Hermitian Inner Product Space (3B)

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Young Won Lim
2/9/12

Please send corrections (or suggestions) to youngwlim@hotmail.com.

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Complex Vector Inner Product

Hermitian inner product

$$\langle \boldsymbol{x} , \boldsymbol{y} \rangle = \boldsymbol{x}^H \cdot \boldsymbol{y} = \sum x_i^* y_i$$

$\boldsymbol{x}^H$: conjugate transpose

Norm of Hermitian inner products

$$\|\boldsymbol{x}\| = \sqrt{\langle \boldsymbol{x} , \boldsymbol{x} \rangle} = \sqrt{\boldsymbol{x}^H \cdot \boldsymbol{x}} = \sqrt{\sum x_i^* x_i}$$

the length of a vector

$$\boldsymbol{x} = \begin{pmatrix} a_1 + j b_1 \\ a_2 + j b_2 \\ \vdots \\ a_n + j b_n \end{pmatrix}$$

$$\langle \boldsymbol{x} , \boldsymbol{x} \rangle = \boldsymbol{x}^H \cdot \boldsymbol{x} = \sum x_i^* x_i$$

$$\begin{pmatrix} a_1 - j b_1 & a_2 - j b_2 & \cdots & a_n - j b_n \end{pmatrix} \begin{pmatrix} a_1 + j b_1 \\ a_2 + j b_2 \\ \vdots \\ a_n + j b_n \end{pmatrix} = \sum_{i=1}^{n} a_i^2 + b_i^2$$

Cauchy-Schwartz Inequality

For all vectors $\boldsymbol{x}$ and $\boldsymbol{y}$ of an inner product space

$$\left| \langle \boldsymbol{x}, \boldsymbol{y} \rangle \right|^2 \le \langle \boldsymbol{x}, \boldsymbol{x} \rangle \cdot \langle \boldsymbol{y}, \boldsymbol{y} \rangle$$

$$\left| \langle \boldsymbol{x}, \boldsymbol{y} \rangle \right| \le \|\boldsymbol{x}\| \cdot \|\boldsymbol{y}\|$$

The equality holds if and only if $\boldsymbol{x}$ and $\boldsymbol{y}$ are linearly dependent ⇨ maximum

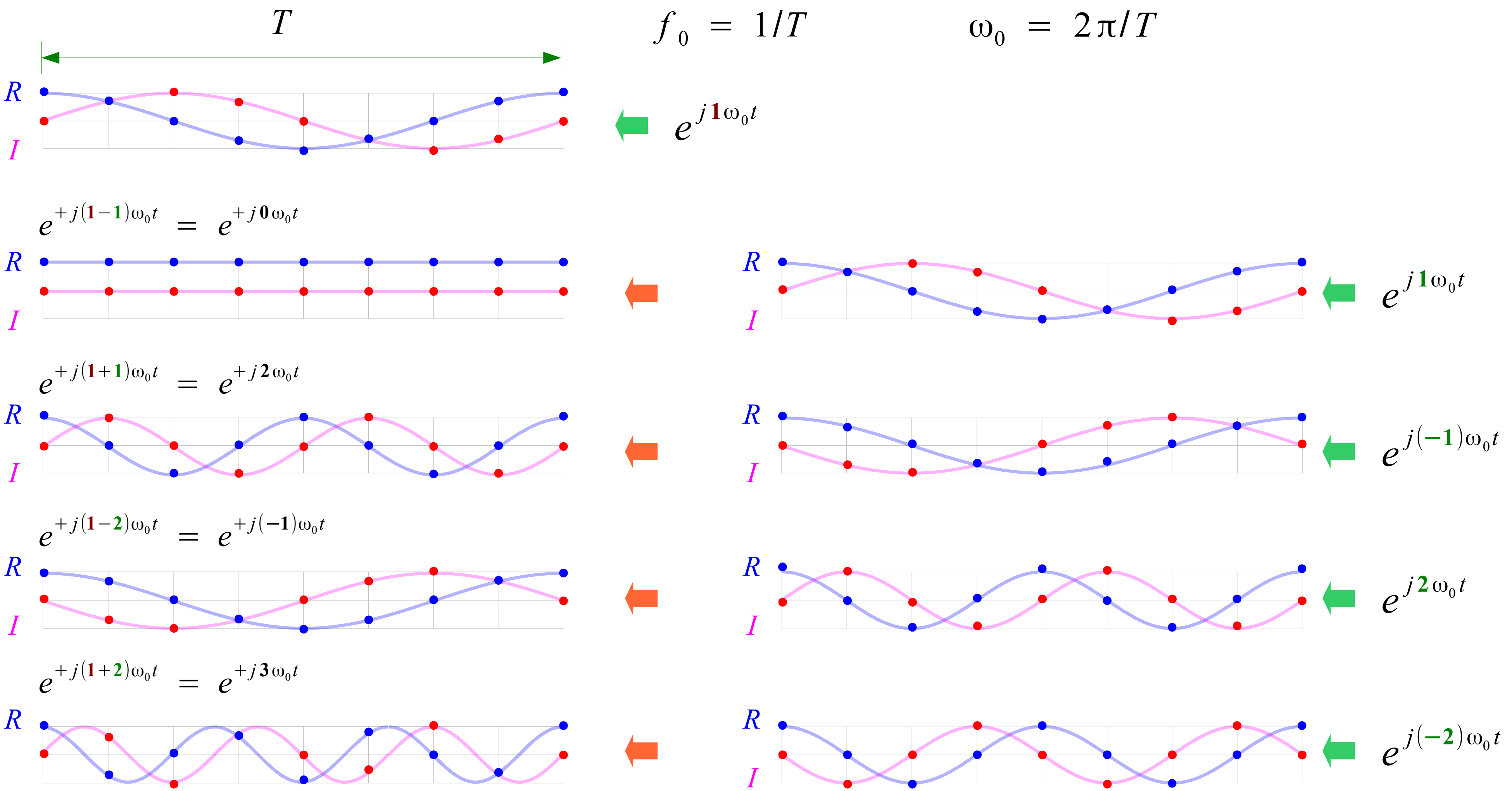
$$\langle \boldsymbol{x}, \boldsymbol{y} \rangle = \boldsymbol{x}^H \cdot \boldsymbol{y} \le \|\boldsymbol{x}\| \cdot \|\boldsymbol{y}\| \qquad \boldsymbol{x} = \begin{pmatrix} a_1 + j b_1 \\ a_2 + j b_2 \\ \vdots \\ a_n + j b_n \end{pmatrix} \qquad \boldsymbol{y} = k \begin{pmatrix} a_1 + j b_1 \\ a_2 + j b_2 \\ \vdots \\ a_n + j b_n \end{pmatrix}$$

Inner product is maximum
when $\boldsymbol{y} = k\,\boldsymbol{x}$

$$\langle \boldsymbol{x}, \boldsymbol{y} \rangle \le k \left(\sum_{i=1}^{n} a_i^2 + b_i^2 \right)$$

Inner Product Examples (1)

$f_0 = 1/T \qquad \omega_0 = 2\pi/T$



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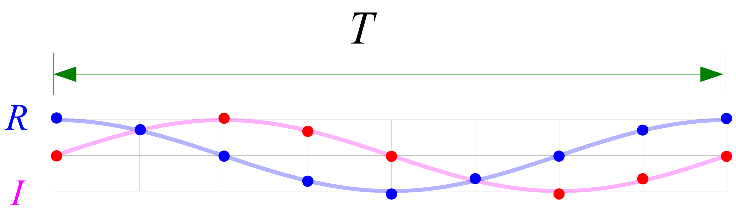
Inner Product Examples (2)



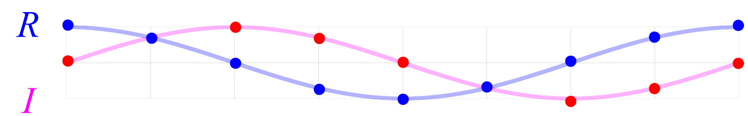
$f_0 = 1/T \qquad \omega_0 = 2\pi/T$

$e^{j\mathbf{1}\omega_0 t}$

$$\langle \boldsymbol{r}_1, \boldsymbol{r}_1 \rangle = \boldsymbol{r}_1^H \cdot \boldsymbol{r}_1 = 8$$

$$\boldsymbol{r}_1^H = \left(1 \quad \tfrac{1+j}{\sqrt{2}} \quad +j \quad \tfrac{-1+j}{\sqrt{2}} \quad -1 \quad \tfrac{-1-j}{\sqrt{2}} \quad -j \quad \tfrac{1-j}{\sqrt{2}}\right)$$

$$\boldsymbol{r}_1 = \left(1 \quad \tfrac{1-j}{\sqrt{2}} \quad -j \quad \tfrac{-1-j}{\sqrt{2}} \quad -1 \quad \tfrac{-1+j}{\sqrt{2}} \quad +j \quad \tfrac{1+j}{\sqrt{2}}\right)^T$$

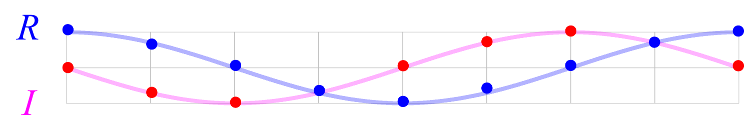


$e^{j\mathbf{1}\omega_0 t}$

$$\langle \boldsymbol{r}_1, \boldsymbol{r}_{-1} \rangle = \boldsymbol{r}_1^H \cdot \boldsymbol{r}_{-1} = 0$$

$$\boldsymbol{r}_1^H = \left(1 \quad \tfrac{1+j}{\sqrt{2}} \quad +j \quad \tfrac{-1+j}{\sqrt{2}} \quad -1 \quad \tfrac{-1-j}{\sqrt{2}} \quad -j \quad \tfrac{1-j}{\sqrt{2}}\right)$$

$$\boldsymbol{r}_1 = \left(1 \quad \tfrac{1+j}{\sqrt{2}} \quad +j \quad \tfrac{-1+j}{\sqrt{2}} \quad -1 \quad \tfrac{-1-j}{\sqrt{2}} \quad -j \quad \tfrac{1-j}{\sqrt{2}}\right)^T$$

