Elementary Matrix

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Please send corrections (or suggestions) to youngwlim@hotmail.com.

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Gauss-Jordan Elimination

Forward Phase - Gaussian Elimination

$$\left(\begin{array}{ccc|c} +2 & +1 & -1 & +8 \\ -3 & -1 & +2 & -11 \\ -2 & +1 & +2 & -3 \end{array}\right) \Rightarrow \left(\begin{array}{ccc|c} +1 & +1/2 & -1/2 & +4 \\ -3 & -1 & +2 & -11 \\ -2 & +1 & +2 & -3 \end{array}\right) \Rightarrow \left(\begin{array}{ccc|c} +1 & +1/2 & -1/2 & +4 \\ 0 & +1/2 & +1/2 & +1 \\ 0 & +2 & +1 & +5 \end{array}\right) \Rightarrow$$

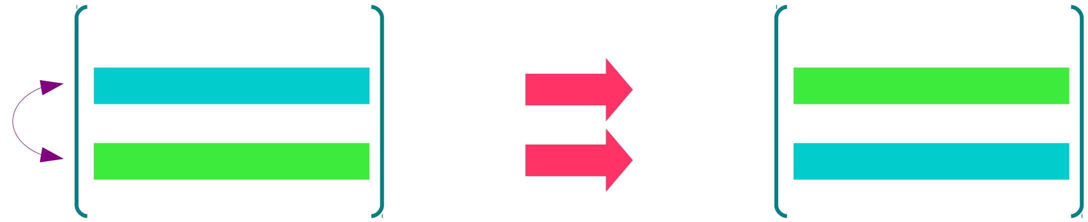
$$\left(\begin{array}{ccc|c} +1 & +1/2 & -1/2 & +4 \\ 0 & +1 & +1 & +2 \\ 0 & +2 & +1 & +5 \end{array}\right) \Rightarrow \left(\begin{array}{ccc|c} +1 & +1/2 & -1/2 & +4 \\ 0 & +1 & +1 & +2 \\ 0 & 0 & -1 & +1 \end{array}\right) \Rightarrow \left(\begin{array}{ccc|c} +1 & +1/2 & -1/2 & +4 \\ 0 & +1 & +1 & +2 \\ 0 & 0 & +1 & -1 \end{array}\right) \Rightarrow$$

Backward Phase

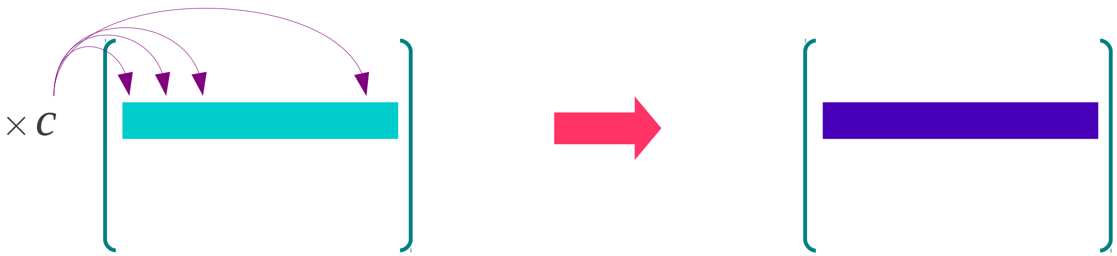
$$\left(\begin{array}{ccc|c} +1 & +1/2 & -1/2 & +4 \\ 0 & +1 & +1 & +2 \\ 0 & 0 & +1 & -1 \end{array}\right) \Rightarrow \left(\begin{array}{ccc|c} +1 & +1/2 & 0 & +7/2 \\ 0 & +1 & 0 & +3 \\ 0 & 0 & +1 & -1 \end{array}\right) \Rightarrow \left(\begin{array}{ccc|c} +1 & 0 & 0 & +2 \\ 0 & +1 & 0 & +3 \\ 0 & 0 & +1 & -1 \end{array}\right)$$

