

# Elementary Matrix

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# Gauss-Jordan Elimination

## Forward Phase - Gaussian Elimination

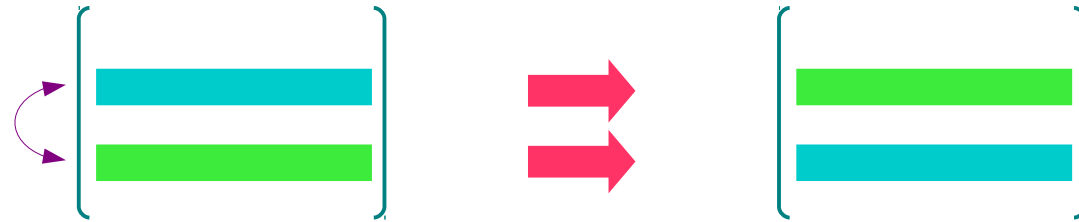
$$\begin{array}{c}
 \left( \begin{array}{ccc|c}
 \textcircled{+2} & +1 & -1 & +8 \\
 -3 & -1 & +2 & -11 \\
 -2 & +1 & +2 & -3
 \end{array} \right) \rightarrow \left( \begin{array}{ccc|c}
 \textcircled{+1} & +1/2 & -1/2 & +4 \\
 -3 & -1 & +2 & -11 \\
 -2 & +1 & +2 & -3
 \end{array} \right) \rightarrow \left( \begin{array}{ccc|c}
 +1 & +1/2 & -1/2 & +4 \\
 \boxed{0} & +1/2 & +1/2 & +1 \\
 \boxed{0} & +2 & +1 & +5
 \end{array} \right) \\
 \\
 \left( \begin{array}{ccc|c}
 +1 & +1/2 & -1/2 & +4 \\
 0 & \textcircled{+1} & +1 & +2 \\
 0 & +2 & +1 & +5
 \end{array} \right) \rightarrow \left( \begin{array}{ccc|c}
 +1 & +1/2 & -1/2 & +4 \\
 0 & +1 & +1 & +2 \\
 0 & \boxed{0} & -1 & +1
 \end{array} \right) \rightarrow \left( \begin{array}{ccc|c}
 +1 & +1/2 & -1/2 & +4 \\
 0 & +1 & +1 & +2 \\
 0 & 0 & \textcircled{+1} & -1
 \end{array} \right)
 \end{array}$$

## Backward Phase

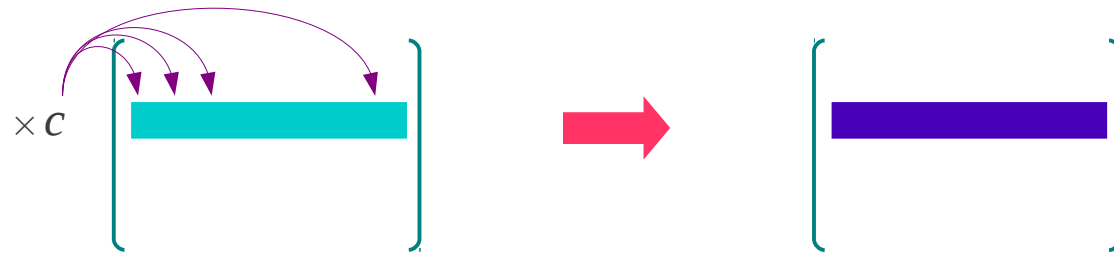
$$\begin{array}{c}
 \left( \begin{array}{ccc|c}
 +1 & +1/2 & \boxed{-1/2} & +4 \\
 0 & +1 & \boxed{+1} & +2 \\
 0 & 0 & +1 & -1
 \end{array} \right) \rightarrow \left( \begin{array}{ccc|c}
 +1 & +1/2 & \boxed{0} & +7/2 \\
 0 & +1 & \boxed{0} & +3 \\
 0 & 0 & +1 & -1
 \end{array} \right) \rightarrow \left( \begin{array}{ccc|c}
 +1 & \boxed{0} & \boxed{0} & +2 \\
 0 & +1 & \boxed{0} & +3 \\
 0 & 0 & +1 & -1
 \end{array} \right)
 \end{array}$$

# Elementary Row Operation

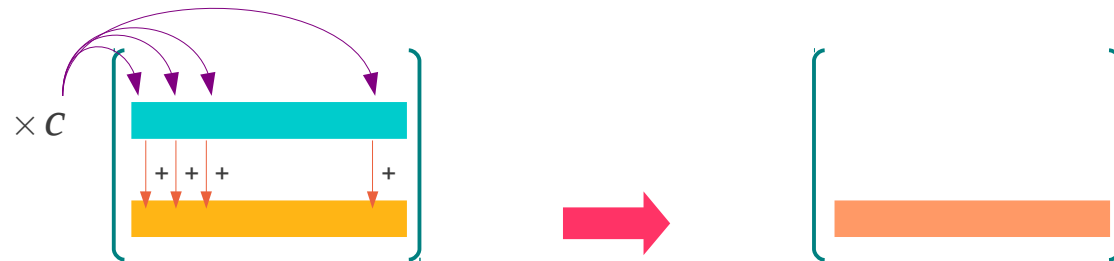
Interchange two rows



Multiply a row by a nonzero constant



Add a multiple of one row to another



# Elementary Matrix

Identity Matrix

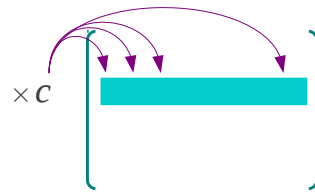
$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Interchange two rows



