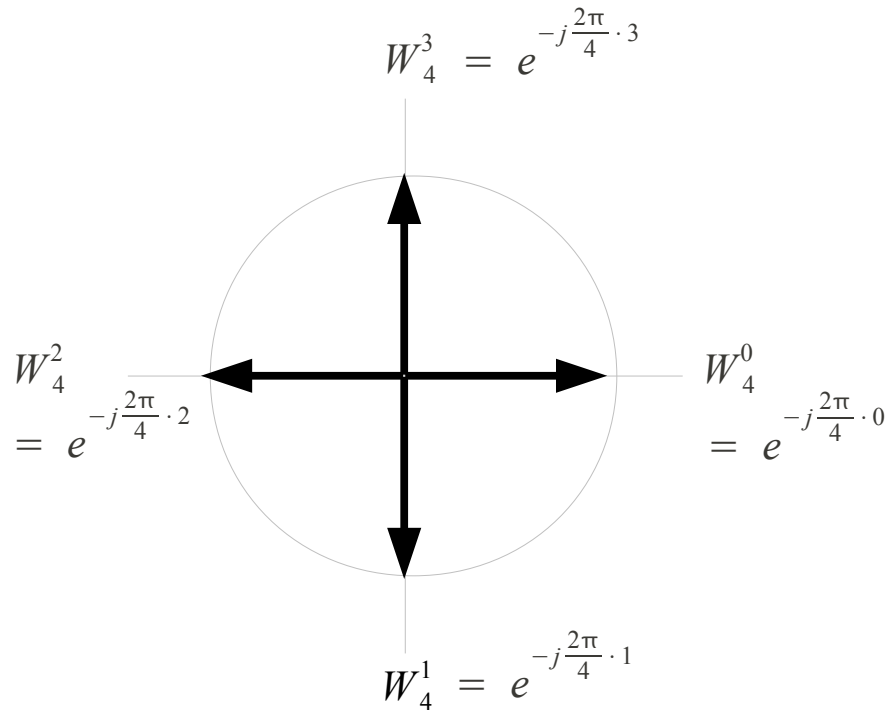
$$X[k] = \sum_{n=0}^{N-1} x[n] W_N^{kn}$$



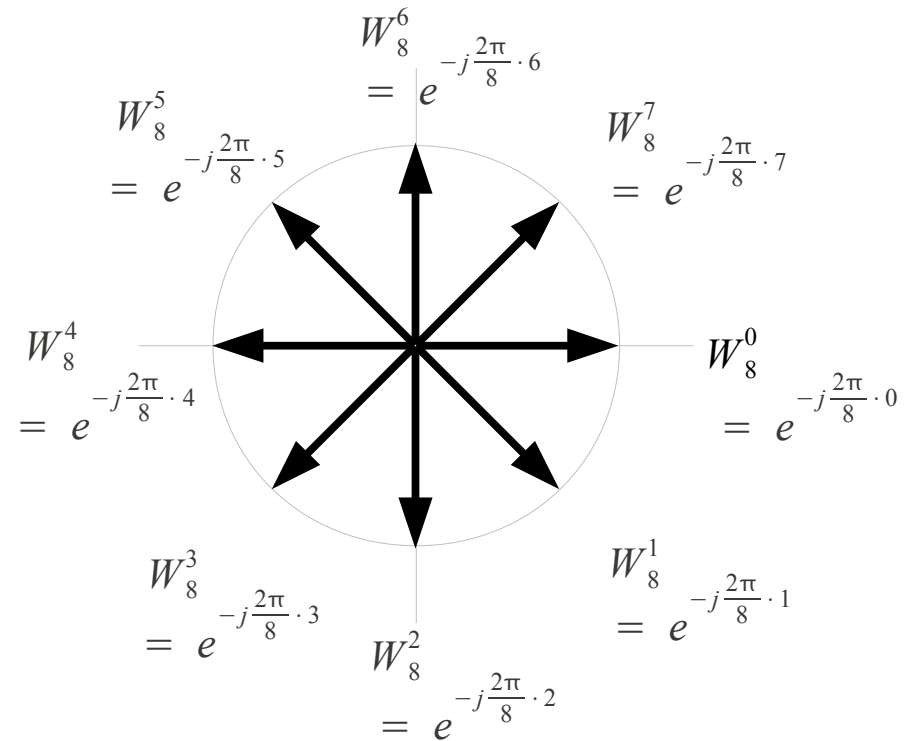
*A Multiplication Table
in Modular N System*