Elementary Row Operation

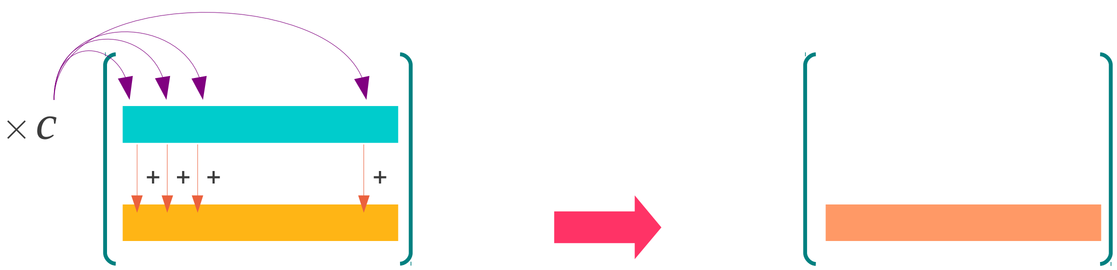
Interchange two rows



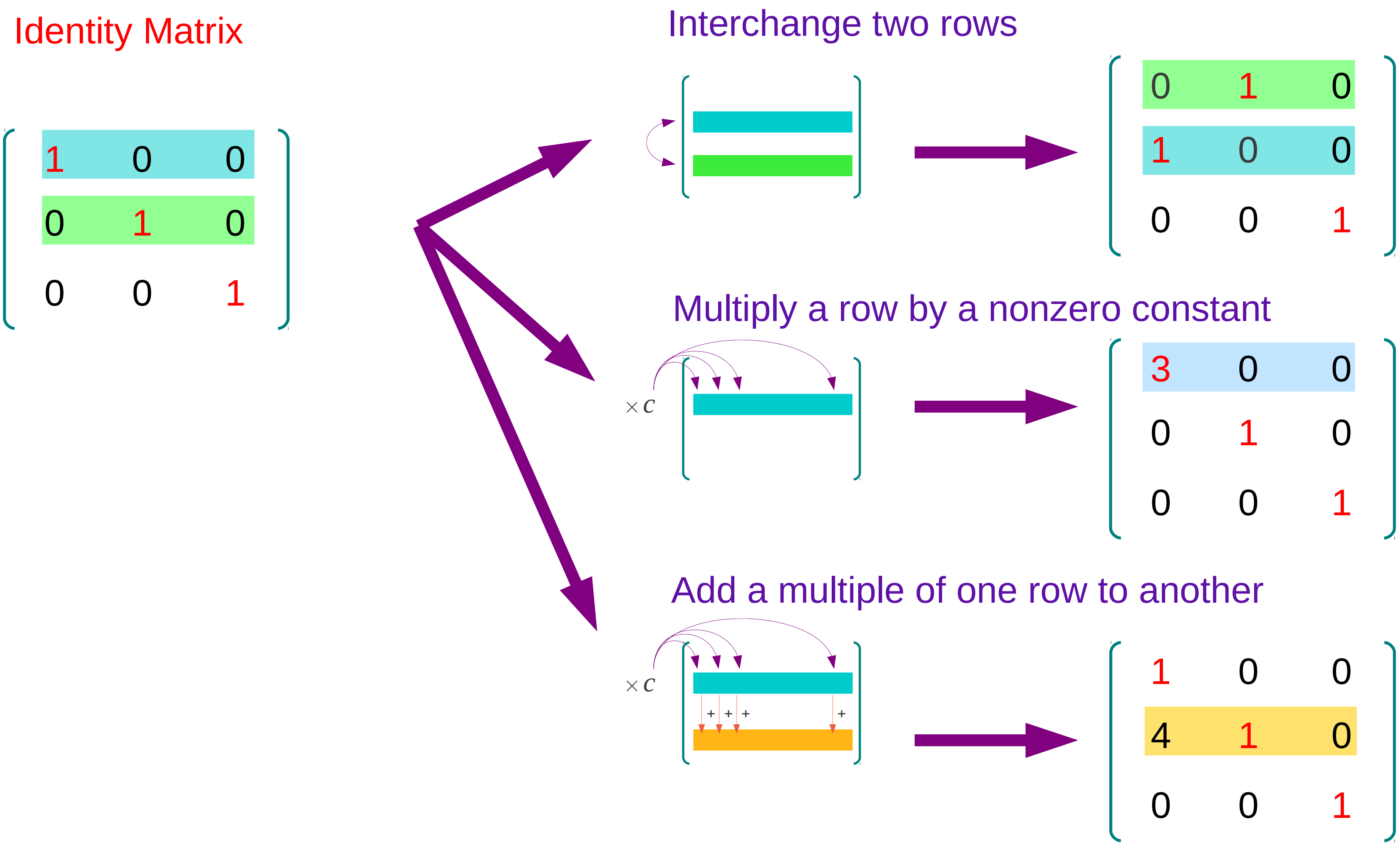
Multiply a row by a nonzero constant



Add a multiple of one row to another



Elementary Matrix



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09/12/2012

Multiplication by an Elementary Matrix

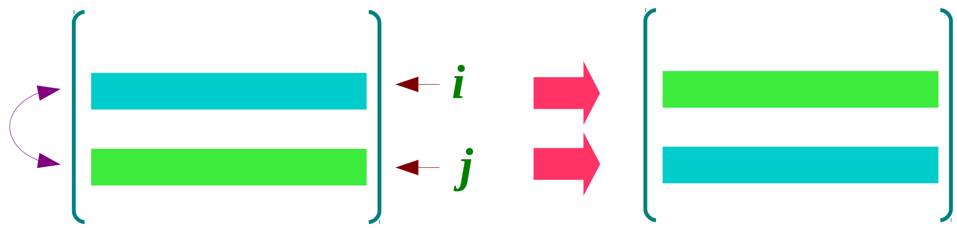
$$\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \qquad \begin{bmatrix} 4 & 5 & 6 \\ 1 & 2 & 3 \\ 7 & 8 & 9 \end{bmatrix}$$

$$\begin{bmatrix} 3 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \qquad \begin{bmatrix} 3 & 6 & 9 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$$

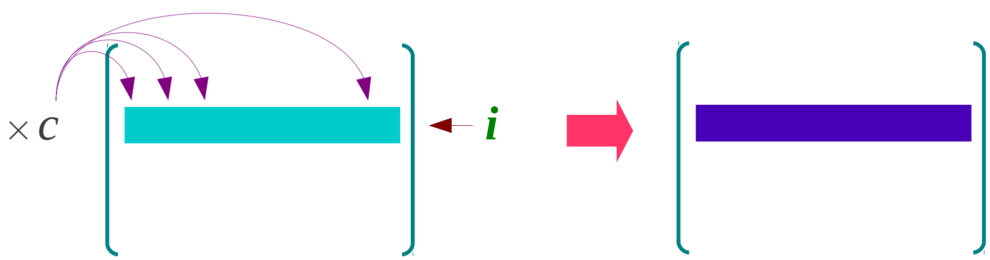
$$\begin{bmatrix} 1 & 0 & 0 \\ 4 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \qquad \begin{bmatrix} 1 & 2 & 3 \\ 8 & 13 & 18 \\ 7 & 8 & 9 \end{bmatrix}$$