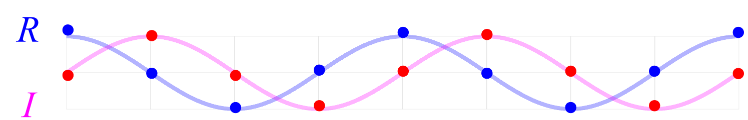


$e^{j(\mathbf{-1})\omega_0 t}$

$$\langle \boldsymbol{r}_1, \boldsymbol{r}_2 \rangle = \boldsymbol{r}_1^H \cdot \boldsymbol{r}_2 = 0$$

$$\boldsymbol{r}_1^H = \left(1 \quad \tfrac{1+j}{\sqrt{2}} \quad +j \quad \tfrac{-1+j}{\sqrt{2}} \quad -1 \quad \tfrac{-1-j}{\sqrt{2}} \quad -j \quad \tfrac{1-j}{\sqrt{2}}\right)$$

$$\boldsymbol{r}_2 = \left(1 \quad +j \quad -1 \quad -j \quad +1 \quad +j \quad -1 \quad -j\right)^T$$

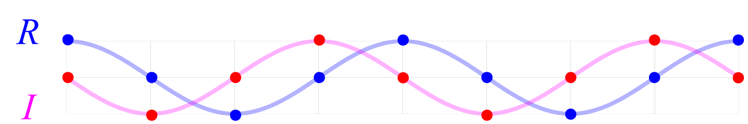


$e^{j\mathbf{2}\omega_0 t}$

$$\langle \boldsymbol{r}_1, \boldsymbol{r}_{-2} \rangle = \boldsymbol{r}_1^H \cdot \boldsymbol{r}_{-2} = 0$$

$$\boldsymbol{r}_1^H = \left(1 \quad \tfrac{1+j}{\sqrt{2}} \quad +j \quad \tfrac{-1+j}{\sqrt{2}} \quad -1 \quad \tfrac{-1-j}{\sqrt{2}} \quad -j \quad \tfrac{1-j}{\sqrt{2}}\right)$$

$$\boldsymbol{r}_2 = \left(1 \quad -j \quad -1 \quad +j \quad +1 \quad -j \quad -1 \quad +j\right)^T$$



$e^{j(\mathbf{-2})\omega_0 t}$

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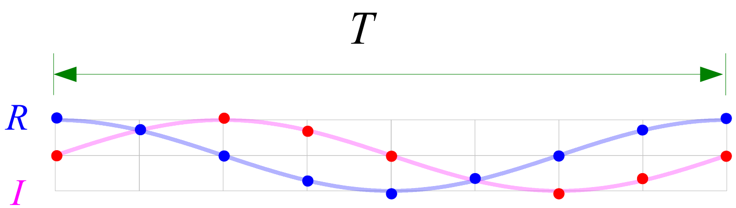
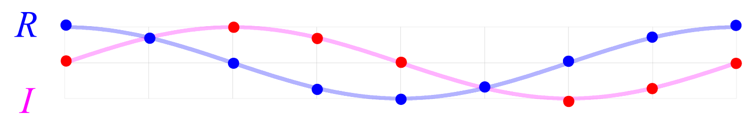
Inner Product Examples (3)



$f_0 = 1/T \qquad \omega_0 = 2\pi/T$

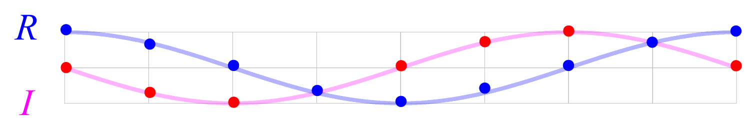
$e^{j\mathbf{1}\omega_0 t}$

$$\left(e^{-j\frac{2\pi}{8}0} \; e^{-j\frac{2\pi}{8}1} \; e^{-j\frac{2\pi}{8}2} \; e^{-j\frac{2\pi}{8}3} \; e^{-j\frac{2\pi}{8}4} \; e^{-j\frac{2\pi}{8}5} \; e^{-j\frac{2\pi}{8}6} \; e^{-j\frac{2\pi}{8}7}\right) \cdot \left(e^{+j\frac{2\pi}{8}0} \; e^{+j\frac{2\pi}{8}1} \; e^{+j\frac{2\pi}{8}2} \; e^{+j\frac{2\pi}{8}3} \; e^{+j\frac{2\pi}{8}4} \; e^{+j\frac{2\pi}{8}5} \; e^{+j\frac{2\pi}{8}6} \; e^{+j\frac{2\pi}{8}7}\right)^T$$



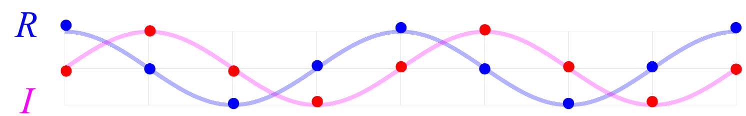
$e^{j\mathbf{1}\omega_0 t}$

$$\left(e^{-j\frac{2\pi}{8}0} \; e^{-j\frac{2\pi}{8}1} \; e^{-j\frac{2\pi}{8}2} \; e^{-j\frac{2\pi}{8}3} \; e^{-j\frac{2\pi}{8}4} \; e^{-j\frac{2\pi}{8}5} \; e^{-j\frac{2\pi}{8}6} \; e^{-j\frac{2\pi}{8}7}\right) \cdot \left(e^{-j\frac{2\pi}{8}0} \; e^{-j\frac{2\pi}{8}1} \; e^{-j\frac{2\pi}{8}2} \; e^{-j\frac{2\pi}{8}3} \; e^{-j\frac{2\pi}{8}4} \; e^{-j\frac{2\pi}{8}5} \; e^{-j\frac{2\pi}{8}6} \; e^{-j\frac{2\pi}{8}7}\right)^T$$



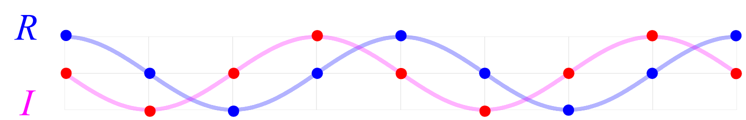
$e^{j(\mathbf{-1})\omega_0 t}$

$$\left(e^{-j\frac{2\pi}{8}0} \; e^{-j\frac{2\pi}{8}1} \; e^{-j\frac{2\pi}{8}2} \; e^{-j\frac{2\pi}{8}3} \; e^{-j\frac{2\pi}{8}4} \; e^{-j\frac{2\pi}{8}5} \; e^{-j\frac{2\pi}{8}6} \; e^{-j\frac{2\pi}{8}7}\right) \cdot \left(e^{+j\frac{2\pi}{8}0} \; e^{+j\frac{2\pi}{8}2} \; e^{+j\frac{2\pi}{8}4} \; e^{+j\frac{2\pi}{8}6} \; e^{+j\frac{2\pi}{8}0} \; e^{+j\frac{2\pi}{8}2} \; e^{+j\frac{2\pi}{8}4} \; e^{+j\frac{2\pi}{8}6}\right)^T$$



$e^{j\mathbf{2}\omega_0 t}$

$$\left(e^{-j\frac{2\pi}{8}0} \; e^{-j\frac{2\pi}{8}1} \; e^{-j\frac{2\pi}{8}2} \; e^{-j\frac{2\pi}{8}3} \; e^{-j\frac{2\pi}{8}4} \; e^{-j\frac{2\pi}{8}5} \; e^{-j\frac{2\pi}{8}6} \; e^{-j\frac{2\pi}{8}7}\right) \cdot \left(e^{-j\frac{2\pi}{8}0} \; e^{-j\frac{2\pi}{8}2} \; e^{-j\frac{2\pi}{8}4} \; e^{-j\frac{2\pi}{8}6} \; e^{-j\frac{2\pi}{8}0} \; e^{-j\frac{2\pi}{8}2} \; e^{-j\frac{2\pi}{8}4} \; e^{-j\frac{2\pi}{8}6}\right)^T$$