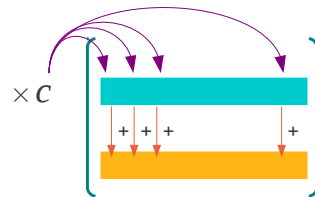
$$\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Multiply a row by a nonzero constant



$$\begin{pmatrix} 3 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Add a multiple of one row to another



$$\begin{pmatrix} 1 & 0 & 0 \\ 4 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

# Multiplication by an Elementary Matrix

$$\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$$

$$\begin{bmatrix} 4 & 5 & 6 \\ 1 & 2 & 3 \\ 7 & 8 & 9 \end{bmatrix}$$

$$\begin{bmatrix} 3 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$$

$$\begin{bmatrix} 3 & 6 & 9 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$$

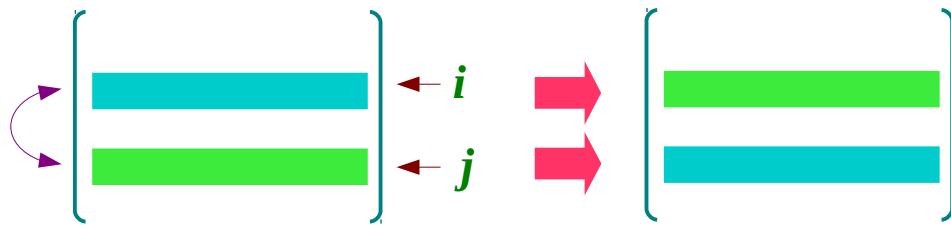
$$\begin{bmatrix} 1 & 0 & 0 \\ 4 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$$

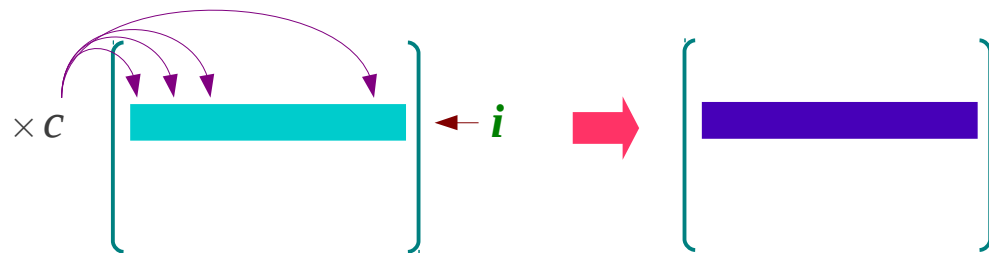
$$\begin{bmatrix} 1 & 2 & 3 \\ 8 & 13 & 18 \\ 7 & 8 & 9 \end{bmatrix}$$

# Elementary Matrix

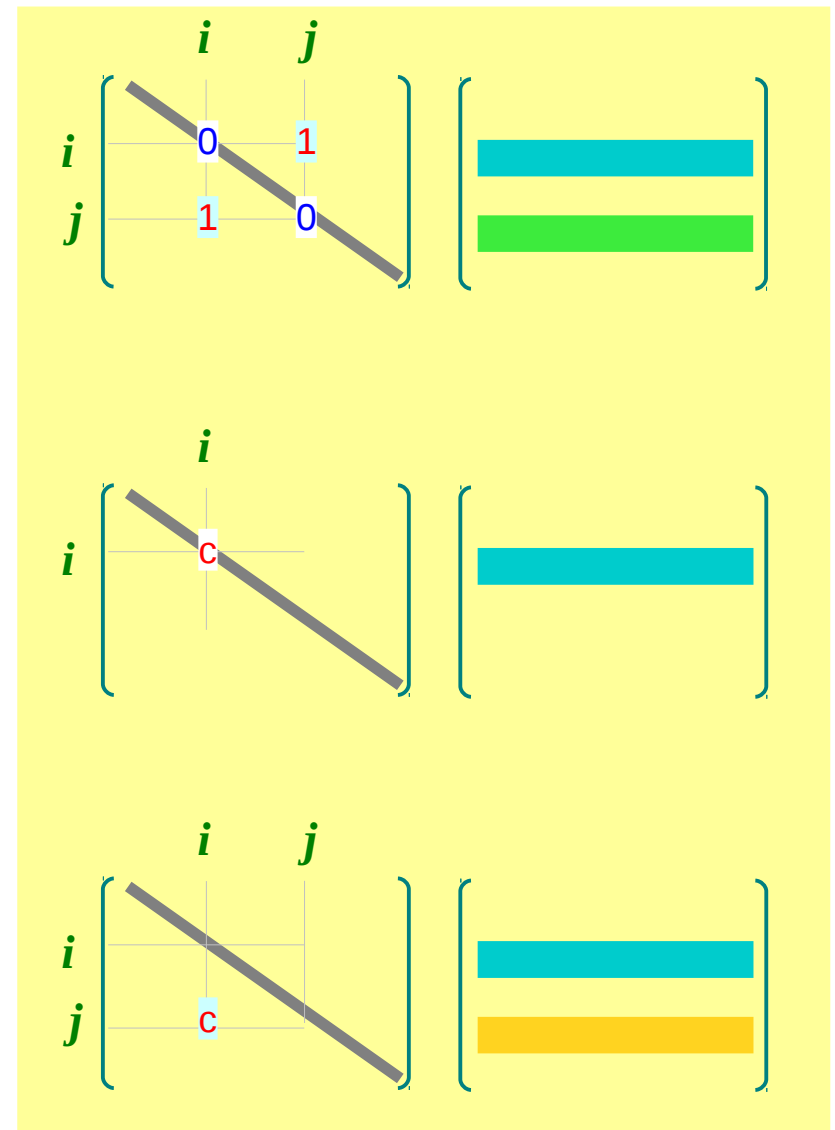
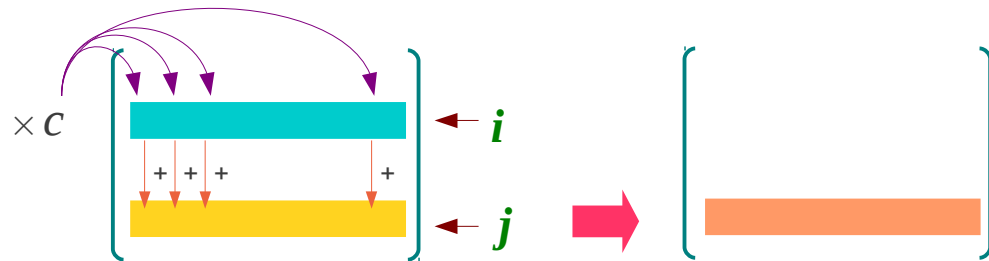
Interchange two rows