Elementary Matrix

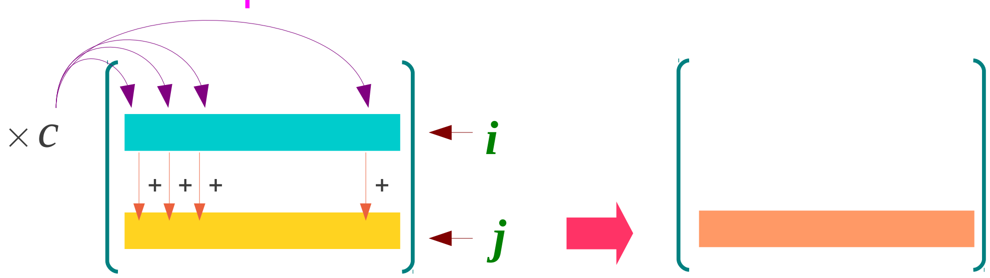
Interchange two rows

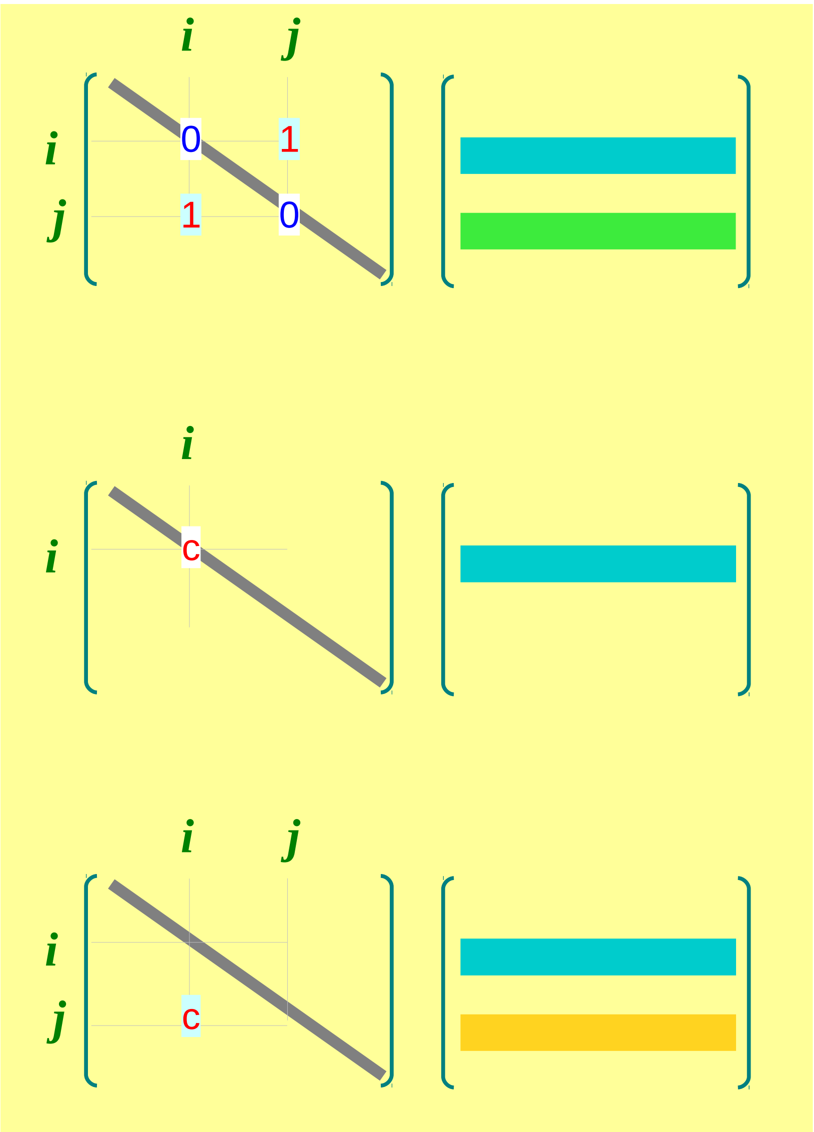


Multiply a row by a nonzero constant



Add a multiple of one row to another





Gauss-Jordan Elimination – Step 1

$$+2x_1 + x_2 - x_3 = 8 \qquad (L_1)$$
$$-3x_1 - x_2 + 2x_3 = -11 \qquad (L_2)$$
$$-2x_1 + x_2 + 2x_3 = -3 \qquad (L_3)$$

$$\left(\begin{array}{ccc|c} +2 & +1 & -1 & +8 \\ -3 & -1 & +2 & -11 \\ -2 & +1 & +2 & -3 \end{array}\right)$$

$$\begin{pmatrix} 1/2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \left(\begin{array}{ccc|c} +2 & +1 & -1 & +8 \\ -3 & -1 & +2 & -11 \\ -2 & +1 & +2 & -3 \end{array}\right)$$

$$+1x_1 + \tfrac{1}{2}x_2 - \tfrac{1}{2}x_3 = 4 \qquad (\tfrac{1}{2} \times L_1)$$
$$-3x_1 - x_2 + 2x_3 = -11 \qquad (L_2)$$
$$-2x_1 + x_2 + 2x_3 = -3 \qquad (L_3)$$

$$\left(\begin{array}{ccc|c} +1 & +1/2 & -1/2 & +4 \\ -3 & -1 & +2 & -11 \\ -2 & +1 & +2 & -3 \end{array}\right)$$

Gauss-Jordan Elimination – Step 2

$$+1x_1 + \tfrac{1}{2}x_2 - \tfrac{1}{2}x_3 = +4 \quad (L_1)$$
$$-3x_1 - x_2 + 2x_3 = -11 \quad (L_2)$$
$$-2x_1 + x_2 + 2x_3 = -3 \quad (L_3)$$

$$\left(\begin{array}{ccc|c} +1 & +1/2 & -1/2 & +4 \\ \boxed{-3} & -1 & +2 & -11 \\ \boxed{-2} & +1 & +2 & -3 \end{array}\right)$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{pmatrix} \quad \begin{pmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \left(\begin{array}{ccc|c} +1 & +1/2 & -1/2 & +4 \\ \boxed{-3} & -1 & +2 & -11 \\ \boxed{-2} & +1 & +2 & -3 \end{array}\right)$$

$$+1x_1 + \tfrac{1}{2}x_2 - \tfrac{1}{2}x_3 = +4 \quad (L_1)$$
$$0x_1 + \tfrac{1}{2}x_2 + \tfrac{1}{2}x_3 = +1 \quad (\boxed{3 \times L_1} + L_2)$$
$$0x_1 + 2x_2 + 1x_3 = +5 \quad (\boxed{2 \times L_1} + L_3)$$

$$\left(\begin{array}{ccc|c} +1 & +1/2 & -1/2 & +4 \\ \boxed{0} & +1/2 & +1/2 & +1 \\ \boxed{0} & +2 & +1 & +5 \end{array}\right)$$

