Line Integrals (4A)

- Line Integral
- Path Independence

Young Won Lim
10/8/12

Please send corrections (or suggestions) to youngwlim@hotmail.com.

This document was produced by using OpenOffice and Octave.

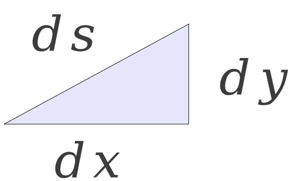
Line Integral In the Plain

$x = f(t)$ ⇒ $\dfrac{dx}{dt} = f'(t)$ ⇒ $dx = f'(t)\,dt$

$y = g(t)$ ⇒ $\dfrac{dy}{dt} = g'(t)$ ⇒ $dy = g'(t)\,dt$

$$ds = \sqrt{[f'(t)]^2 + [g'(t)]^2}\,dt$$

Curve C $\quad a \le t \le b$



$$\int_C G(x, y)\,dx = \int_a^b G(f(t), g(t))\,f'(t)\,dt$$

$$\int_C G(x, y)\,dy = \int_a^b G(f(t), g(t))\,g'(t)\,dt$$

$$\int_C G(x, y)\,ds = \int_a^b G(f(t), g(t))\sqrt{[f'(t)]^2 + [g(t)]^2}\,dt$$

Line Integral In Space

$x = f(t)$ ➡ $\frac{dx}{dt} = f'(t)$ ➡ $dx = f'(t)\,dt$

$y = g(t)$ ➡ $\frac{dy}{dt} = g'(t)$ ➡ $dy = g'(t)\,dt$

$z = h(t)$ ➡ $\frac{dz}{dt} = h'(t)$ ➡ $dz = h'(t)\,dt$

Curve C $\quad a \le t \le b$ $\quad ds = \sqrt{[f'(t)]^2 + [g'(t)]^2 + [h'(t)]^2}\,dt$

$$\int_C G(x, y, z)\,dz = \int_a^b G(f(t), g(t), h(t))\,h'(t)\,dt$$

$$\int_C G(x, y, z)\,ds = \int_a^b G(f(t), g(t), h(t))\sqrt{[f'(t)]^2 + [g(t)]^2 + [h(t)]^2}\,dt$$

Line Integral using ***r***(t)

Arc Length Parameter

s increases in the direction of increasing t

$$s(t) = \int_{t_0}^{t} |\mathbf{v}(\tau)|\, d\tau = \int_{t_0}^{t} \|\mathbf{r}'(\tau)\|\, d\tau = \int_{t_0}^{t} \sqrt{[f'(\tau)]^2 + [g'(\tau)]^2 + [h'(\tau)]^2}\, d\tau$$

$$ds = |\mathbf{v}(t)|\, dt \qquad ds = \sqrt{[f'(t)]^2 + [g'(t)]^2 + [h'(t)]^2}\, dt$$