Multiply a row by a nonzero constant



Add a multiple of one row to another



# Gauss-Jordan Elimination – Step 1

$$\begin{array}{rcl} +2x_1 + x_2 - x_3 = 8 & (L_1) & \\ -3x_1 - x_2 + 2x_3 = -11 & (L_2) & \\ -2x_1 + x_2 + 2x_3 = -3 & (L_3) & \end{array} \quad \left[ \begin{array}{ccc|c} \textcircled{+2} & +1 & -1 & +8 \\ -3 & -1 & +2 & -11 \\ -2 & +1 & +2 & -3 \end{array} \right]$$

$$\left[ \begin{array}{ccc} 1/2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array} \right] \quad \left[ \begin{array}{ccc|c} \textcircled{+2} & +1 & -1 & +8 \\ -3 & -1 & +2 & -11 \\ -2 & +1 & +2 & -3 \end{array} \right]$$

$$\begin{array}{rcl} +1x_1 + \frac{1}{2}x_2 - \frac{1}{2}x_3 = 4 & (\frac{1}{2} \times L_1) & \\ -3x_1 - x_2 + 2x_3 = -11 & (L_2) & \\ -2x_1 + x_2 + 2x_3 = -3 & (L_3) & \end{array} \quad \left[ \begin{array}{ccc|c} \textcircled{+1} & +1/2 & -1/2 & +4 \\ -3 & -1 & +2 & -11 \\ -2 & +1 & +2 & -3 \end{array} \right]$$



# Gauss-Jordan Elimination – Step 2

$$+1x_1 + \frac{1}{2}x_2 - \frac{1}{2}x_3 = +4 \quad (L_1)$$

$$-3x_1 - x_2 + 2x_3 = -11 \quad (L_2)$$

$$-2x_1 + x_2 + 2x_3 = -3 \quad (L_3)$$

$$\left[ \begin{array}{ccc|c} +1 & +1/2 & -1/2 & +4 \\ -3 & -1 & +2 & -11 \\ -2 & +1 & +2 & -3 \end{array} \right]$$

$$\left[ \begin{array}{ccc} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{array} \right]$$

$$\left[ \begin{array}{ccc} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 0 & 0 & 1 \end{array} \right]$$

$$\left[ \begin{array}{ccc|c} +1 & +1/2 & -1/2 & +4 \\ -3 & -1 & +2 & -11 \\ -2 & +1 & +2 & -3 \end{array} \right]$$

$$+1x_1 + \frac{1}{2}x_2 - \frac{1}{2}x_3 = +4 \quad (L_1)$$

$$0x_1 + \frac{1}{2}x_2 + \frac{1}{2}x_3 = +1 \quad (3 \times L_1 + L_2)$$

$$0x_1 + 2x_2 + 1x_3 = +5 \quad (2 \times L_1 + L_3)$$

$$\left[ \begin{array}{ccc|c} +1 & +1/2 & -1/2 & +4 \\ 0 & +1/2 & +1/2 & +1 \\ 0 & +2 & +1 & +5 \end{array} \right]$$