Gauss-Jordan Elimination – Step 3

$$+1x_1 + \tfrac{1}{2}x_2 - \tfrac{1}{2}x_3 = +4 \qquad (L_1)$$
$$0x_1 + \tfrac{1}{2}x_2 + \tfrac{1}{2}x_3 = +1 \qquad (L_2)$$
$$0x_1 + 2x_2 + 1x_3 = +5 \qquad (L_3)$$

$$\left[\begin{array}{ccc|c} +1 & +1/2 & -1/2 & +4 \\ 0 & +1/2 & +1/2 & +1 \\ 0 & +2 & +1 & +5 \end{array}\right]$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \left[\begin{array}{ccc|c} +1 & +1/2 & -1/2 & +4 \\ 0 & +1/2 & +1/2 & +1 \\ 0 & +2 & +1 & +5 \end{array}\right]$$

$$+1x_1 + \tfrac{1}{2}x_2 - \tfrac{1}{2}x_3 = +4 \qquad (L_1)$$
$$0x_1 + 1x_2 + 1x_3 = +2 \qquad (2 \times L_2)$$
$$0x_1 + 2x_2 + 1x_3 = +5 \qquad (L_3)$$

$$\left[\begin{array}{ccc|c} +1 & +1/2 & -1/2 & +4 \\ 0 & +1 & +1 & +2 \\ 0 & +2 & +1 & +5 \end{array}\right]$$

Gauss-Jordan Elimination – Step 4

$$+1x_1 + \tfrac{1}{2}x_2 - \tfrac{1}{2}x_3 = +4 \qquad (L_1)$$
$$0x_1 + 1x_2 + 1x_3 = +2 \qquad (L_2)$$
$$0x_1 + 2x_2 + 1x_3 = +5 \qquad (L_3)$$

$$\left(\begin{array}{ccc|c} +1 & +1/2 & -1/2 & +4 \\ 0 & +1 & +1 & +2 \\ 0 & \boxed{+2} & +1 & +5 \end{array}\right)$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{bmatrix} \left(\begin{array}{ccc|c} +1 & +1/2 & -1/2 & +4 \\ 0 & +1 & +1 & +2 \\ 0 & \boxed{+2} & +1 & +5 \end{array}\right)$$

$$+1x_1 + \tfrac{1}{2}x_2 - \tfrac{1}{2}x_3 = +4 \qquad (L_1)$$
$$0x_1 + 1x_2 + 1x_3 = +2 \qquad (L_2)$$
$$0x_1 + 0x_2 - 1x_3 = +1 \qquad (\boxed{-2 \times L_2} + L_3)$$

$$\left(\begin{array}{ccc|c} +1 & +1/2 & -1/2 & +4 \\ 0 & +1 & +1 & +2 \\ 0 & \boxed{0} & -1 & +1 \end{array}\right)$$

Gauss-Jordan Elimination – Step 5

$$+1x_1 + \tfrac{1}{2}x_2 - \tfrac{1}{2}x_3 = +4 \qquad (L_1)$$
$$0x_1 + 1x_2 + 1x_3 = +2 \qquad (L_2)$$
$$0x_1 + 0x_2 - 1x_3 = +1 \qquad (L_3)$$

$$\left[\begin{array}{ccc|c} +1 & +1/2 & -1/2 & +4 \\ 0 & +1 & +1 & +2 \\ 0 & 0 & -1 & +1 \end{array}\right]$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \quad \left[\begin{array}{ccc|c} +1 & +1/2 & -1/2 & +4 \\ 0 & +1 & +1 & +2 \\ 0 & 0 & -1 & +1 \end{array}\right]$$

$$+1x_1 + \tfrac{1}{2}x_2 - \tfrac{1}{2}x_3 = +4 \qquad (L_1)$$
$$0x_1 + 1x_2 + 1x_3 = +2 \qquad (L_2)$$
$$0x_1 + 0x_2 + 1x_3 = -1 \qquad (-1 \times L_3)$$

$$\left[\begin{array}{ccc|c} +1 & +1/2 & -1/2 & +4 \\ 0 & +1 & +1 & +2 \\ 0 & 0 & +1 & -1 \end{array}\right]$$

Forward Phase

$$\left(\begin{array}{ccc|c} +2 & +1 & -1 & +8 \\ -3 & -1 & +2 & -11 \\ -2 & +1 & +2 & -3 \end{array}\right) \Rightarrow \left(\begin{array}{ccc|c} +1 & +1/2 & -1/2 & +4 \\ -3 & -1 & +2 & -11 \\ -2 & +1 & +2 & -3 \end{array}\right) \Rightarrow \left(\begin{array}{ccc|c} +1 & +1/2 & -1/2 & +4 \\ 0 & +1/2 & +1/2 & +1 \\ 0 & +2 & +1 & +5 \end{array}\right) \Rightarrow$$

$$\left(\begin{array}{ccc|c} +1 & +1/2 & -1/2 & +4 \\ 0 & +1 & +1 & +2 \\ 0 & +2 & +1 & +5 \end{array}\right) \Rightarrow \left(\begin{array}{ccc|c} +1 & +1/2 & -1/2 & +4 \\ 0 & +1 & +1 & +2 \\ 0 & 0 & -1 & +1 \end{array}\right) \Rightarrow \left(\begin{array}{ccc|c} +1 & +1/2 & -1/2 & +4 \\ 0 & +1 & +1 & +2 \\ 0 & 0 & +1 & -1 \end{array}\right) \Rightarrow$$