$e^{j(\mathbf{-2})\omega_0 t}$

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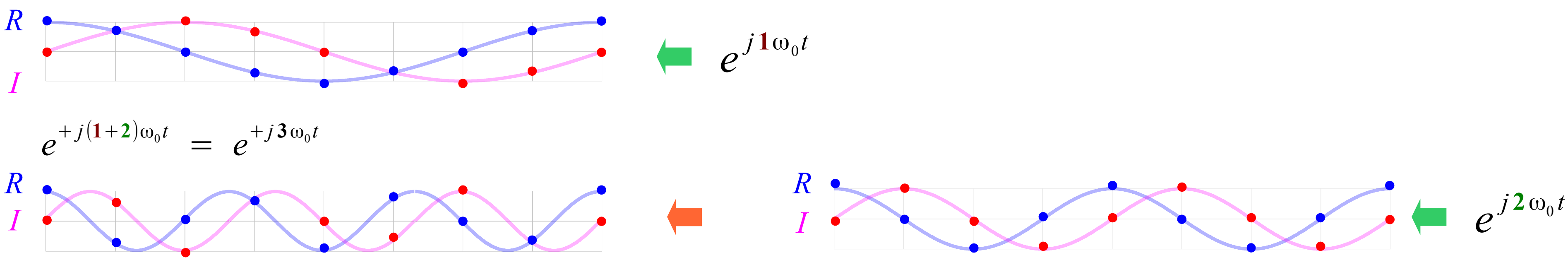
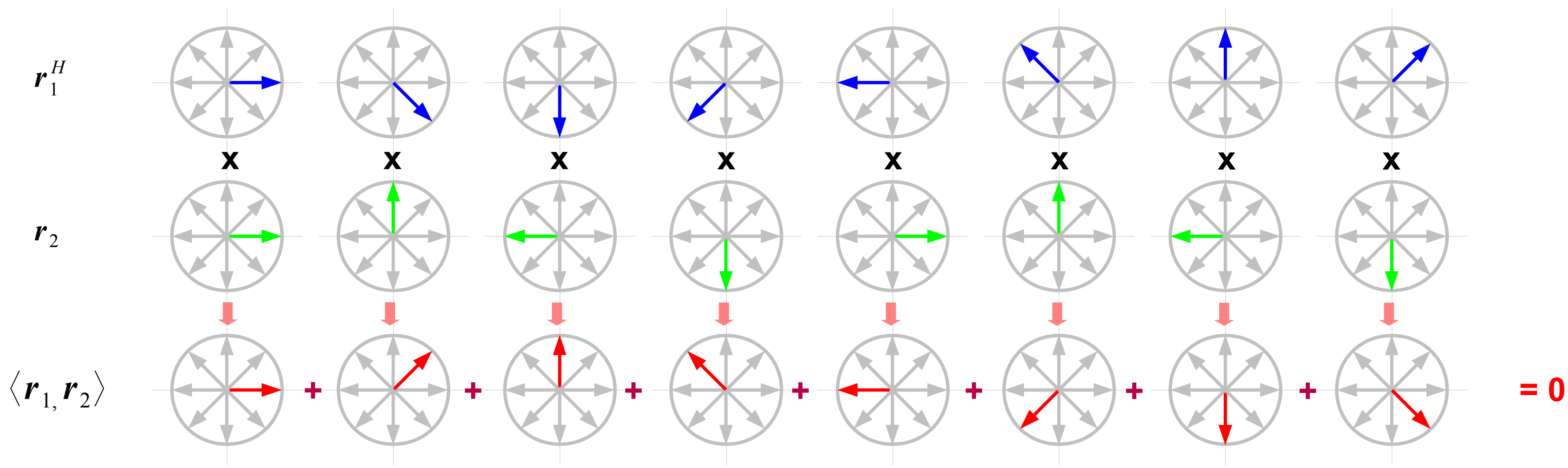
Inner Product Examples (4)



$$\langle \boldsymbol{r}_1, \boldsymbol{r}_2 \rangle = \boldsymbol{r}_1^H \cdot \boldsymbol{r}_2 = 0$$

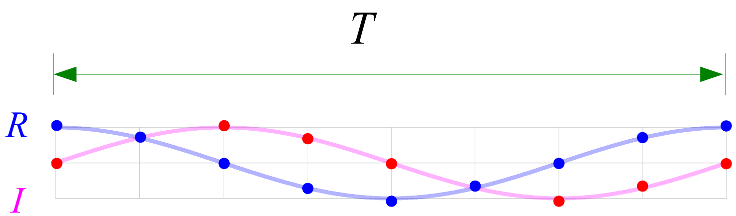
$$\boldsymbol{r}_1^H = \left(e^{-j\frac{2\pi}{8}0} \quad e^{-j\frac{2\pi}{8}1} \quad e^{-j\frac{2\pi}{8}2} \quad e^{-j\frac{2\pi}{8}3} \quad e^{-j\frac{2\pi}{8}4} \quad e^{-j\frac{2\pi}{8}5} \quad e^{-j\frac{2\pi}{8}6} \quad e^{-j\frac{2\pi}{8}7}\right) = \left(1 \quad \frac{1+j}{\sqrt{2}} \quad +j \quad \frac{-1+j}{\sqrt{2}} \quad -1 \quad \frac{-1-j}{\sqrt{2}} \quad -j \quad \frac{1-j}{\sqrt{2}}\right)$$

$$\boldsymbol{r}_2 = \left(e^{+j\frac{2\pi}{8}0} \quad e^{+j\frac{2\pi}{8}2} \quad e^{+j\frac{2\pi}{8}4} \quad e^{+j\frac{2\pi}{8}6} \quad e^{+j\frac{2\pi}{8}0} \quad e^{+j\frac{2\pi}{8}2} \quad e^{+j\frac{2\pi}{8}4} \quad e^{+j\frac{2\pi}{8}6}\right)^T = \left(1 \quad +j \quad -1 \quad -j \quad +1 \quad +j \quad -1 \quad -j\right)^T$$



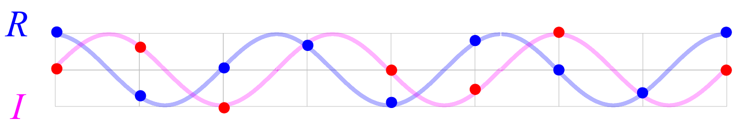
Inner Product Examples (5)

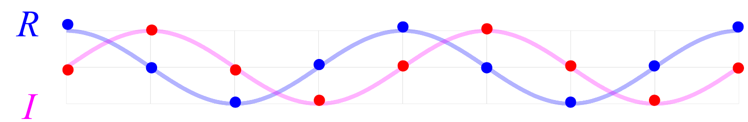
$f_0 = 1/T \qquad \omega_0 = 2\pi/T$



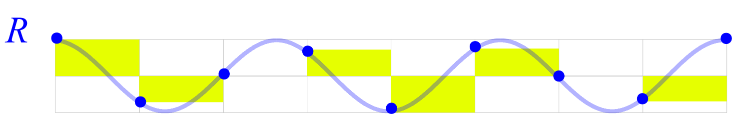
$e^{j\mathbf{1}\omega_0 t}$

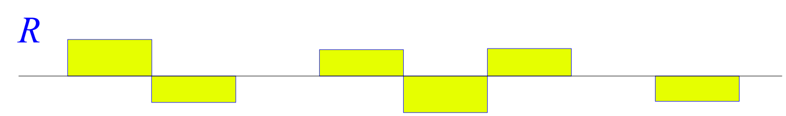
$e^{+j(\mathbf{1}+\mathbf{2})\omega_0 t} = e^{+j\mathbf{3}\omega_0 t}$



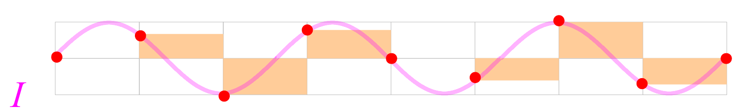


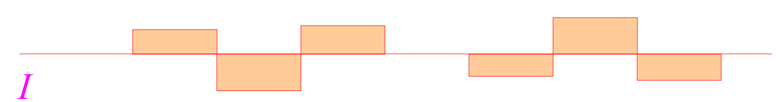
$e^{j\mathbf{2}\omega_0 t}$





Approx area = 0





Approx area = 0

$$\langle \mathbf{r}_1, \mathbf{r}_2 \rangle = \mathbf{r}_1^H \cdot \mathbf{r}_2 = 0$$

$$\mathbf{r}_1^H = \left(e^{-j\frac{2\pi}{8}0} \;\; e^{-j\frac{2\pi}{8}1} \;\; e^{-j\frac{2\pi}{8}2} \;\; e^{-j\frac{2\pi}{8}3} \;\; e^{-j\frac{2\pi}{8}4} \;\; e^{-j\frac{2\pi}{8}5} \;\; e^{-j\frac{2\pi}{8}6} \;\; e^{-j\frac{2\pi}{8}7}\right) = \left(1 \;\; \tfrac{1+j}{\sqrt{2}} \;\; +j \;\; \tfrac{-1+j}{\sqrt{2}} \;\; -1 \;\; \tfrac{-1-j}{\sqrt{2}} \;\; -j \;\; \tfrac{1-j}{\sqrt{2}}\right)$$

$$\mathbf{r}_2 = \left(e^{+j\frac{2\pi}{8}0} \;\; e^{+j\frac{2\pi}{8}2} \;\; e^{+j\frac{2\pi}{8}4} \;\; e^{+j\frac{2\pi}{8}6} \;\; e^{+j\frac{2\pi}{8}0} \;\; e^{+j\frac{2\pi}{8}2} \;\; e^{+j\frac{2\pi}{8}4} \;\; e^{+j\frac{2\pi}{8}6}\right)^T = \left(1 \;\; +j \;\; -1 \;\; -j \;\; +1 \;\; +j \;\; -1 \;\; -j\right)^T$$

N=8 DFT Matrix in Cosine and Sine Terms

$$W_8^{kn} = e^{-j\left(\frac{2\pi}{8}\right)kn} = \cos\left(\frac{\pi}{4}\cdot k\cdot n\right) - j\sin\left(\frac{\pi}{4}\cdot k\cdot n\right)$$