# Gauss-Jordan Elimination – Step 3

$$\begin{array}{lcl}
 +1x_1 + \frac{1}{2}x_2 - \frac{1}{2}x_3 = +4 & (L_1) & \\
 0x_1 + \frac{1}{2}x_2 + \frac{1}{2}x_3 = +1 & (L_2) & \\
 0x_1 + 2x_2 + 1x_3 = +5 & (L_3) & 
 \end{array}
 \left[ \begin{array}{ccc|c}
 +1 & +1/2 & -1/2 & +4 \\
 0 & +1/2 & +1/2 & +1 \\
 0 & +2 & +1 & +5
 \end{array} \right]$$

$$\left[ \begin{array}{ccc}
 1 & 0 & 0 \\
 0 & 2 & 0 \\
 0 & 0 & 1
 \end{array} \right]
 \left[ \begin{array}{ccc|c}
 +1 & +1/2 & -1/2 & +4 \\
 0 & +1/2 & +1/2 & +1 \\
 0 & +2 & +1 & +5
 \end{array} \right]$$

$$\begin{array}{lcl}
 +1x_1 + \frac{1}{2}x_2 - \frac{1}{2}x_3 = +4 & (L_1) & \\
 0x_1 + 1x_2 + 1x_3 = +2 & (2 \times L_2) & \\
 0x_1 + 2x_2 + 1x_3 = +5 & (L_3) & 
 \end{array}
 \left[ \begin{array}{ccc|c}
 +1 & +1/2 & -1/2 & +4 \\
 0 & +1 & +1 & +2 \\
 0 & +2 & +1 & +5
 \end{array} \right]$$

# Gauss-Jordan Elimination – Step 4

$$\begin{array}{rcl}
 +1x_1 + \frac{1}{2}x_2 - \frac{1}{2}x_3 = +4 & (L_1) & \\
 0x_1 + 1x_2 + 1x_3 = +2 & (L_2) & \\
 0x_1 + 2x_2 + 1x_3 = +5 & (L_3) & 
 \end{array}
 \left[ \begin{array}{ccc|c}
 +1 & +1/2 & -1/2 & +4 \\
 0 & +1 & +1 & +2 \\
 0 & +2 & +1 & +5 
 \end{array} \right]$$

$$\left[ \begin{array}{ccc}
 1 & 0 & 0 \\
 0 & 1 & 0 \\
 0 & -2 & 1 
 \end{array} \right]
 \left[ \begin{array}{ccc|c}
 +1 & +1/2 & -1/2 & +4 \\
 0 & +1 & +1 & +2 \\
 0 & +2 & +1 & +5 
 \end{array} \right]$$

$$\begin{array}{rcl}
 +1x_1 + \frac{1}{2}x_2 - \frac{1}{2}x_3 = +4 & (L_1) & \\
 0x_1 + 1x_2 + 1x_3 = +2 & (L_2) & \\
 0x_1 + 0x_2 - 1x_3 = +1 & (-2 \times L_2 + L_3) & 
 \end{array}
 \left[ \begin{array}{ccc|c}
 +1 & +1/2 & -1/2 & +4 \\
 0 & +1 & +1 & +2 \\
 0 & 0 & -1 & +1 
 \end{array} \right]$$

# Gauss-Jordan Elimination – Step 5

$$+1x_1 + \frac{1}{2}x_2 - \frac{1}{2}x_3 = +4 \quad (L_1)$$

$$0x_1 + 1x_2 + 1x_3 = +2 \quad (L_2)$$

$$0x_1 + 0x_2 - 1x_3 = +1 \quad (L_3)$$

$$\left[ \begin{array}{ccc|c} +1 & +1/2 & -1/2 & +4 \\ 0 & +1 & +1 & +2 \\ 0 & 0 & -1 & +1 \end{array} \right]$$

$$\left[ \begin{array}{ccc} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{array} \right]$$

$$\left[ \begin{array}{ccc|c} +1 & +1/2 & -1/2 & +4 \\ 0 & +1 & +1 & +2 \\ 0 & 0 & -1 & +1 \end{array} \right]$$

$$+1x_1 + \frac{1}{2}x_2 - \frac{1}{2}x_3 = +4 \quad (L_1)$$

$$0x_1 + 1x_2 + 1x_3 = +2 \quad (L_2)$$

$$0x_1 + 0x_2 + 1x_3 = -1 \quad (-1 \times L_3)$$

$$\left[ \begin{array}{ccc|c} +1 & +1/2 & -1/2 & +4 \\ 0 & +1 & +1 & +2 \\ 0 & 0 & +1 & -1 \end{array} \right]$$

# Forward Phase

$$\begin{array}{c}
 \left( \begin{array}{ccc|c}
 \textcircled{+2} & +1 & -1 & +8 \\
 -3 & -1 & +2 & -11 \\
 -2 & +1 & +2 & -3
 \end{array} \right) \rightarrow \left( \begin{array}{ccc|c}
 \textcircled{+1} & +1/2 & -1/2 & +4 \\
 -3 & -1 & +2 & -11 \\
 -2 & +1 & +2 & -3
 \end{array} \right) \rightarrow \left( \begin{array}{ccc|c}
 +1 & +1/2 & -1/2 & +4 \\
 \boxed{0} & +1/2 & +1/2 & +1 \\
 \boxed{0} & +2 & +1 & +5
 \end{array} \right) \\
 \\
 \left( \begin{array}{ccc|c}
 +1 & +1/2 & -1/2 & +4 \\
 0 & \textcircled{+1} & +1 & +2 \\
 0 & +2 & +1 & +5
 \end{array} \right) \rightarrow \left( \begin{array}{ccc|c}
 +1 & +1/2 & -1/2 & +4 \\
 0 & +1 & +1 & +2 \\
 0 & \boxed{0} & -1 & +1
 \end{array} \right) \rightarrow \left( \begin{array}{ccc|c}
 +1 & +1/2 & -1/2 & +4 \\
 0 & +1 & +1 & +2 \\
 0 & 0 & \textcircled{+1} & -1
 \end{array} \right)
 \end{array}$$