Forward Phase - Gaussian Elimination

Gauss-Jordan Elimination – Step 6

$$+1x_1 + \tfrac{1}{2}x_2 - \tfrac{1}{2}x_3 = +4 \qquad (L_1)$$
$$0x_1 + 1x_2 + 1x_3 = +2 \qquad (L_2)$$
$$0x_1 + 0x_2 + 1x_3 = -1 \qquad (L_3)$$

$$\left(\begin{array}{ccc|c} +1 & +1/2 & -1/2 & +4 \\ 0 & +1 & +1 & +2 \\ 0 & 0 & +1 & -1 \end{array}\right)$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix} \qquad \begin{pmatrix} 1 & 0 & 1/2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad \left(\begin{array}{ccc|c} +1 & +1/2 & -1/2 & +4 \\ 0 & +1 & +1 & +2 \\ 0 & 0 & +1 & -1 \end{array}\right)$$

$$+1x_1 + \tfrac{1}{2}x_2 + 0x_3 = +\tfrac{7}{2} \qquad (+\tfrac{1}{2} \times L_3 + L_1)$$
$$0x_1 + 1x_2 + 0x_3 = +3 \qquad (-1 \times L_3 + L_2)$$
$$0x_1 + 0x_2 + 1x_3 = -1 \qquad (L_3)$$

$$\left(\begin{array}{ccc|c} +1 & +1/2 & 0 & +7/2 \\ 0 & +1 & 0 & +3 \\ 0 & 0 & +1 & -1 \end{array}\right)$$

Gauss-Jordan Elimination – Step 7

$$+1x_1 + \tfrac{1}{2}x_2 + 0x_3 = +\tfrac{7}{2} \qquad (L_1)$$
$$0x_1 + 1x_2 + 0x_3 = +3 \qquad (L_2)$$
$$0x_1 + 0x_2 + 1x_3 = -1 \qquad (L_3)$$

$$\left(\begin{array}{ccc|c} +1 & \boxed{+1/2} & 0 & +7/2 \\ 0 & +1 & 0 & +3 \\ 0 & 0 & +1 & -1 \end{array}\right)$$

$$\begin{pmatrix} 1 & -1/2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \left(\begin{array}{ccc|c} +1 & \boxed{+1/2} & 0 & +7/2 \\ 0 & +1 & 0 & +3 \\ 0 & 0 & +1 & -1 \end{array}\right)$$

$$+1x_1 + 0x_2 - 0x_3 = +2 \qquad (-\tfrac{1}{2} \times L_2 + L_1)$$
$$0x_1 + 1x_2 + 0x_3 = +3 \qquad (L_2)$$
$$0x_1 + 0x_2 + 1x_3 = -1 \qquad (L_3)$$

$$\left(\begin{array}{ccc|c} +1 & \boxed{0} & 0 & +2 \\ 0 & +1 & 0 & +3 \\ 0 & 0 & +1 & -1 \end{array}\right)$$

Backward Phase

$$\left(\begin{array}{ccc|c} +1 & +1/2 & \boxed{-1/2} & +4 \\ 0 & +1 & \boxed{+1} & +2 \\ 0 & 0 & +1 & -1 \end{array}\right) \Rightarrow \left(\begin{array}{ccc|c} +1 & +1/2 & \boxed{0} & +7/2 \\ 0 & +1 & \boxed{0} & +3 \\ 0 & 0 & +1 & -1 \end{array}\right) \Rightarrow \left(\begin{array}{ccc|c} +1 & \boxed{0} & 0 & +2 \\ 0 & +1 & 0 & +3 \\ 0 & 0 & +1 & -1 \end{array}\right) \Rightarrow$$

Gauss-Jordan Elimination

Forward Phase - Gaussian Elimination

$$\left(\begin{array}{ccc|c} +2 & +1 & -1 & +8 \\ -3 & -1 & +2 & -11 \\ -2 & +1 & +2 & -3 \end{array}\right) \Rightarrow \left(\begin{array}{ccc|c} +1 & +1/2 & -1/2 & +4 \\ -3 & -1 & +2 & -11 \\ -2 & +1 & +2 & -3 \end{array}\right) \Rightarrow \left(\begin{array}{ccc|c} +1 & +1/2 & -1/2 & +4 \\ 0 & +1/2 & +1/2 & +1 \\ 0 & +2 & +1 & +5 \end{array}\right) \Rightarrow$$

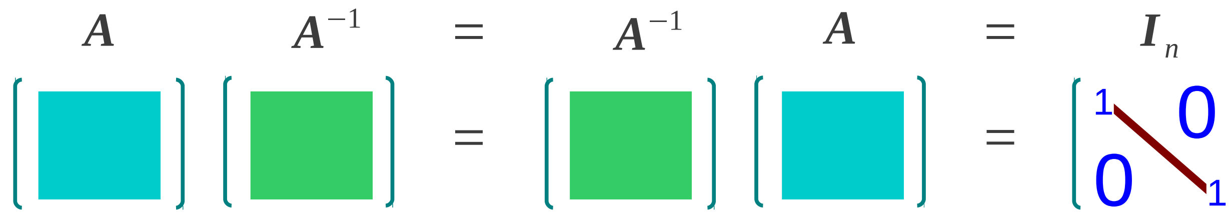
$$\left(\begin{array}{ccc|c} +1 & +1/2 & -1/2 & +4 \\ 0 & +1 & +1 & +2 \\ 0 & +2 & +1 & +5 \end{array}\right) \Rightarrow \left(\begin{array}{ccc|c} +1 & +1/2 & -1/2 & +4 \\ 0 & +1 & +1 & +2 \\ 0 & 0 & -1 & +1 \end{array}\right) \Rightarrow \left(\begin{array}{ccc|c} +1 & +1/2 & -1/2 & +4 \\ 0 & +1 & +1 & +2 \\ 0 & 0 & +1 & -1 \end{array}\right) \Rightarrow$$