$$\begin{bmatrix}
\cos(\pi/4)\cdot 0 - j\sin(\pi/4)\cdot 0 & \cos(\pi/4)\cdot 0 - j\sin(\pi/4)\cdot 0 & \cos(\pi/4)\cdot 0 - j\sin(\pi/4)\cdot 0 & \cos(\pi/4)\cdot 0 - j\sin(\pi/4)\cdot 0 & \cos(\pi/4)\cdot 0 - j\sin(\pi/4)\cdot 0 & \cos(\pi/4)\cdot 0 - j\sin(\pi/4)\cdot 0 & \cos(\pi/4)\cdot 0 - j\sin(\pi/4)\cdot 0 & \cos(\pi/4)\cdot 0 - j\sin(\pi/4)\cdot 0 \\
\cos(\pi/4)\cdot 0 - j\sin(\pi/4)\cdot 0 & \cos(\pi/4)\cdot 1 - j\sin(\pi/4)\cdot 1 & \cos(\pi/4)\cdot 2 - j\sin(\pi/4)\cdot 2 & \cos(\pi/4)\cdot 3 - j\sin(\pi/4)\cdot 3 & \cos(\pi/4)\cdot 4 - j\sin(\pi/4)\cdot 4 & \cos(\pi/4)\cdot 5 - j\sin(\pi/4)\cdot 5 & \cos(\pi/4)\cdot 6 - j\sin(\pi/4)\cdot 6 & \cos(\pi/4)\cdot 7 - j\sin(\pi/4)\cdot 7 \\
\cos(\pi/4)\cdot 0 - j\sin(\pi/4)\cdot 0 & \cos(\pi/4)\cdot 2 - j\sin(\pi/4)\cdot 2 & \cos(\pi/4)\cdot 4 - j\sin(\pi/4)\cdot 4 & \cos(\pi/4)\cdot 6 - j\sin(\pi/4)\cdot 6 & \cos(\pi/4)\cdot 0 - j\sin(\pi/4)\cdot 0 & \cos(\pi/4)\cdot 2 - j\sin(\pi/4)\cdot 2 & \cos(\pi/4)\cdot 4 - j\sin(\pi/4)\cdot 4 & \cos(\pi/4)\cdot 6 - j\sin(\pi/4)\cdot 6 \\
\cos(\pi/4)\cdot 0 - j\sin(\pi/4)\cdot 0 & \cos(\pi/4)\cdot 3 - j\sin(\pi/4)\cdot 3 & \cos(\pi/4)\cdot 6 - j\sin(\pi/4)\cdot 6 & \cos(\pi/4)\cdot 1 - j\sin(\pi/4)\cdot 1 & \cos(\pi/4)\cdot 4 - j\sin(\pi/4)\cdot 4 & \cos(\pi/4)\cdot 7 - j\sin(\pi/4)\cdot 7 & \cos(\pi/4)\cdot 2 - j\sin(\pi/4)\cdot 2 & \cos(\pi/4)\cdot 5 - j\sin(\pi/4)\cdot 5 \\
\cos(\pi/4)\cdot 0 - j\sin(\pi/4)\cdot 0 & \cos(\pi/4)\cdot 4 - j\sin(\pi/4)\cdot 4 & \cos(\pi/4)\cdot 0 - j\sin(\pi/4)\cdot 0 & \cos(\pi/4)\cdot 4 - j\sin(\pi/4)\cdot 4 & \cos(\pi/4)\cdot 0 - j\sin(\pi/4)\cdot 0 & \cos(\pi/4)\cdot 4 - j\sin(\pi/4)\cdot 4 & \cos(\pi/4)\cdot 0 - j\sin(\pi/4)\cdot 0 & \cos(\pi/4)\cdot 4 - j\sin(\pi/4)\cdot 4 \\
\cos(\pi/4)\cdot 0 - j\sin(\pi/4)\cdot 0 & \cos(\pi/4)\cdot 5 - j\sin(\pi/4)\cdot 5 & \cos(\pi/4)\cdot 2 - j\sin(\pi/4)\cdot 2 & \cos(\pi/4)\cdot 7 - j\sin(\pi/4)\cdot 7 & \cos(\pi/4)\cdot 4 - j\sin(\pi/4)\cdot 4 & \cos(\pi/4)\cdot 1 - j\sin(\pi/4)\cdot 1 & \cos(\pi/4)\cdot 6 - j\sin(\pi/4)\cdot 6 & \cos(\pi/4)\cdot 3 - j\sin(\pi/4)\cdot 3 \\
\cos(\pi/4)\cdot 0 - j\sin(\pi/4)\cdot 0 & \cos(\pi/4)\cdot 6 - j\sin(\pi/4)\cdot 6 & \cos(\pi/4)\cdot 4 - j\sin(\pi/4)\cdot 4 & \cos(\pi/4)\cdot 2 - j\sin(\pi/4)\cdot 2 & \cos(\pi/4)\cdot 0 - j\sin(\pi/4)\cdot 0 & \cos(\pi/4)\cdot 6 - j\sin(\pi/4)\cdot 6 & \cos(\pi/4)\cdot 4 - j\sin(\pi/4)\cdot 4 & \cos(\pi/4)\cdot 2 - j\sin(\pi/4)\cdot 2 \\
\cos(\pi/4)\cdot 0 - j\sin(\pi/4)\cdot 0 & \cos(\pi/4)\cdot 7 - j\sin(\pi/4)\cdot 7 & \cos(\pi/4)\cdot 6 - j\sin(\pi/4)\cdot 6 & \cos(\pi/4)\cdot 5 - j\sin(\pi/4)\cdot 5 & \cos(\pi/4)\cdot 4 - j\sin(\pi/4)\cdot 4 & \cos(\pi/4)\cdot 3 - j\sin(\pi/4)\cdot 3 & \cos(\pi/4)\cdot 2 - j\sin(\pi/4)\cdot 2 & \cos(\pi/4)\cdot 1 - j\sin(\pi/4)\cdot 1
\end{bmatrix}$$

N=8 DFT Matrix Real and Imaginary Terms

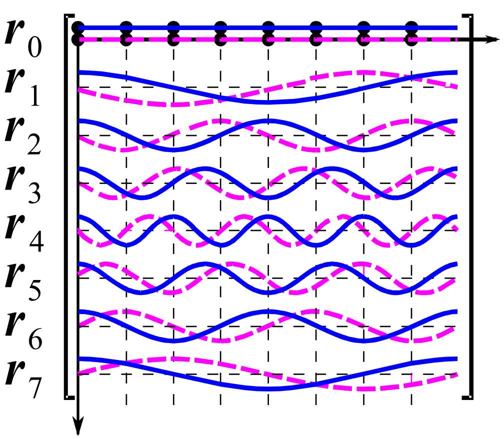
$$W_8^{kn} = e^{-j\left(\frac{2\pi}{8}\right)kn} = \cos\left(\frac{\pi}{4}\cdot k\cdot n\right) - j\sin\left(\frac{\pi}{4}\cdot k\cdot n\right)$$

$$\begin{bmatrix}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & +\frac{1}{\sqrt{2}} - j\frac{1}{\sqrt{2}} & -j & -\frac{1}{\sqrt{2}} - j\frac{1}{\sqrt{2}} & -1 & -\frac{1}{\sqrt{2}} + j\frac{1}{\sqrt{2}} & +j & +\frac{1}{\sqrt{2}} + j\frac{1}{\sqrt{2}} \\
1 & -j & -1 & +j & 1 & -j & -1 & +j \\
1 & -\frac{1}{\sqrt{2}} - j\frac{1}{\sqrt{2}} & +j & +\frac{1}{\sqrt{2}} - j\frac{1}{\sqrt{2}} & -1 & +\frac{1}{\sqrt{2}} + j\frac{1}{\sqrt{2}} & -j & -\frac{1}{\sqrt{2}} + j\frac{1}{\sqrt{2}} \\
1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \\
1 & -\frac{1}{\sqrt{2}} + j\frac{1}{\sqrt{2}} & -j & +\frac{1}{\sqrt{2}} + j\frac{1}{\sqrt{2}} & -1 & +\frac{1}{\sqrt{2}} - j\frac{1}{\sqrt{2}} & +j & -\frac{1}{\sqrt{2}} - j\frac{1}{\sqrt{2}} \\
1 & +j & -1 & -j & 1 & +j & -1 & -j \\
1 & +\frac{1}{\sqrt{2}} + j\frac{1}{\sqrt{2}} & +j & -\frac{1}{\sqrt{2}} + j\frac{1}{\sqrt{2}} & -1 & -\frac{1}{\sqrt{2}} - j\frac{1}{\sqrt{2}} & -j & +\frac{1}{\sqrt{2}} - j\frac{1}{\sqrt{2}}
\end{bmatrix}
\begin{matrix}
\Leftarrow \boldsymbol{r}_0 \\
\Leftarrow \boldsymbol{r}_1 \\
\Leftarrow \boldsymbol{r}_2 \\
\Leftarrow \boldsymbol{r}_3 \\
\Leftarrow \boldsymbol{r}_4 \\
\Leftarrow \boldsymbol{r}_5 = \boldsymbol{r}_{-3} \\
\Leftarrow \boldsymbol{r}_4 = \boldsymbol{r}_{-2} \\
\Leftarrow \boldsymbol{r}_5 = \boldsymbol{r}_{-1}
\end{matrix}$$

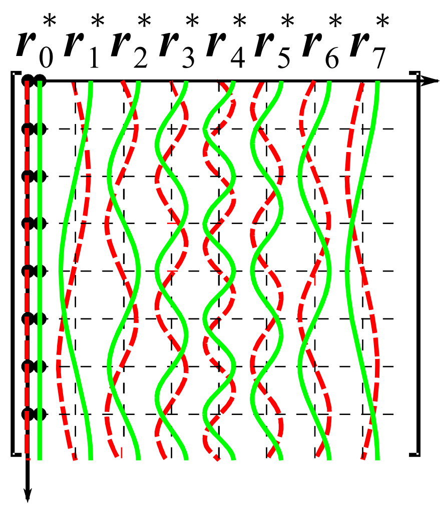
Orthogonality

$$X[k] = \sum_{n=0}^{7} W_8^{kn}\, x[n] \qquad W_8^{kn} = e^{-j\left(\frac{2\pi}{8}\right)kn}$$

$$\begin{cases} A^H = B \\ B^H = A \end{cases} \quad \begin{cases} AB = N\,I \\ BA = N\,I \end{cases} \quad \Rightarrow \quad \begin{cases} A^H A = A\,A^H = N\,I \\ B^H B = B\,B^H = N\,I \end{cases}$$