Forward Phase - Gaussian Elimination

# Gauss-Jordan Elimination – Step 6

$$+1x_1 + \frac{1}{2}x_2 - \frac{1}{2}x_3 = +4 \quad (L_1)$$

$$0x_1 + 1x_2 + 1x_3 = +2 \quad (L_2)$$

$$0x_1 + 0x_2 + 1x_3 = -1 \quad (L_3)$$

$$\left[ \begin{array}{ccc|c} +1 & +1/2 & -1/2 & +4 \\ 0 & +1 & +1 & +2 \\ 0 & 0 & +1 & -1 \end{array} \right]$$

$$\left[ \begin{array}{ccc} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{array} \right]$$

$$\left[ \begin{array}{ccc} 1 & 0 & 1/2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array} \right]$$

$$\left[ \begin{array}{ccc|c} +1 & +1/2 & -1/2 & +4 \\ 0 & +1 & +1 & +2 \\ 0 & 0 & +1 & -1 \end{array} \right]$$

$$+1x_1 + \frac{1}{2}x_2 + 0x_3 = +\frac{7}{2} \quad \left(+\frac{1}{2} \times L_3 + L_1\right)$$

$$0x_1 + 1x_2 + 0x_3 = +3 \quad (-1 \times L_3 + L_2)$$

$$0x_1 + 0x_2 + 1x_3 = -1 \quad (L_3)$$

$$\left[ \begin{array}{ccc|c} +1 & +1/2 & 0 & +7/2 \\ 0 & +1 & 0 & +3 \\ 0 & 0 & +1 & -1 \end{array} \right]$$

# Gauss-Jordan Elimination – Step 7

$$\begin{array}{rcl}
 +1x_1 + \frac{1}{2}x_2 + 0x_3 = +\frac{7}{2} & (L_1) & \\
 0x_1 + 1x_2 + 0x_3 = +3 & (L_2) & \\
 0x_1 + 0x_2 + 1x_3 = -1 & (L_3) & 
 \end{array}
 \left[ \begin{array}{ccc|c}
 +1 & \boxed{+1/2} & 0 & +7/2 \\
 0 & +1 & 0 & +3 \\
 0 & 0 & +1 & -1
 \end{array} \right]$$

$$\left[ \begin{array}{ccc}
 1 & -1/2 & 0 \\
 0 & 1 & 0 \\
 0 & 0 & 1
 \end{array} \right]
 \left[ \begin{array}{ccc|c}
 +1 & \boxed{+1/2} & 0 & +7/2 \\
 0 & +1 & 0 & +3 \\
 0 & 0 & +1 & -1
 \end{array} \right]$$

$$\begin{array}{rcl}
 +1x_1 + 0x_2 - 0x_3 = +2 & (-\frac{1}{2} \times L_2 + L_1) & \\
 0x_1 + 1x_2 + 0x_3 = +3 & (L_2) & \\
 0x_1 + 0x_2 + 1x_3 = -1 & (L_3) & 
 \end{array}
 \left[ \begin{array}{ccc|c}
 +1 & \boxed{0} & 0 & +2 \\
 0 & +1 & 0 & +3 \\
 0 & 0 & +1 & -1
 \end{array} \right]$$

# Backward Phase

$$\left( \begin{array}{ccc|c} +1 & +1/2 & -1/2 & +4 \\ 0 & +1 & +1 & +2 \\ 0 & 0 & +1 & -1 \end{array} \right) \rightarrow \left( \begin{array}{ccc|c} +1 & +1/2 & 0 & +7/2 \\ 0 & +1 & 0 & +3 \\ 0 & 0 & +1 & -1 \end{array} \right) \rightarrow \left( \begin{array}{ccc|c} +1 & 0 & 0 & +2 \\ 0 & +1 & 0 & +3 \\ 0 & 0 & +1 & -1 \end{array} \right) \rightarrow$$



# Gauss-Jordan Elimination

## Forward Phase - Gaussian Elimination

$$\begin{array}{c}
 \left( \begin{array}{ccc|c}
 \textcircled{+2} & +1 & -1 & +8 \\
 -3 & -1 & +2 & -11 \\
 -2 & +1 & +2 & -3
 \end{array} \right) \rightarrow \left( \begin{array}{ccc|c}
 \textcircled{+1} & +1/2 & -1/2 & +4 \\
 -3 & -1 & +2 & -11 \\
 -2 & +1 & +2 & -3
 \end{array} \right) \rightarrow \left( \begin{array}{ccc|c}
 +1 & +1/2 & -1/2 & +4 \\
 \boxed{0} & +1/2 & +1/2 & +1 \\
 \boxed{0} & +2 & +1 & +5
 \end{array} \right) \rightarrow \\
 \left( \begin{array}{ccc|c}
 +1 & +1/2 & -1/2 & +4 \\
 0 & \textcircled{+1} & +1 & +2 \\
 0 & +2 & +1 & +5
 \end{array} \right) \rightarrow \left( \begin{array}{ccc|c}
 +1 & +1/2 & -1/2 & +4 \\
 0 & +1 & +1 & +2 \\
 0 & \boxed{0} & -1 & +1
 \end{array} \right) \rightarrow \left( \begin{array}{ccc|c}
 +1 & +1/2 & -1/2 & +4 \\
 0 & +1 & +1 & +2 \\
 0 & 0 & \textcircled{+1} & -1
 \end{array} \right)
 \end{array}$$