Backward Phase

$$\left(\begin{array}{ccc|c} +1 & +1/2 & -1/2 & +4 \\ 0 & +1 & +1 & +2 \\ 0 & 0 & +1 & -1 \end{array}\right) \Rightarrow \left(\begin{array}{ccc|c} +1 & +1/2 & 0 & +7/2 \\ 0 & +1 & 0 & +3 \\ 0 & 0 & +1 & -1 \end{array}\right) \Rightarrow \left(\begin{array}{ccc|c} +1 & 0 & 0 & +2 \\ 0 & +1 & 0 & +3 \\ 0 & 0 & +1 & -1 \end{array}\right) \Rightarrow$$

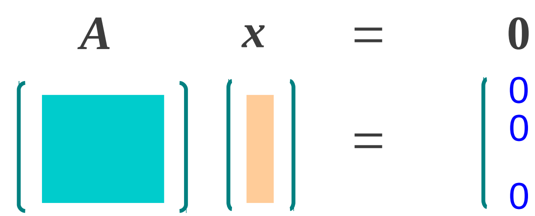
Equivalent Statements

A : invertible



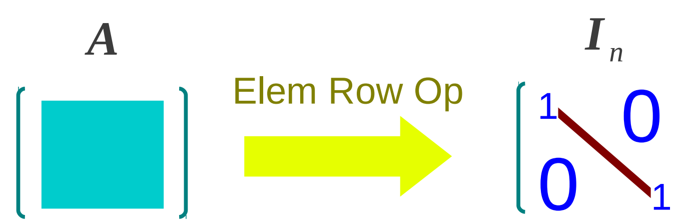
$Ax = 0$

only the trivial solution



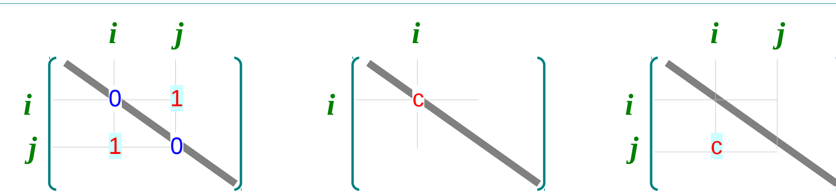
A the RREF is I_n

(Reduced Row Echelon Form)



A can be written as a product of E_k

(Elementary Matrices)



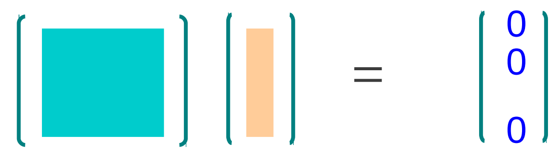
A : invertible

$$A \quad A^{-1} = A^{-1} \quad A = I_n$$

$Ax = 0$

only the trivial solution

$$A \quad x = \mathbf{0}$$



A : invertible

x_0 a solution of $Ax = 0$

$$Ax_0 = 0$$

$$A^{-1}Ax_0 = A^{-1}0$$

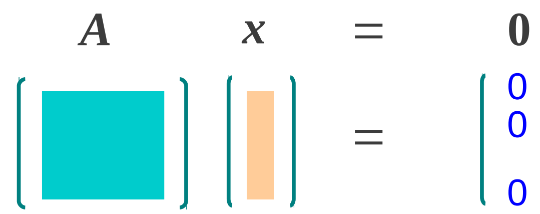
$$I_n x_0 = 0$$

$$x_0 = 0 \quad \text{trivial}$$

Proof (2)

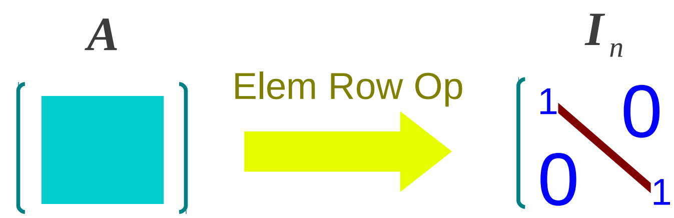
$\boldsymbol{A}\boldsymbol{x} = 0$

only the trivial solution



$\boldsymbol{A}$ the RREF is $\boldsymbol{I}_n$

(Reduced Row Echelon Form)



only the trivial solution

After the forward and backward phases of Gauss-Jordan Elimination

$$\left(\begin{array}{cccc|c} 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{array}\right)$$

$$\begin{array}{ccccccccc} a_{11} x_1 & + & a_{12} x_2 & + & \cdots & + & a_{1n} x_n & = & 0 \\ a_{21} x_1 & + & a_{22} x_2 & + & \cdots & + & a_{2n} x_n & = & 0 \\ \vdots & & \vdots & & & & \vdots & & \vdots \\ a_{m1} x_1 & + & a_{m2} x_2 & + & \cdots & + & a_{mn} x_n & = & 0 \end{array}$$

$$\begin{array}{ccccccc} 1\,x_1 & & & & & = & 0 \\ & + & 1\,x_2 & & & = & 0 \\ & & & \ddots & & & \vdots \\ & & & & 1\,x_n & = & 0 \end{array}$$

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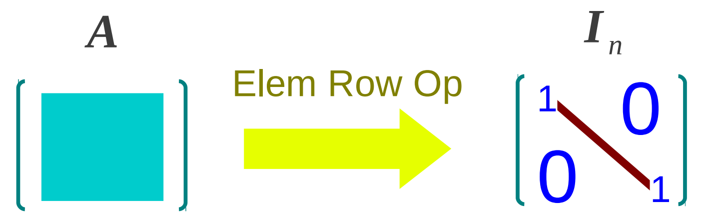
09/12/2012