$\bullet$



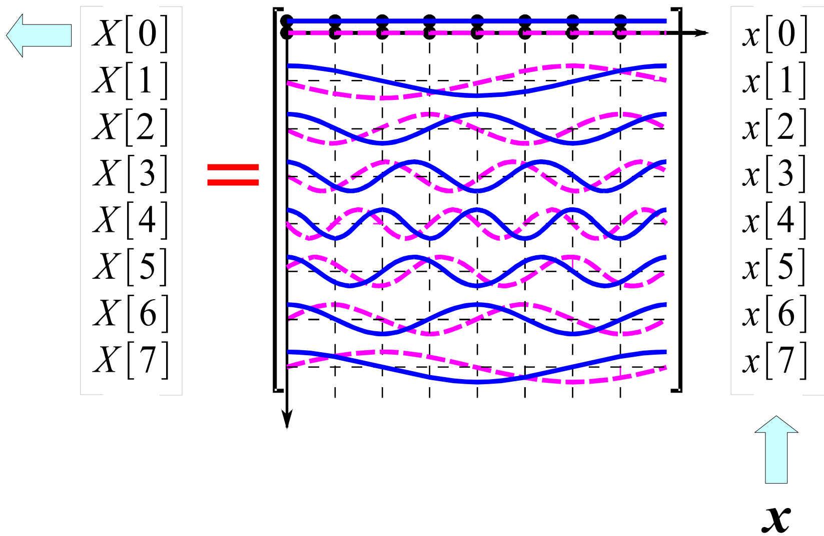
$=$

$r_0 \cdot r_0^*$	$r_0 \cdot r_1^*$	$r_0 \cdot r_2^*$	$r_0 \cdot r_3^*$	$r_0 \cdot r_4^*$	$r_0 \cdot r_5^*$	$r_0 \cdot r_6^*$	$r_0 \cdot r_7^*$
$r_1 \cdot r_0^*$	$r_1 \cdot r_1^*$	$r_1 \cdot r_2^*$	$r_1 \cdot r_3^*$	$r_1 \cdot r_4^*$	$r_1 \cdot r_5^*$	$r_1 \cdot r_6^*$	$r_1 \cdot r_7^*$
$r_2 \cdot r_0^*$	$r_2 \cdot r_1^*$	$r_2 \cdot r_2^*$	$r_2 \cdot r_3^*$	$r_2 \cdot r_4^*$	$r_2 \cdot r_5^*$	$r_2 \cdot r_6^*$	$r_2 \cdot r_7^*$
$r_3 \cdot r_0^*$	$r_3 \cdot r_1^*$	$r_3 \cdot r_2^*$	$r_3 \cdot r_3^*$	$r_3 \cdot r_4^*$	$r_3 \cdot r_5^*$	$r_3 \cdot r_6^*$	$r_3 \cdot r_7^*$
$r_4 \cdot r_0^*$	$r_4 \cdot r_1^*$	$r_4 \cdot r_2^*$	$r_4 \cdot r_3^*$	$r_4 \cdot r_4^*$	$r_4 \cdot r_5^*$	$r_4 \cdot r_6^*$	$r_4 \cdot r_7^*$
$r_5 \cdot r_0^*$	$r_5 \cdot r_1^*$	$r_5 \cdot r_2^*$	$r_5 \cdot r_3^*$	$r_5 \cdot r_4^*$	$r_5 \cdot r_5^*$	$r_5 \cdot r_6^*$	$r_5 \cdot r_7^*$
$r_6 \cdot r_0^*$	$r_6 \cdot r_1^*$	$r_6 \cdot r_2^*$	$r_6 \cdot r_3^*$	$r_6 \cdot r_4^*$	$r_6 \cdot r_5^*$	$r_6 \cdot r_6^*$	$r_6 \cdot r_7^*$
$r_7 \cdot r_0^*$	$r_7 \cdot r_1^*$	$r_7 \cdot r_2^*$	$r_7 \cdot r_3^*$	$r_7 \cdot r_4^*$	$r_7 \cdot r_5^*$	$r_7 \cdot r_6^*$	$r_7 \cdot r_7^*$

$$\langle r_i^H, r_i^H \rangle = r_i \cdot r_i^* = N$$

$$\langle r_i^H, r_j^H \rangle = r_i \cdot r_j^* = 0 \quad (i \neq j)$$

N=8 DFT : Inner Product X[0]

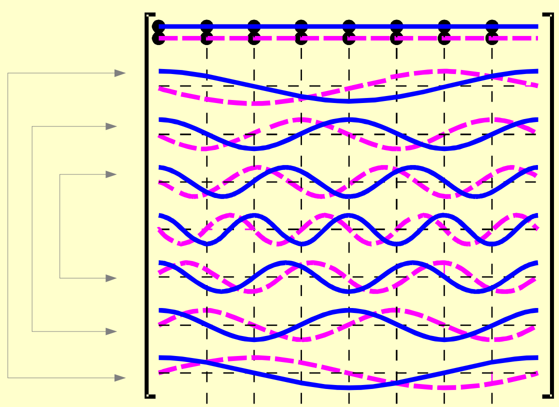


X[0] measures "0 cycle" component in x

$$\langle \boldsymbol{r}_0^H, \boldsymbol{x} \rangle = \boldsymbol{r}_0 \cdot \boldsymbol{x} \;\le\; \|\boldsymbol{r}_0^H\| \cdot \|\boldsymbol{x}\|$$

maximum when $\boldsymbol{x} = k\, \boldsymbol{r}_0^H$

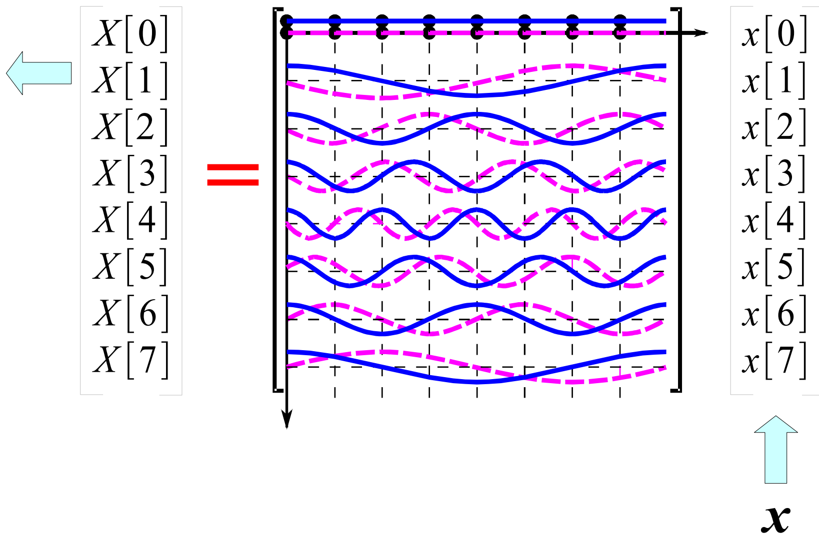
When $\boldsymbol{x}$ *looks like this,* $\boldsymbol{X}[0]$ *is max.*



$$Re\left\{ e^{-j\frac{2\pi}{8}kn} \right\} = \cos\left(-\frac{2\pi}{8}kn\right)$$

$$Im\left\{ e^{-j\frac{2\pi}{8}kn} \right\} = \sin\left(-\frac{2\pi}{8}kn\right)$$

N=8 DFT : Inner Product X[1]

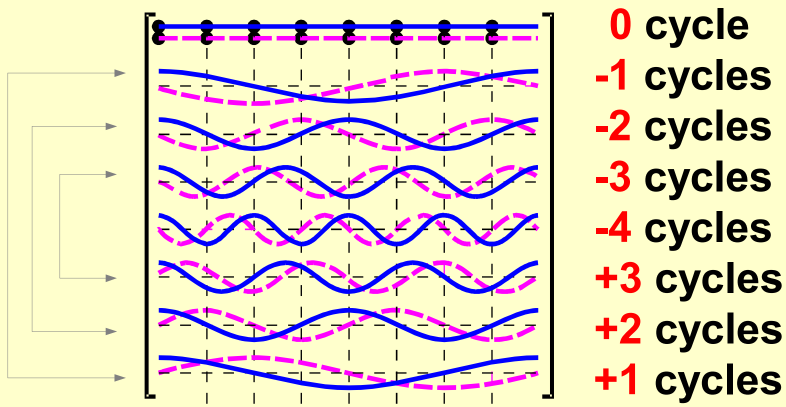


X[1] measures "+1 cycle" component in x

$$\langle \boldsymbol{r}_1^H , \boldsymbol{x} \rangle = \boldsymbol{r}_1 \cdot \boldsymbol{x} \;\le\; \|\boldsymbol{r}_1^H\| \cdot \|\boldsymbol{x}\|$$

maximum when $\boldsymbol{x} = k\,\boldsymbol{r}_1^H$

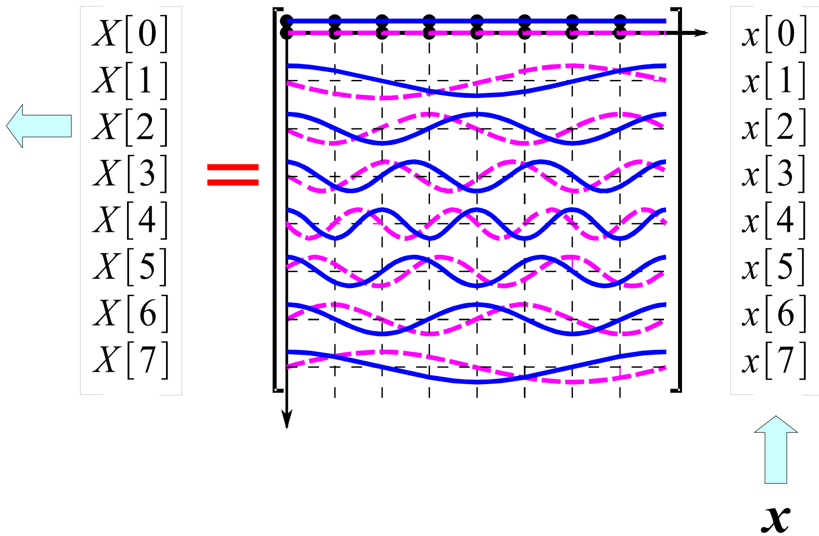
*When **x** looks like this, **X**[1] is max.*



$$Re\left\{e^{-j\frac{2\pi}{8}kn}\right\} = \cos\left(-\frac{2\pi}{8}kn\right)$$

$$Im\left\{e^{-j\frac{2\pi}{8}kn}\right\} = \sin\left(-\frac{2\pi}{8}kn\right)$$

N=8 DFT : Inner Product X[2]