## Backward Phase

$$\begin{array}{c}
 \left( \begin{array}{ccc|c}
 +1 & +1/2 & \boxed{-1/2} & +4 \\
 0 & +1 & \boxed{+1} & +2 \\
 0 & 0 & +1 & -1
 \end{array} \right) \rightarrow \left( \begin{array}{ccc|c}
 +1 & +1/2 & \boxed{0} & +7/2 \\
 0 & +1 & \boxed{0} & +3 \\
 0 & 0 & +1 & -1
 \end{array} \right) \rightarrow \left( \begin{array}{ccc|c}
 +1 & \boxed{0} & \boxed{0} & +2 \\
 0 & +1 & \boxed{0} & +3 \\
 0 & 0 & +1 & -1
 \end{array} \right) \rightarrow
 \end{array}$$

# Equivalent Statements

$A$  : invertible

$$\begin{bmatrix} A \\ \text{cyan square} \end{bmatrix} \begin{bmatrix} A^{-1} \\ \text{green square} \end{bmatrix} = \begin{bmatrix} A^{-1} \\ \text{green square} \end{bmatrix} \begin{bmatrix} A \\ \text{cyan square} \end{bmatrix} = \begin{bmatrix} I_n \\ \text{identity matrix} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$Ax = 0$   
only the trivial solution

$$\begin{bmatrix} A \\ \text{cyan square} \end{bmatrix} \begin{bmatrix} x \\ \text{orange column} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$A$  the RREF is  $I_n$   
(Reduced Row Echelon Form)

$$\begin{bmatrix} A \\ \text{cyan square} \end{bmatrix} \xrightarrow{\text{Elem Row Op}} \begin{bmatrix} I_n \\ \text{identity matrix} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$A$  can be written as a product of  $E_k$   
(Elementary Matrices)

$$\begin{matrix} i & j \\ \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \\ j \end{matrix} \quad \begin{matrix} i \\ \begin{bmatrix} c \\ \text{row } i \end{bmatrix} \end{matrix} \quad \begin{matrix} i & j \\ \begin{bmatrix} c \\ \text{row } j \end{bmatrix} \\ j \end{matrix}$$

# Proof (1)

$A$  : invertible

$$\begin{bmatrix} A \\ \text{cyan square} \end{bmatrix} \begin{bmatrix} A^{-1} \\ \text{green square} \end{bmatrix} = \begin{bmatrix} A^{-1} \\ \text{green square} \end{bmatrix} \begin{bmatrix} A \\ \text{cyan square} \end{bmatrix} = \begin{bmatrix} I_n \\ \text{cyan square} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$Ax = 0$   
only the trivial solution

$$\begin{bmatrix} A \\ \text{cyan square} \end{bmatrix} \begin{bmatrix} x \\ \text{orange square} \end{bmatrix} = \begin{bmatrix} 0 \\ \text{cyan square} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$A$  : invertible  
 $x_0$  a solution of  $Ax = 0$  }

$$Ax_0 = 0$$

$$A^{-1}Ax_0 = A^{-1}0$$

$$I_n x_0 = 0$$

$$x_0 = 0 \quad \text{trivial}$$

# Proof (2)

$$Ax = 0$$

only the **trivial** solution

$$A \quad x = 0$$

$$\left[ \begin{array}{c} \square \\ \square \\ \square \end{array} \right] \left[ \begin{array}{c} \square \\ \square \end{array} \right] = \left[ \begin{array}{c} 0 \\ 0 \\ 0 \end{array} \right]$$

$A$  the RREF is  $I_n$   
(Reduced Row Echelon Form)

$$A \xrightarrow{\text{Elem Row Op}} I_n$$

$$\left[ \begin{array}{c} \square \\ \square \\ \square \end{array} \right] \rightarrow \left[ \begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array} \right]$$

only the **trivial** solution

After the forward and backward phases of Gauss-Jordan Elimination

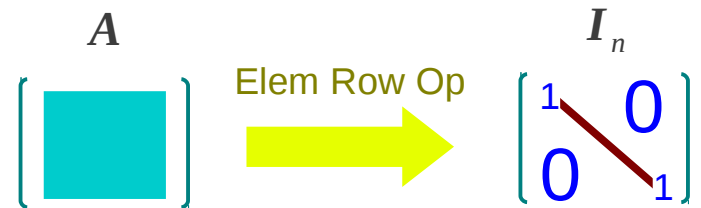
$$\left( \begin{array}{cccc|c} 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \dots & 1 & 0 \end{array} \right)$$