$$\int_C G(x, y, z)\, dz = \int_a^b G(\mathbf{r}(t))\, |\mathbf{r}'(t)|\, dt$$

$$= \int_a^b G(f(t), g(t), h(t))\, |\mathbf{v}(t)|\, dt$$

$$= \int_a^b G(f(t), g(t), h(t)) \sqrt{[f'(t)]^2 + [g(t)]^2 + [h(t)]^2}\, dt$$

Line Integral

In many applications

$$\int_C G(x, y)\, ds = \int_C P(x, y)\, dx + \int_C Q(x, y)\, dy$$

$$= \int_C P(x, y)\, dx + Q(x, y)\, dy$$

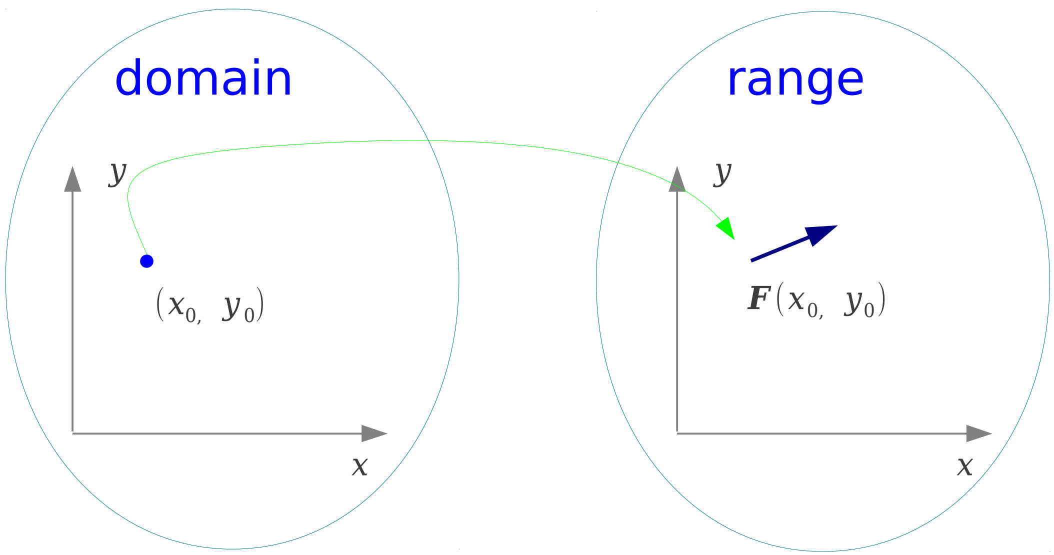
$$= \int_C P\, dx + Q\, dy$$

$$\int_C G(x, y, z)\, ds = \int_C P(x, y, z)\, dx + Q(x, y, z)\, dy + R(x, y, z)\, dz$$

Line Integral over a 2-D Vector Field (1)

a given point in a 2-d space ➡ A vector

$(x_0, \; y_0)$ $\qquad$ $\langle \, P(x_0, y_0), \; Q(x_0, y_0) \, \rangle$



2 functions

$(x_0, \; y_0) \longrightarrow P(x_0, y_0)$

$(x_0, \; y_0) \longrightarrow Q(x_0, y_0)$

$(x_0, \; y_0)$

only points that are
on the curve

$\boldsymbol{r}(t) \; = \; f(t)\boldsymbol{i} \; + \; g(t)\boldsymbol{j} \longrightarrow \boldsymbol{F}(x_0, \; y_0) \; = \; P(x_0, y_0)\boldsymbol{i} \; + \; Q(x_0, y_0)\boldsymbol{j}$

$x = f(t) \quad y = g(t) \quad a \le t \le b$

Line Integral over a 2-D Vector Field (2)

$$\boldsymbol{r}(t) = f(t)\,\boldsymbol{i} + g(t)\,\boldsymbol{j}$$

$$\frac{d\boldsymbol{r}}{dt} = f'(t)\boldsymbol{i} + g'(t)\,\boldsymbol{j} = \frac{dx}{dt}\boldsymbol{i} + \frac{dy}{dt}\,\boldsymbol{j}$$

$$\frac{d\boldsymbol{r}}{dt}dt = \frac{dx}{dt}dt\,\boldsymbol{i} + \frac{dy}{dt}dt\,\boldsymbol{j} = dx\,\boldsymbol{i} + dy\,\boldsymbol{j}$$

$$d\boldsymbol{r} = dx\,\boldsymbol{i} + dy\,\boldsymbol{j}$$

$$\boldsymbol{F}(x, y) = P(x, y)\,\boldsymbol{i} + Q(x, y)\,\boldsymbol{j} \qquad d\boldsymbol{r} = dx\,\boldsymbol{i} + dy\,\boldsymbol{j}$$