Proof (3)

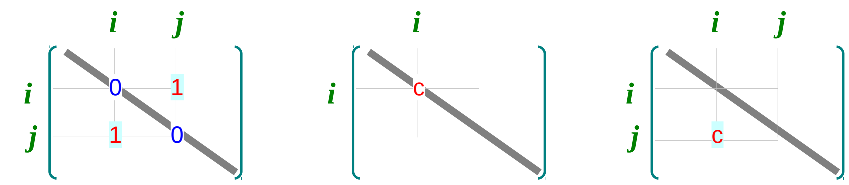
A the RREF is I_n

(Reduced Row Echelon Form)



A can be written as a product of E_k

(Elementary Matrices)



$$E_k \cdots E_2 E_1 A = I_n$$

$$E_k^{-1} E_k E_{k-1} \cdots E_2 E_1 A = E_k^{-1} I_n$$

$$E_{k-1} \cdots E_2 E_1 A = E_k^{-1}$$

$$E_{k-1}^{-1} E_{k-1} \cdots E_2 E_1 A = E_{k-1}^{-1} E_k^{-1}$$

$$A = E_1^{-1} E_2^{-1} \cdots E_k^{-1}$$

(Elementary Matrices)

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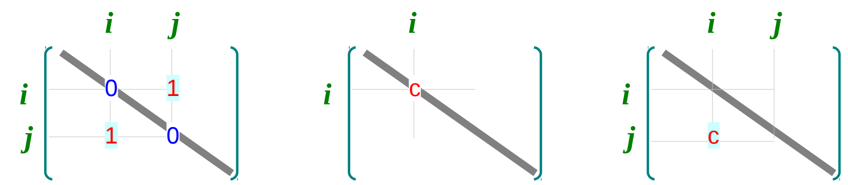
Proof (4)

A can be written as a product of E_k

(Elementary Matrices)