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n &= 0 \\ a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n &= 0 \\ \vdots & \\ a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n &= 0 \end{aligned}$$

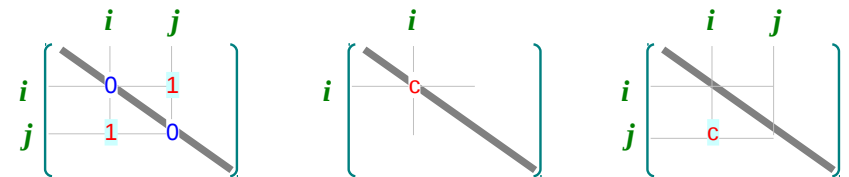
$$\begin{aligned} 1x_1 &= 0 \\ + 1x_2 &= 0 \\ & \vdots \\ 1x_n &= 0 \end{aligned}$$

# Proof (3)

$A$  the RREF is  $I_n$   
(Reduced Row Echelon Form)



$A$  can be written as a product of  $E_k$   
(Elementary Matrices)



$$E_k \cdots E_2 E_1 A = I_n$$



$$E_{k-1} \cdots E_2 E_1 A = E_k^{-1}$$



$$A = E_1^{-1} E_2^{-1} \cdots E_k^{-1}$$

$$E_k^{-1} E_k E_{k-1} \cdots E_2 E_1 A = E_k^{-1} I_n$$

$$E_{k-1}^{-1} E_{k-1} \cdots E_2 E_1 A = E_{k-1}^{-1} E_k^{-1}$$

(Elementary Matrices)

# Proof (4)

$A$  can be written as a product of  $E_k$   
(Elementary Matrices)

Three elementary matrices are shown in a row:

- 1. A row swap matrix: A square matrix with a diagonal line from top-left to bottom-right. The  $i$ -th and  $j$ -th rows are swapped. The diagonal elements are 1, and the off-diagonal elements at  $(i, j)$  and  $(j, i)$  are 1.
- 2. A row scaling matrix: A square matrix with a diagonal line from top-left to bottom-right. The  $i$ -th row is scaled by a factor  $c$ . The diagonal element at  $(i, i)$  is  $c$ , and all other diagonal elements are 1.
- 3. A row addition matrix: A square matrix with a diagonal line from top-left to bottom-right. The  $i$ -th row is added to the  $j$ -th row. The diagonal elements are 1, and the element at  $(j, i)$  is  $c$ .

$A$  : invertible

$$\begin{pmatrix} \blacksquare \end{pmatrix} \begin{pmatrix} \blacksquare \end{pmatrix} = \begin{pmatrix} \blacksquare \end{pmatrix} \begin{pmatrix} \blacksquare \end{pmatrix} = \begin{pmatrix} 1 & & \\ & \ddots & \\ & & 1 \end{pmatrix}$$

$$E_k \cdots E_2 E_1 A = I_n$$

$$A^{-1} A = I_n$$

$$A^{-1} = E_k \cdots E_2 E_1$$

# Inversion Algorithm (1)

$$\begin{array}{c}
 \mathbf{A} \\
 \left[ \begin{array}{c} \text{cyan square} \end{array} \right]
 \end{array}
 \begin{array}{c}
 \mathbf{A}^{-1} \\
 \left[ \begin{array}{c} \text{green square with blue vertical lines} \end{array} \right] \\
 \begin{array}{c} \swarrow \quad \searrow \quad \searrow \\
 \mathbf{x}_1 \mid \mathbf{x}_2 \mid \cdots \mid \mathbf{x}_n
 \end{array}
 \end{array}
 =
 \begin{array}{c}
 \left[ \begin{array}{c} \text{cyan square with blue vertical lines} \end{array} \right] \\
 \begin{array}{c} \swarrow \quad \searrow \quad \searrow \\
 \mathbf{b}_1 \mid \mathbf{b}_2 \mid \cdots \mid \mathbf{b}_n
 \end{array}
 \end{array}$$

$$\begin{array}{c}
 \mathbf{A} \\
 \left[ \begin{array}{c} \text{cyan square} \end{array} \right]
 \end{array}
 \begin{array}{c}
 \mathbf{x}_1 \\
 \left[ \begin{array}{c} \text{green vertical bar} \end{array} \right]
 \end{array}
 =
 \begin{array}{c}
 \mathbf{b}_1 \\
 \left[ \begin{array}{c} 1 \\ 0 \\ 0 \end{array} \right]
 \end{array}$$

$$\begin{array}{c}
 \mathbf{A} \\
 \left[ \begin{array}{c} \text{cyan square} \end{array} \right]
 \end{array}
 \begin{array}{c}
 \mathbf{x}_n \\
 \left[ \begin{array}{c} \text{green vertical bar} \end{array} \right]
 \end{array}
 =
 \begin{array}{c}
 \mathbf{b}_n \\
 \left[ \begin{array}{c} 0 \\ 0 \\ 1 \end{array} \right]
 \end{array}$$