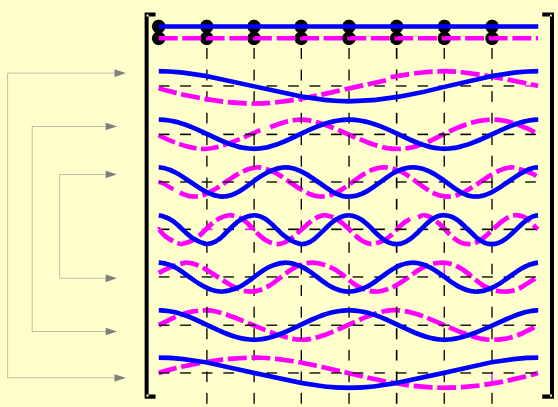


X[2] measures "+2 cycle" component in x

$$\langle \boldsymbol{r}_2^H , \boldsymbol{x} \rangle = \boldsymbol{r}_2 \cdot \boldsymbol{x} \;\le\; \|\boldsymbol{r}_2^H\| \cdot \|\boldsymbol{x}\|$$

maximum when $\boldsymbol{x} = k\,\boldsymbol{r}_2^H$

When $\boldsymbol{x}$ *looks like this,* $\boldsymbol{X}[2]$ *is max.*

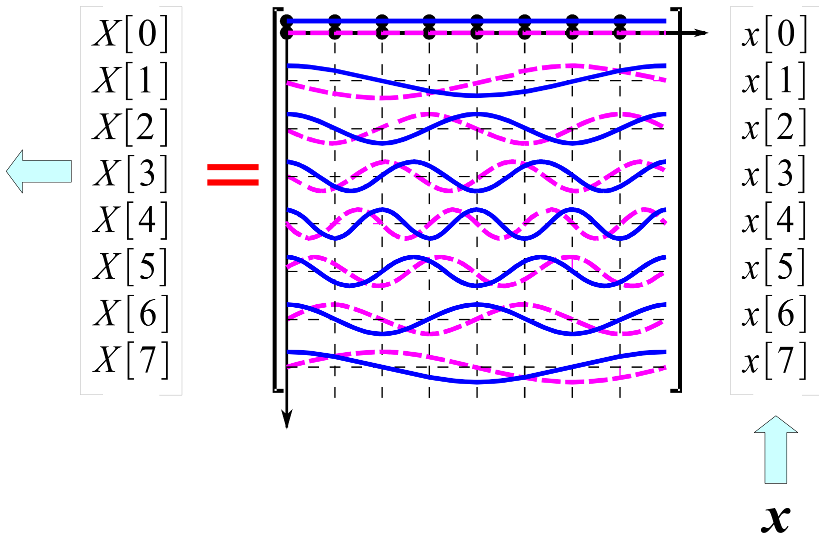


0 cycle
-1 cycles
-2 cycles
-3 cycles
-4 cycles
+3 cycles
+2 cycles
+1 cycles

$$Re\left\{ e^{-j\frac{2\pi}{8}kn} \right\} = \cos\left(-\frac{2\pi}{8}kn\right)$$

$$Im\left\{ e^{-j\frac{2\pi}{8}kn} \right\} = \sin\left(-\frac{2\pi}{8}kn\right)$$

N=8 DFT : Inner Product X[3]

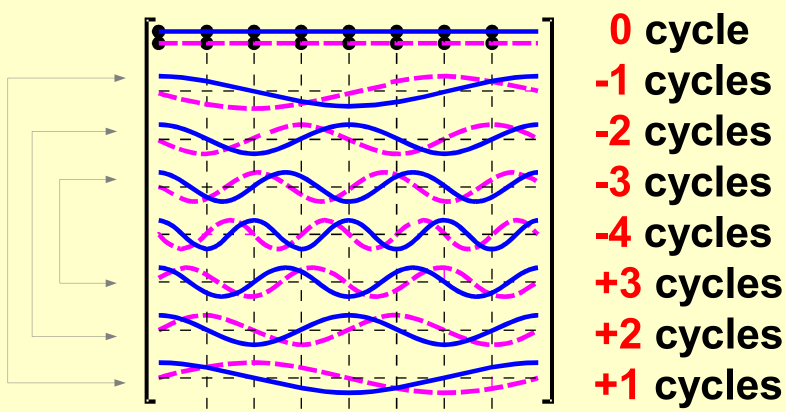


X[3] measures "+3 cycle" component in x

$$\langle \boldsymbol{r}_3^H , \boldsymbol{x} \rangle = \boldsymbol{r}_3 \cdot \boldsymbol{x} \;\le\; \|\boldsymbol{r}_3^H\| \cdot \|\boldsymbol{x}\|$$

maximum when $\boldsymbol{x} = k\,\boldsymbol{r}_3^H$

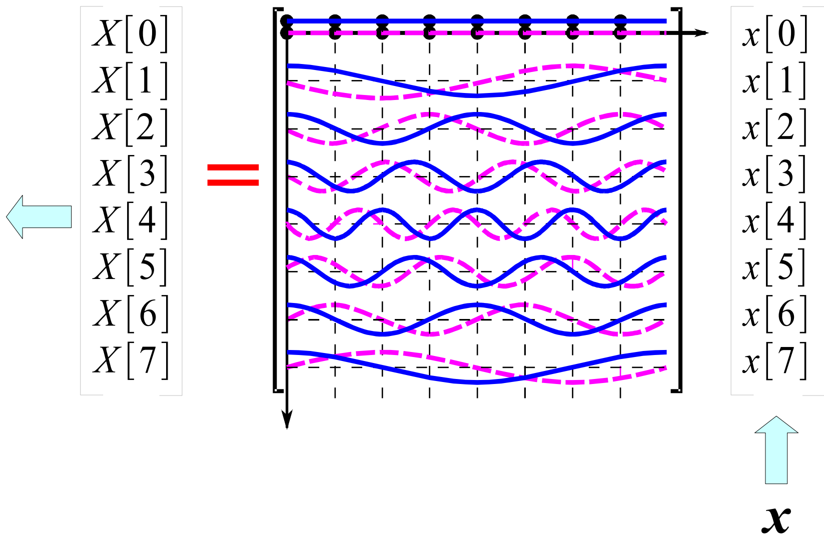
*When **x** looks like this, **X**[3] is max.*



$$Re\left\{ e^{-j\frac{2\pi}{8}kn} \right\} = \cos\left(-\frac{2\pi}{8}kn\right)$$

$$Im\left\{ e^{-j\frac{2\pi}{8}kn} \right\} = \sin\left(-\frac{2\pi}{8}kn\right)$$

N=8 DFT : Inner Product X[4]

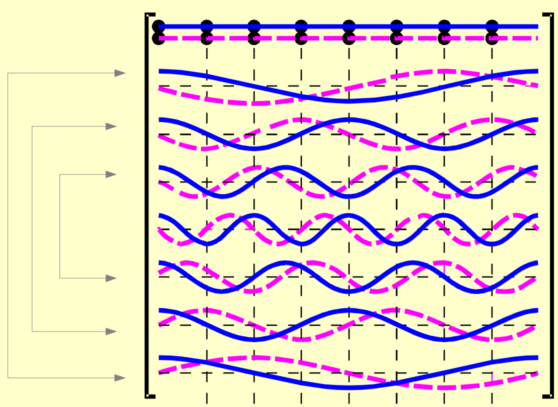


X[4] measures "+4 cycle" component in x

$$\langle \boldsymbol{r}_4^H, \boldsymbol{x} \rangle = \boldsymbol{r}_4 \cdot \boldsymbol{x} \le \|\boldsymbol{r}_4^H\| \cdot \|\boldsymbol{x}\|$$

maximum when $\boldsymbol{x} = k\,\boldsymbol{r}_4^H$

*When **x** looks like this, **X**[4] is max.*



0 cycle
-1 cycles
-2 cycles
-3 cycles
-4 cycles
+3 cycles
+2 cycles
+1 cycles

$$Re\left\{ e^{-j\frac{2\pi}{8}kn} \right\} = \cos\left(-\frac{2\pi}{8}kn\right)$$

$$Im\left\{ e^{-j\frac{2\pi}{8}kn} \right\} = \sin\left(-\frac{2\pi}{8}kn\right)$$

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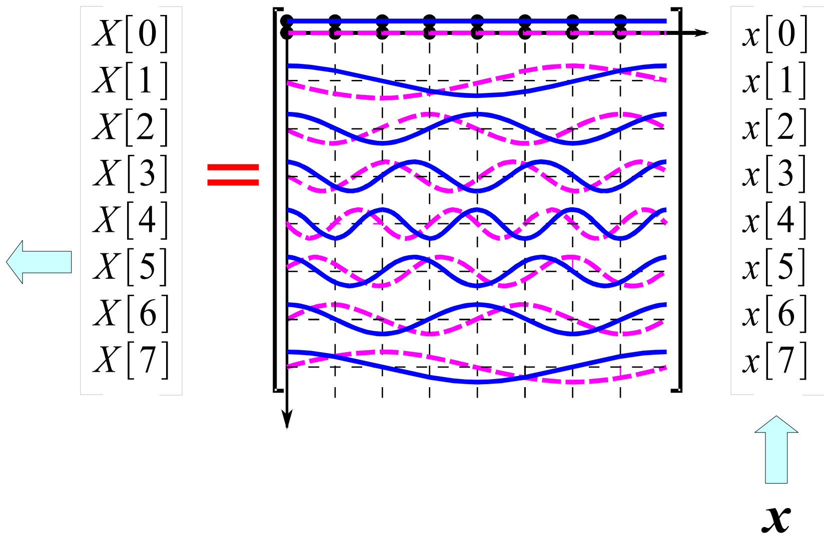
N=8 DFT : Inner Product X[5]