Complex Phase Factor

Modular 4 System



Modular 8 System



Complex Phase Factor Symmetry

$$W_N^{+kN} = 1$$

$$W_N^{-kN} = 1$$

$$W_N^{\pm N} = e^{-j\left(\frac{2\pi}{N}\right)(\pm N)}$$

$$W_N^{k \pm N} = W_N^k$$

$$W_N^{-k \pm N} = W_N^{-k}$$

$$W_N^{\pm k \pm N} = W_N^{\pm k} \cdot W_N^{\pm N}$$

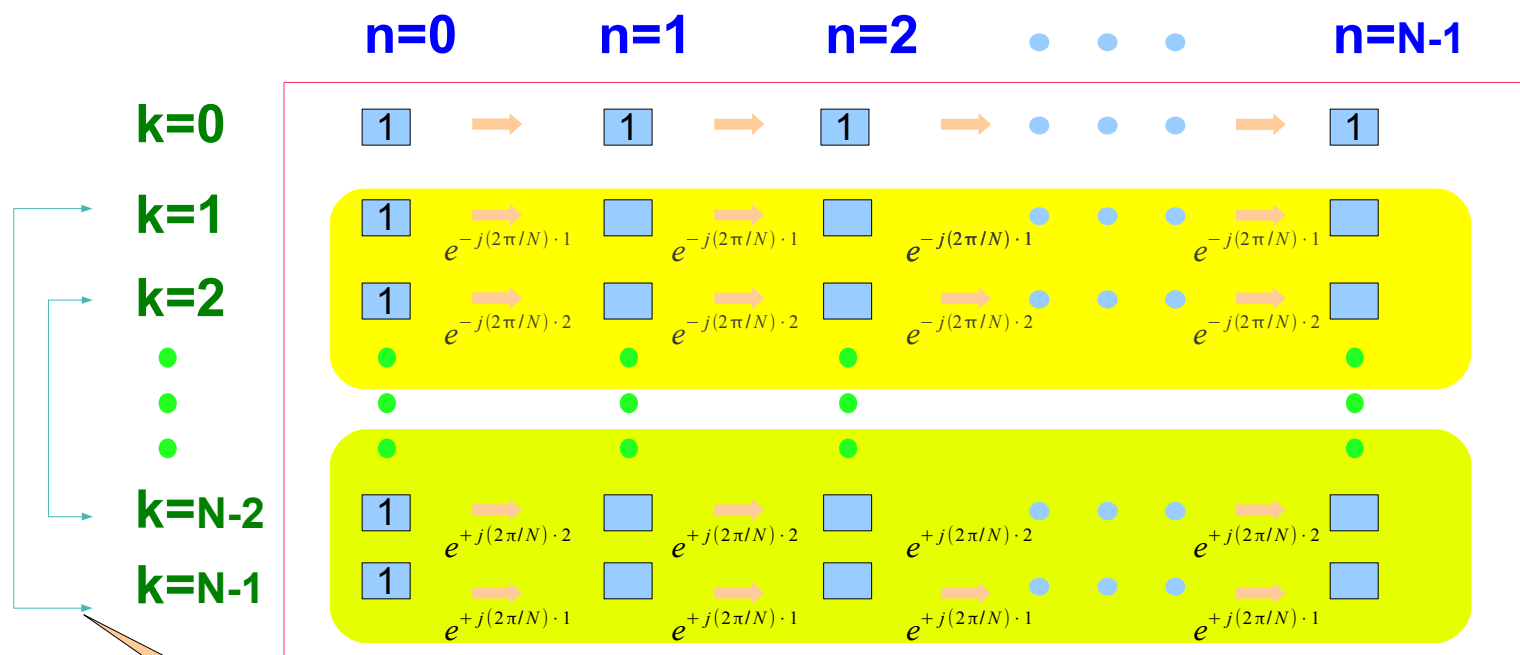
$$W_N^{N-k} = W_N^{-k} = \{W_N^k\}^*$$

$$e^{-j\frac{2\pi}{N}(N-k)} = e^{j\frac{2\pi}{N}k} = \left\{ e^{-j\frac{2\pi}{N}k} \right\}^*$$

DFT Matrix Symmetry

$$X[k] = \sum_{n=0}^{N-1} x[n] e^{-j(2\pi/N)kn}$$

$$X[k] = \sum_{n=0}^{N-1} x[n] W_N^{kn}$$



$$e^{-j\left(\frac{2\pi}{N}\right) \cdot (N-2)} = e^{+j\left(\frac{2\pi}{N}\right) \cdot 2}$$

$$e^{-j\left(\frac{2\pi}{N}\right) \cdot (N-1)} = e^{+j\left(\frac{2\pi}{N}\right) \cdot 1}$$

complex conjugate



$$e^{-j\left(\frac{2\pi}{N}\right) \cdot (N-i)} = e^{+j\left(\frac{2\pi}{N}\right) \cdot i}$$

References

- [1] <http://en.wikipedia.org/>
- [2] J.H. McClellan, et al., Signal Processing First, Pearson Prentice Hall, 2003