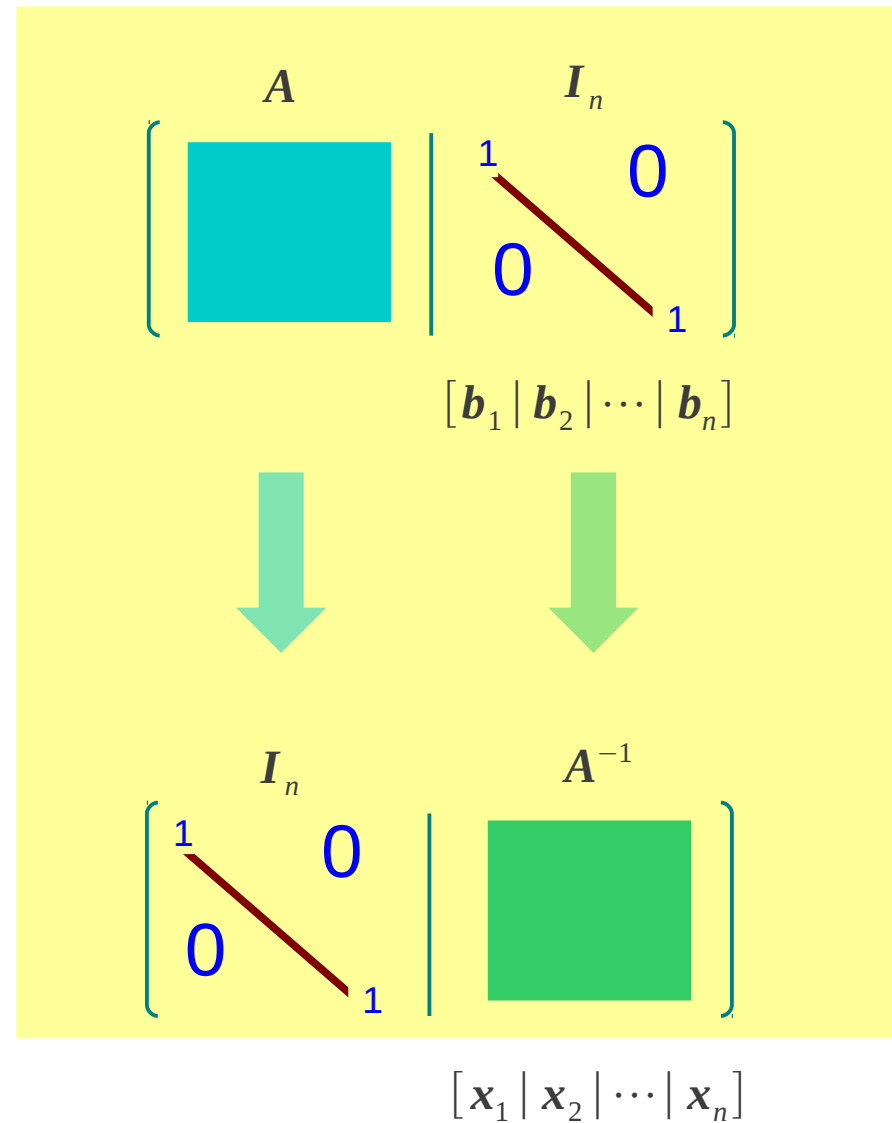
$$\begin{array}{c}
 \mathbf{A} \\
 \left[ \begin{array}{c} \text{cyan square} \end{array} \right]
 \end{array}
 \begin{array}{c}
 \mathbf{x}_2 \\
 \left[ \begin{array}{c} \text{green vertical bar} \end{array} \right]
 \end{array}
 =
 \begin{array}{c}
 \mathbf{b}_2 \\
 \left[ \begin{array}{c} 0 \\ 1 \\ 0 \end{array} \right]
 \end{array}$$

# Inversion Algorithm (2)

$$\left[ \begin{array}{c|c} A & \mathbf{x}_1 \end{array} \right] = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$\left[ \begin{array}{c|c} A & \mathbf{x}_2 \end{array} \right] = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$$\left[ \begin{array}{c|c} A & \mathbf{x}_n \end{array} \right] = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$





# Homogeneous System

$$\begin{array}{ccccccccccc} a_{11} & x_1 & + & a_{12} & x_2 & + & \cdots & + & a_{1n} & x_n & = & 0 \\ a_{21} & x_1 & + & a_{22} & x_2 & + & \cdots & + & a_{2n} & x_n & = & 0 \\ \vdots & & & \vdots & & & & & \vdots & & & \vdots \\ a_{m1} & x_1 & + & a_{m2} & x_2 & + & \cdots & + & a_{mn} & x_n & = & 0 \end{array}$$

All constant terms are zero

# Homogeneous System

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All constant terms  
are zero



$$\begin{matrix} & & i & j \\ i & & 0 & 1 \\ j & & 1 & 0 \end{matrix} \left[ \begin{array}{c} \diagdown \\ \diagup \end{array} \right]$$

$$\begin{matrix} & & i \\ i & & c \end{matrix} \left[ \begin{array}{c} \diagdown \\ \diagup \end{array} \right]$$

$$\begin{matrix} & & i & j \\ i & & & \\ j & & c & \end{matrix} \left[ \begin{array}{c} \diagdown \\ \diagup \end{array} \right]$$

## References

- [1] <http://en.wikipedia.org/>
- [2] Anton & Busby, "Contemporary Linear Algebra"