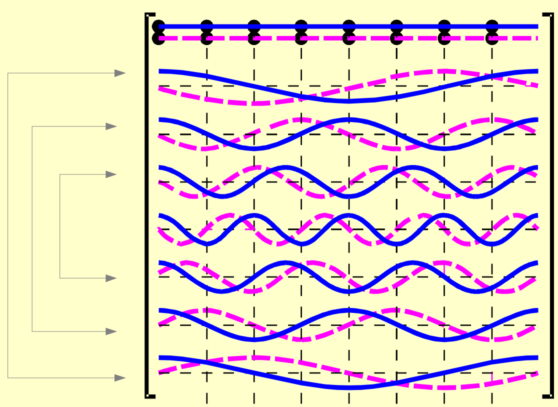


X[5] measures "-3 cycle" component in x

$$\langle \boldsymbol{r}_5^H , \boldsymbol{x} \rangle = \boldsymbol{r}_5 \cdot \boldsymbol{x} \;\le\; \|\boldsymbol{r}_5^H\| \cdot \|\boldsymbol{x}\|$$

maximum when $\boldsymbol{x} = k\,\boldsymbol{r}_5^H$

*When **x** looks like this, **X**[5] is max.*

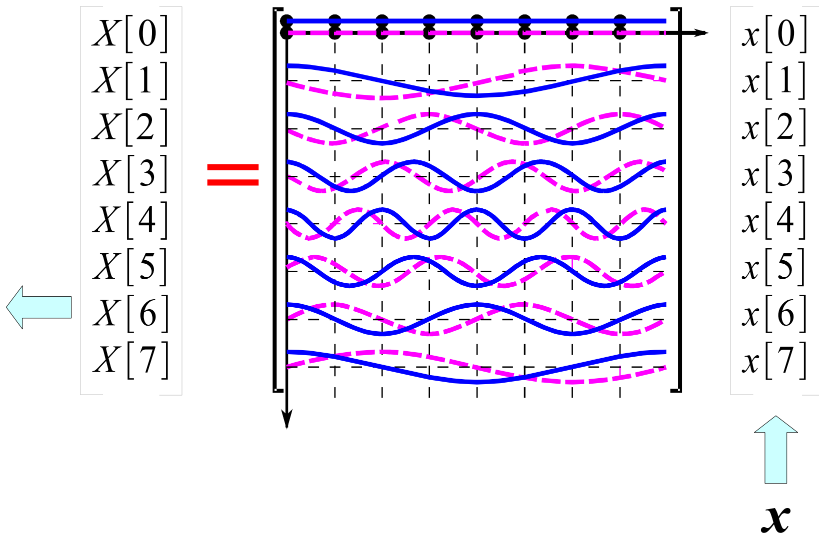


0 cycle
-1 cycles
-2 cycles
-3 cycles
-4 cycles
+3 cycles
+2 cycles
+1 cycles

$$Re\left\{ e^{-j\frac{2\pi}{8}kn} \right\} = \cos\left(-\frac{2\pi}{8}kn\right)$$

$$Im\left\{ e^{-j\frac{2\pi}{8}kn} \right\} = \sin\left(-\frac{2\pi}{8}kn\right)$$

N=8 DFT : Inner Product X[6]

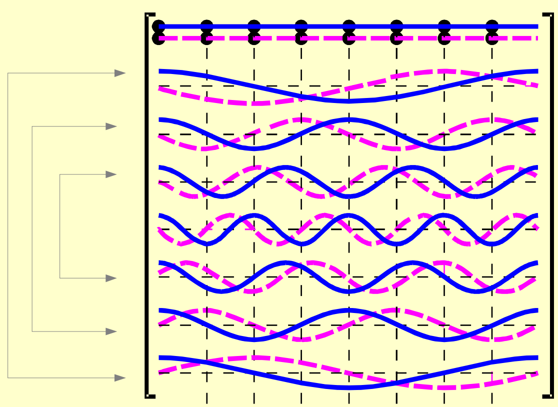


X[6] measures "-2 cycle" component in x

$$\langle \boldsymbol{r}_6^H , \boldsymbol{x} \rangle = \boldsymbol{r}_6 \cdot \boldsymbol{x} \;\le\; \|\boldsymbol{r}_6^H\| \cdot \|\boldsymbol{x}\|$$

maximum when $\boldsymbol{x} = k\, \boldsymbol{r}_6^H$

*When **x** looks like this, **X**[6] is max.*



0 cycle
-1 cycles
-2 cycles
-3 cycles
-4 cycles
+3 cycles
+2 cycles
+1 cycles

$$Re\left\{ e^{-j\frac{2\pi}{8}kn} \right\} = \cos\left(-\frac{2\pi}{8}kn\right)$$

$$Im\left\{ e^{-j\frac{2\pi}{8}kn} \right\} = \sin\left(-\frac{2\pi}{8}kn\right)$$

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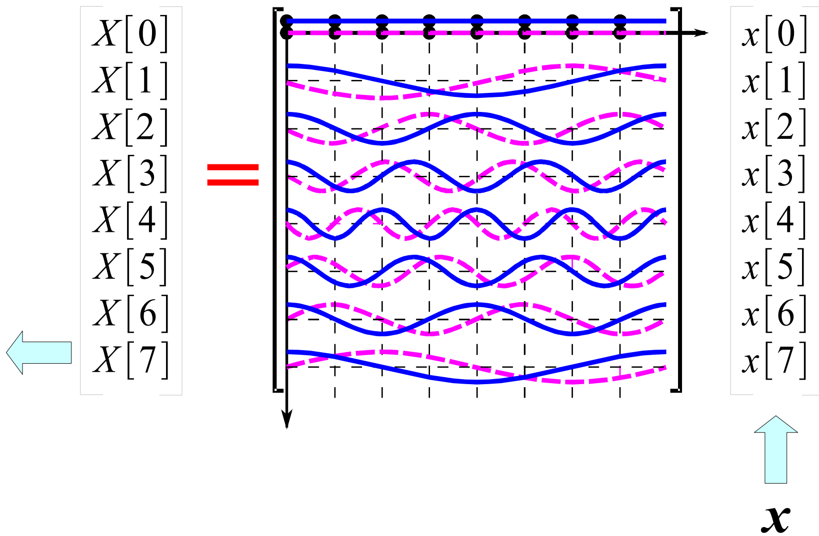
N=8 DFT : Inner Product X[7]

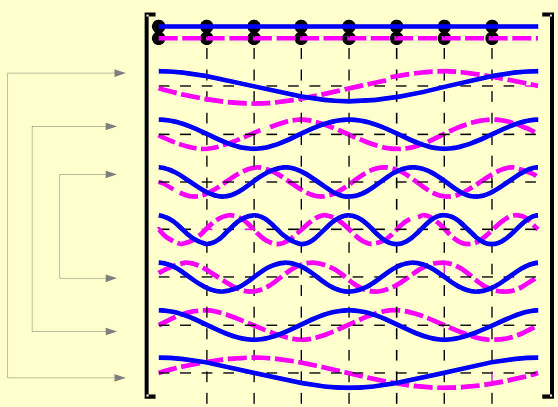


X[7] measures "-1 cycle" component in x

$$\langle \boldsymbol{r}_7^H , \boldsymbol{x} \rangle = \boldsymbol{r}_7 \cdot \boldsymbol{x} \le \|\boldsymbol{r}_7^H\| \cdot \|\boldsymbol{x}\|$$

maximum when $\boldsymbol{x} = k\,\boldsymbol{r}_7^H$

*When **x** looks like this, **X**[7] is max.*



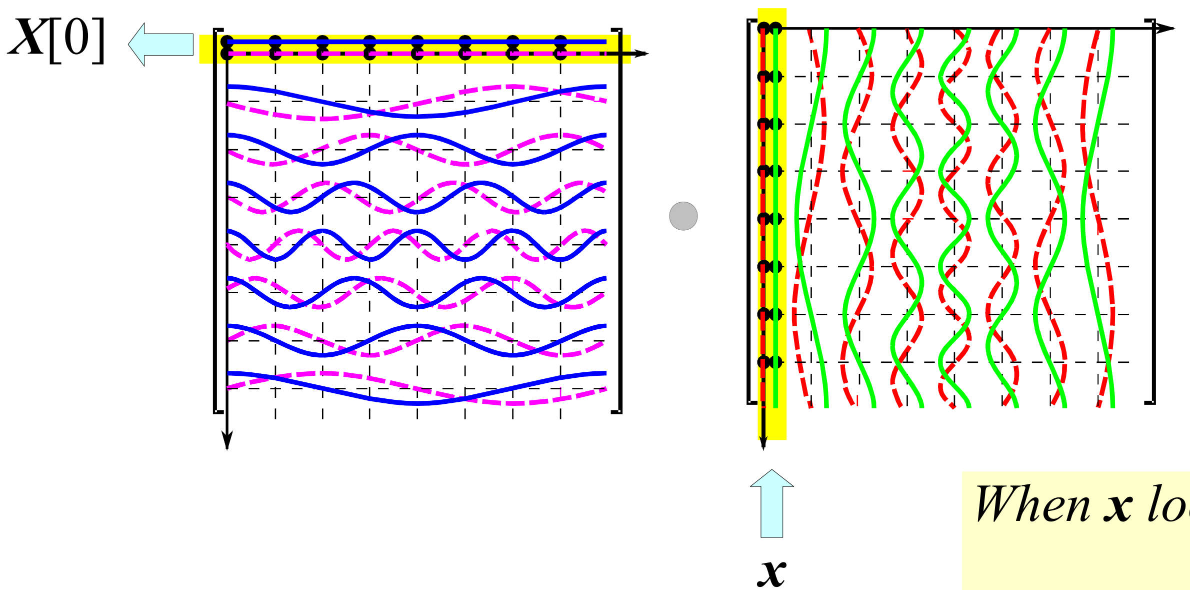
0 cycle
-1 cycles
-2 cycles
-3 cycles
-4 cycles
+3 cycles
+2 cycles
+1 cycles