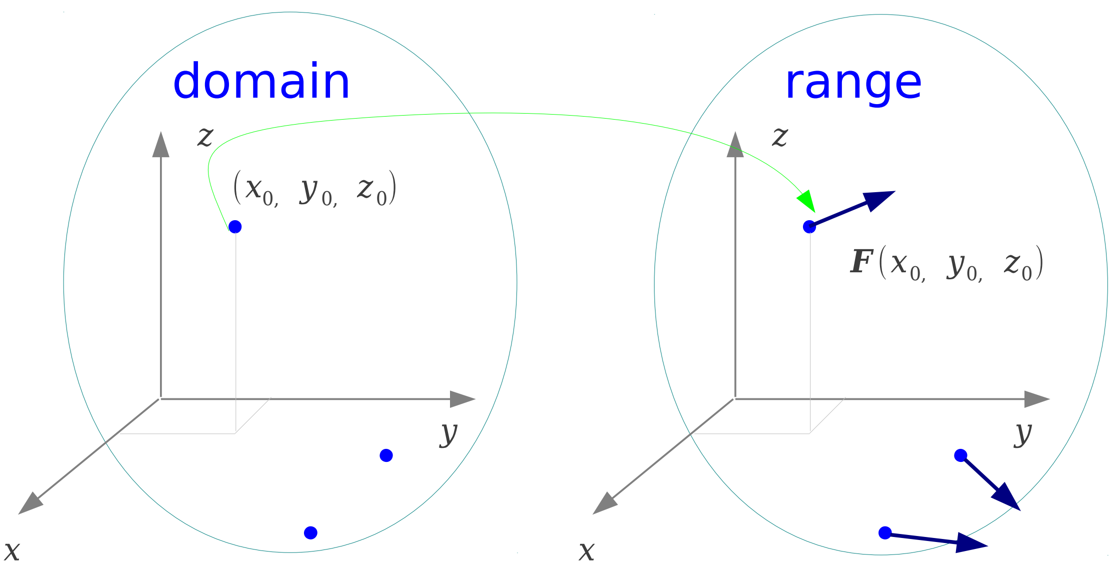
$$\boldsymbol{F}\cdot d\boldsymbol{r} = P(x, y)\,dx + Q(x, y)\,dy$$

$$\int_C \boldsymbol{F}\cdot d\boldsymbol{r} = \int_C P(x, y)\,dx + Q(x, y)\,dy$$

Line Integral over a 2-D Vector Field (1)

A given point in a 3-d space ➡ A vector

$(x_0, \; y_0, \; z_0)$ $\quad \langle\, P(x_0, y_0, z_0), \; Q(x_0, y_0, z_0), \; R(x_0, y_0, z_0) \,\rangle$



3 functions

$(x_0, \; y_0, \; z_0) \longrightarrow P(x_0, y_0, Z_0)$

$(x_0, \; y_0, \; z_0) \longrightarrow Q(x_0, y_0, z_0)$

$(x_0, \; y_0, \; z_0) \longrightarrow R(x_0, y_0, z_0)$

$(x_0, \; y_0, \; z_0)$

only points that are
on the curve $\longrightarrow$ $\boldsymbol{F}(x_0, \; y_0, \; z_0) \; = \; P(x_0, y_0, z_0)\boldsymbol{i} \; + \; Q(x_0, y_0, z_0)\boldsymbol{j} \; + \; R(x_0, y_0, z_0)\boldsymbol{k}$

$\boldsymbol{r}(t) \; = \; f(t)\boldsymbol{i} \; + \; g(t)\boldsymbol{j} \; + \; h(t)\boldsymbol{j}$