A : invertible

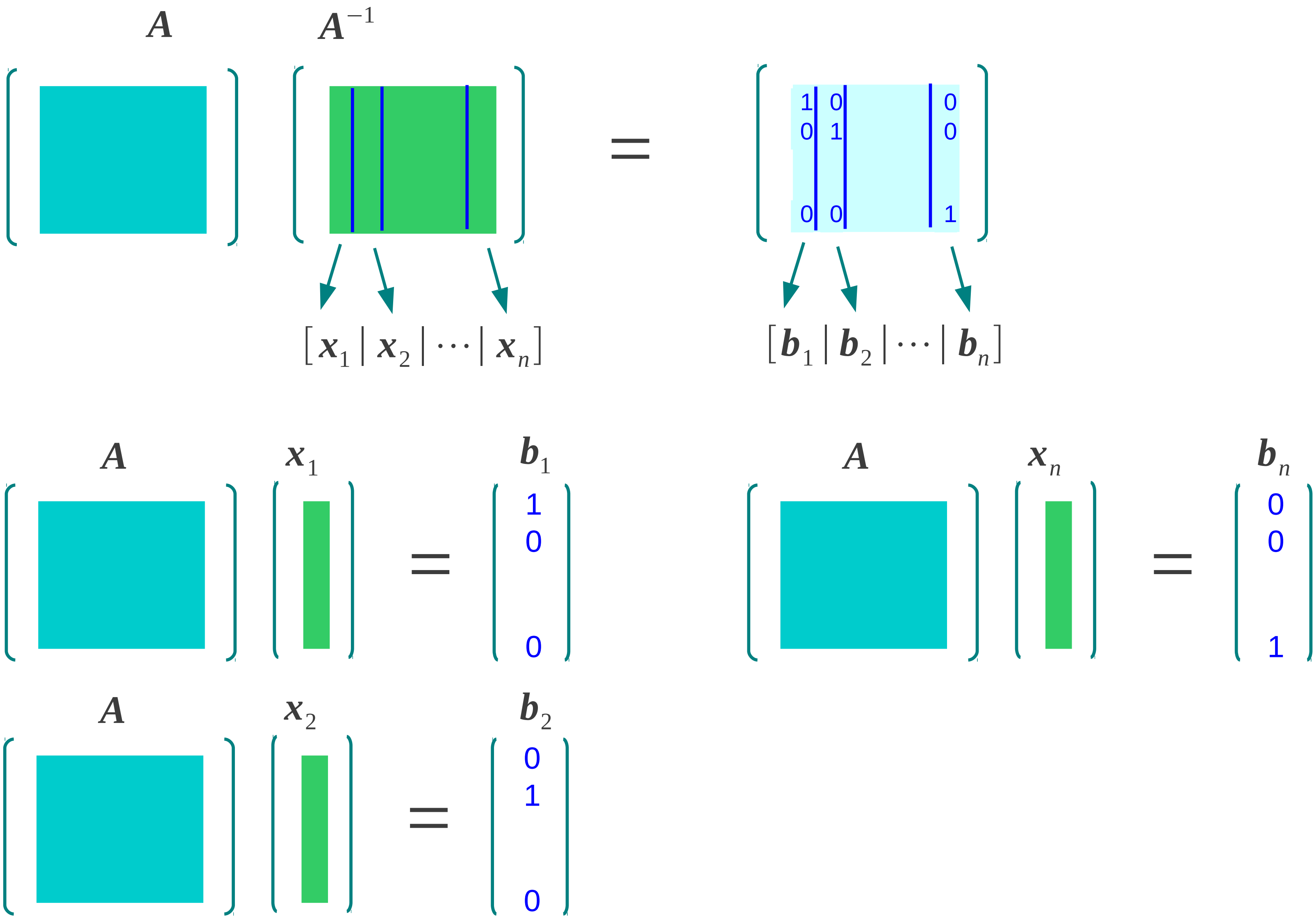
$$A \; A^{-1} = A^{-1} \; A = I_n$$

$$E_k \cdots E_2 E_1 A = I_n$$

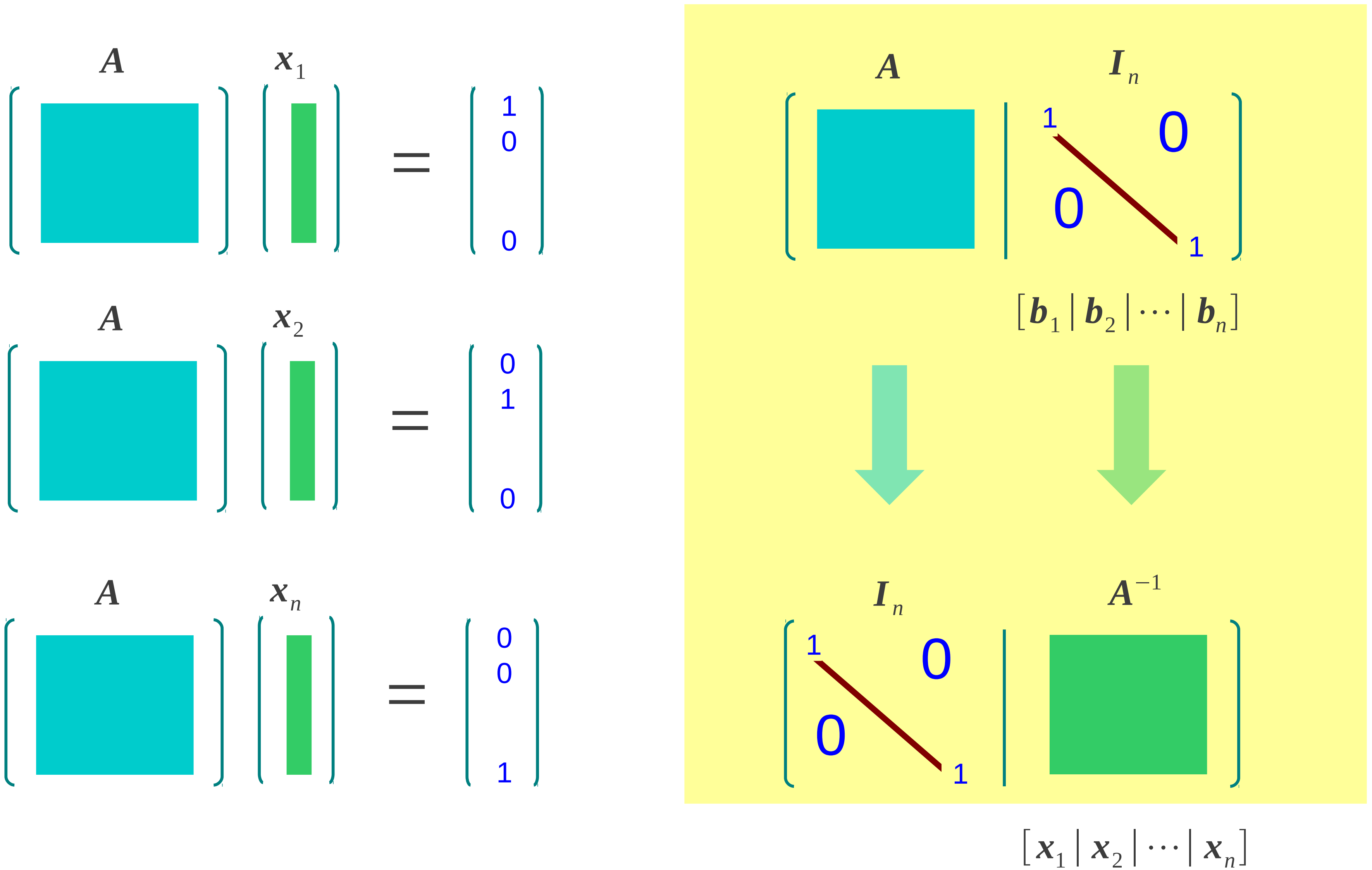
$$A^{-1} A = I_n$$

$$A^{-1} = E_k \cdots E_2 E_1$$

Inversion Algorithm (1)



Inversion Algorithm (2)



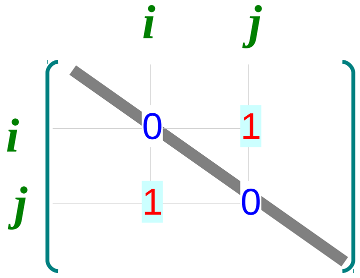
Homogeneous System

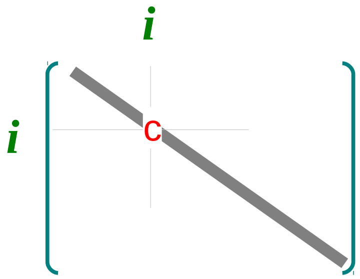
$$
\begin{array}{ccccccccc}
a_{11} x_1 & + & a_{12} x_2 & + & \cdots & + & a_{1n} x_n & = & 0 \\
a_{21} x_1 & + & a_{22} x_2 & + & \cdots & + & a_{2n} x_n & = & 0 \\
\vdots & & \vdots & & & & \vdots & & \vdots \\
a_{m1} x_1 & + & a_{m2} x_2 & + & \cdots & + & a_{mn} x_n & = & 0
\end{array}
$$

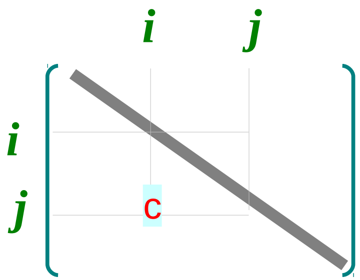
All constant terms are zero

Homogeneous System

All constant terms
are zero


i j
i 0 1
j 1 0


i
i c


i j
i
j c

References

[1] http://en.wikipedia.org/
[2] Anton & Busby, “Contemporary Linear Algebra”