$x = f(t) \quad y = g(t) \quad z = h(t) \quad a \le t \le b$

Line Integral over a 3-D Vector Field (2)

$$\boldsymbol{r}(t) = f(t)\,\boldsymbol{i} + g(t)\,\boldsymbol{j} + h(t)\,\boldsymbol{k}$$

$$\frac{d\boldsymbol{r}}{dt} = f'(t)\,\boldsymbol{i} + g'(t)\,\boldsymbol{j} + h'(t)\,\boldsymbol{k} = \frac{dx}{dt}\boldsymbol{i} + \frac{dy}{dt}\boldsymbol{j} + \frac{dz}{dt}\boldsymbol{k}$$

$$\frac{d\boldsymbol{r}}{dt}dt = \frac{dx}{dt}dt\,\boldsymbol{i} + \frac{dy}{dt}dt\,\boldsymbol{j} + \frac{dz}{dt}dt\,\boldsymbol{k} = dx\,\boldsymbol{i} + dy\,\boldsymbol{j} + dz\,\boldsymbol{k}$$

$$d\boldsymbol{r} = dx\,\boldsymbol{i} + dy\,\boldsymbol{j} + dz\,\boldsymbol{k}$$

$$\boldsymbol{F}(x,y,z) = P(x,y,z)\,\boldsymbol{i} + Q(x,y,z)\,\boldsymbol{j} + R(x,y,z)\,\boldsymbol{k}$$

$$\boldsymbol{F}\cdot d\boldsymbol{r} = P(x,y,z)\,dx + Q(x,y,z)\,dy + R(x,y,z)\,dz$$

$$\int_c \boldsymbol{F}\cdot d\boldsymbol{r} = \int_C P(x,y,z)\,dx + Q(x,y,z)\,dy + R(x,y,z)\,dy$$

Work (1)

$$W = \int_c \boldsymbol{F}(x, y) \cdot \boldsymbol{r}(t) = \int_C P(x, y)\, dx + Q(x, y)\, dy$$

$$\frac{d\boldsymbol{r}}{dt} = \frac{d\boldsymbol{r}}{ds}\frac{ds}{dt} \qquad d\boldsymbol{r} = \frac{d\boldsymbol{r}}{ds} ds \qquad d\boldsymbol{r} = \boldsymbol{T} ds$$

$$W = \int_c \boldsymbol{F} \cdot d\boldsymbol{r} = \int_c \boldsymbol{F} \cdot \boldsymbol{T}\, ds$$

$$= \int_{t_1}^{t_0} \boldsymbol{F} \cdot \frac{d\boldsymbol{r}}{dt} dt$$

$$= \int_{t_0}^{t_1} \left(P\frac{df}{dt} + Q\frac{dg}{dt} + R\frac{dh}{dt} \right) dt$$

$$= \int_{t_0}^{t_1} \left(P\frac{dx}{dt} + Q\frac{dy}{dt} + R\frac{dz}{dt} \right) dt$$

$$= \int_{t_0}^{t_1} (P\,dx + Q\,dy + R\,dz)\, dt$$

Work (2)

$$\frac{d\boldsymbol{r}}{dt} = \frac{d\boldsymbol{r}}{ds}\frac{ds}{dt} \qquad d\boldsymbol{r} = \frac{d\boldsymbol{r}}{ds}ds \qquad d\boldsymbol{r} = \boldsymbol{T}ds$$

$$W = \int_c \boldsymbol{F}\cdot d\boldsymbol{r} = \int_c \boldsymbol{F}\cdot\boldsymbol{T}\,ds$$

$$= \int_{t_1}^{t_0} \boldsymbol{F}\cdot\frac{d\boldsymbol{r}}{dt}dt$$

$$= \int_{t_0}^{t_1} \left(P\frac{df}{dt} + Q\frac{dg}{dt} + R\frac{dh}{dt}\right)dt$$

$$= \int_{t_0}^{t_1} \left(P\frac{dx}{dt} + Q\frac{dy}{dt} + R\frac{dz}{dt}\right)dt$$

$$= \int_{t_0}^{t_1} P\,dx + Q\,dy + R\,dz$$

Circulation

A Simple Closed Curve C → Circulation

$$circulation = \oint_C \boldsymbol{F} \cdot d\boldsymbol{r} = \oint_C \boldsymbol{F} \cdot \boldsymbol{T}\, d\boldsymbol{s}$$

Conservative Vector Field

A vector function $\boldsymbol{F}$ in 2-d or 3-d space is conservative

⇐ $\boldsymbol{F}$ can be written as the gradient of a scalar function Φ $\quad$ $\boldsymbol{F} = \nabla \Phi$

$$\boldsymbol{r}(t) = x(t)\,\boldsymbol{i} + y(t)\,\boldsymbol{j} \qquad a \le t \le b$$

$$\boldsymbol{F}(x, y) = P(x, y)\,\boldsymbol{i} + Q(x, y)\,\boldsymbol{j}$$ conservative

$$\int_C \boldsymbol{F} \cdot d\boldsymbol{r} = \int_C \nabla \Phi \cdot d\boldsymbol{r} = \Phi(B) - \Phi(A)$$

$$A = (x(a), y(a))$$
$$B = (x(b), y(b))$$

Path Independence

$$\int_a^b f'(x)\,dx = f(b) - f(a)$$

Connected

Connected

Simply Connected

Disconnected

Multiply Connected

Open Connected

Equivalence

In an open connected region

Path Independence	$\int_C \boldsymbol{F} \cdot d\,\boldsymbol{r}$	⟺
Conservative	$\boldsymbol{F}$	⟺
Closed path C	$\oint_C \boldsymbol{F} \cdot d\,\boldsymbol{r} = 0$	⟺

Equivalence in 2-D

In an open connected region

Path Independence $\int_C \boldsymbol{F} \cdot d\boldsymbol{r}$ $\Leftrightarrow$

Conservative $\boldsymbol{F}$ $\Leftrightarrow$

Closed path C $\oint_C \boldsymbol{F} \cdot d\boldsymbol{r} = 0$ $\Leftrightarrow$ $\dfrac{\partial P}{\partial y} = \dfrac{\partial Q}{\partial x}$

$$\boldsymbol{F} = P\,\boldsymbol{i} + Q\,\boldsymbol{j} \qquad \boldsymbol{F} = \nabla\Phi = \frac{\partial \Phi}{\partial x}\boldsymbol{i} + \frac{\partial \Phi}{\partial y}\boldsymbol{j}$$

Equivalence in 3-D

In an open connected region

Path Independence $\int_C \boldsymbol{F} \cdot d\boldsymbol{r}$ ⟺

Conservative $\boldsymbol{F}$ ⟺

Closed path C $\oint_C \boldsymbol{F} \cdot d\boldsymbol{r} = 0$ ⟺

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x} \qquad \frac{\partial P}{\partial z} = \frac{\partial R}{\partial x} \qquad \frac{\partial Q}{\partial z} = \frac{\partial R}{\partial y}$$

$$curl\,\boldsymbol{F} = \left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right) + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right) + \left(\frac{\partial Q}{\partial z} - \frac{\partial R}{\partial y}\right)$$

$$\boldsymbol{F} = P\,\boldsymbol{i} + Q\,\boldsymbol{j} + R\,\boldsymbol{k} \qquad \boldsymbol{F} = \nabla\Phi = \frac{\partial \Phi}{\partial x}\,\boldsymbol{i} + \frac{\partial \Phi}{\partial y}\,\boldsymbol{j} + \frac{\partial \Phi}{\partial z}\,\boldsymbol{k}$$

2-Divergence

Flux across rectangle boundary

$$\approx \left(\frac{\partial M}{\partial x} \Delta x \right) \Delta y + \left(\frac{\partial N}{\partial y} \Delta y \right) \Delta x \qquad = \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} \right) \Delta x \Delta y$$

Flux density

$$= \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} \right)$$

Divergence of **F** Flux Density

References

[1] http://en.wikipedia.org/
[2] http://planetmath.org/
[3] M.L. Boas, “Mathematical Methods in the Physical Sciences”
[4] D.G. Zill, “Advanced Engineering Mathematics”