$$Re\left\{e^{-j\frac{2\pi}{8}kn}\right\} = \cos\left(-\frac{2\pi}{8}kn\right)$$

$$Im\left\{e^{-j\frac{2\pi}{8}kn}\right\} = \sin\left(-\frac{2\pi}{8}kn\right)$$

N=8 DFT : X[0] in A·B

$$X[k] = \sum_{n=0}^{7} W_8^{kn}\, x[n] \qquad W_8^{kn} = e^{-j(\frac{2\pi}{8})kn}$$



$$\boldsymbol{A} \cdot \boldsymbol{B} = \boldsymbol{A} \cdot \boldsymbol{A}^H = N\,\boldsymbol{I}$$

$$\boldsymbol{U} \cdot \boldsymbol{U}^H = \boldsymbol{I}$$

Unitary Matrix

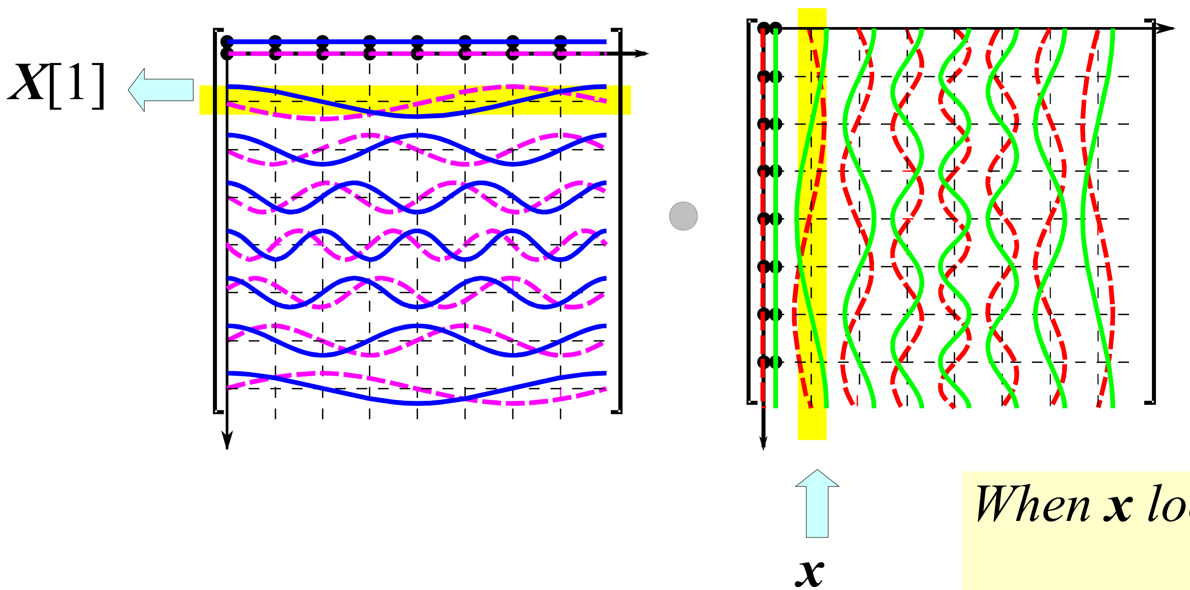
When $\boldsymbol{x}$ *looks like this,* $\boldsymbol{X}[0]$ *is max.* (= $\boldsymbol{N}$)

$\boldsymbol{X}[k] = 0 \quad for \;\; k \neq 0$

$$\boldsymbol{X}[0] = \begin{pmatrix} e^{-j\cdot\frac{\pi}{4}\cdot 0} & e^{-j\cdot\frac{\pi}{4}\cdot 0} & e^{-j\cdot\frac{\pi}{4}\cdot 0} & e^{-j\cdot\frac{\pi}{4}\cdot 0} & e^{-j\cdot\frac{\pi}{4}\cdot 0} & e^{-j\cdot\frac{\pi}{4}\cdot 0} & e^{-j\cdot\frac{\pi}{4}\cdot 0} & e^{-j\cdot\frac{\pi}{4}\cdot 0} \end{pmatrix} \bullet \begin{pmatrix} x[0] & x[1] & x[2] & x[3] & x[4] & x[5] & x[6] & x[7] \end{pmatrix}^T$$

N=8 DFT : X[1] in A·B

$$X[k] = \sum_{n=0}^{7} W_8^{kn}\, x[n] \qquad W_8^{kn} = e^{-j(\frac{2\pi}{8})kn}$$



$$\boldsymbol{A} \cdot \boldsymbol{B} = \boldsymbol{A} \cdot \boldsymbol{A}^H = N\,\boldsymbol{I}$$

$$\boldsymbol{U} \cdot \boldsymbol{U}^H = \boldsymbol{I}$$

Unitary Matrix

When $\boldsymbol{x}$ *looks like this,* $\boldsymbol{X}[1]$ *is max.* (= $\boldsymbol{N}$)

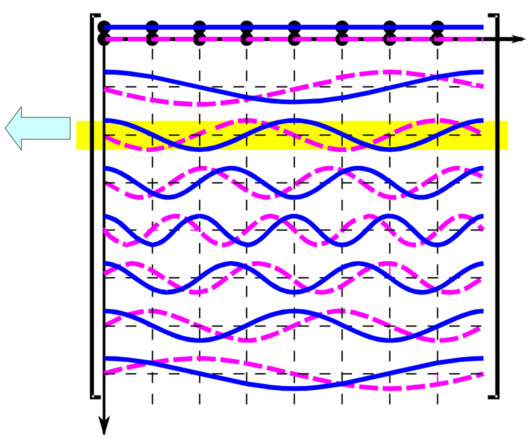
$\boldsymbol{X}[k] = 0 \quad for \;\; k \neq 1$

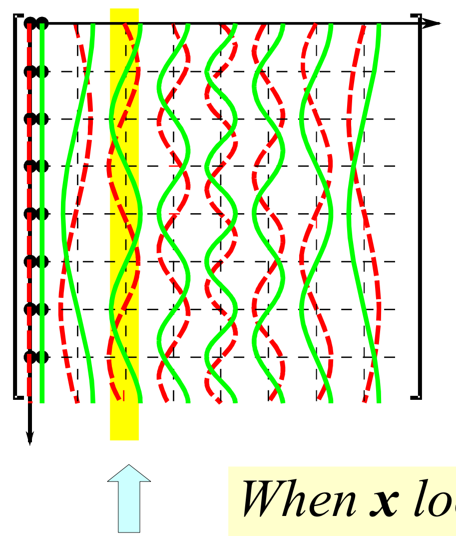
$$X[1] = \begin{pmatrix} e^{-j\cdot\frac{\pi}{4}\cdot 0} & e^{-j\cdot\frac{\pi}{4}\cdot 1} & e^{-j\cdot\frac{\pi}{4}\cdot 2} & e^{-j\cdot\frac{\pi}{4}\cdot 3} & e^{-j\cdot\frac{\pi}{4}\cdot 4} & e^{-j\cdot\frac{\pi}{4}\cdot 5} & e^{-j\cdot\frac{\pi}{4}\cdot 6} & e^{-j\cdot\frac{\pi}{4}\cdot 7} \end{pmatrix} \bullet \begin{pmatrix} x[0] & x[1] & x[2] & x[3] & x[4] & x[5] & x[6] & x[7] \end{pmatrix}^T$$

N=8 DFT : X[2] in A·B

$$X[k] = \sum_{n=0}^{7} W_8^{kn}\, x[n] \qquad W_8^{kn} = e^{-j(\frac{2\pi}{8})kn}$$

$\boldsymbol{X}[2]$





$\boldsymbol{x}$

$$\boldsymbol{A}\cdot\boldsymbol{B} = \boldsymbol{A}\cdot\boldsymbol{A}^H = N\,\boldsymbol{I}$$

$$\boldsymbol{U}\cdot\boldsymbol{U}^H = \boldsymbol{I}$$

Unitary Matrix

When $\boldsymbol{x}$ *looks like this,* $\boldsymbol{X}[2]$ *is max.* (=$\boldsymbol{N}$)

$\boldsymbol{X}[k] = 0 \quad for \;\; k \neq 2$

$$X[2] = \begin{pmatrix} e^{-j\cdot\frac{\pi}{4}\cdot 0} & e^{-j\cdot\frac{\pi}{4}\cdot 2} & e^{-j\cdot\frac{\pi}{4}\cdot 4} & e^{-j\cdot\frac{\pi}{4}\cdot 6} & e^{-j\cdot\frac{\pi}{4}\cdot 0} & e^{-j\cdot\frac{\pi}{4}\cdot 2} & e^{-j\cdot\frac{\pi}{4}\cdot 4} & e^{-j\cdot\frac{\pi}{4}\cdot 6} \end{pmatrix} \bullet \begin{pmatrix} x[0] & x[1] & x[2] & x[3] & x[4] & x[5] & x[6] & x[7] \end{pmatrix}^T$$

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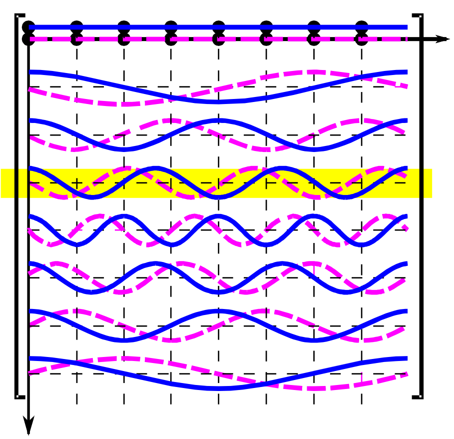
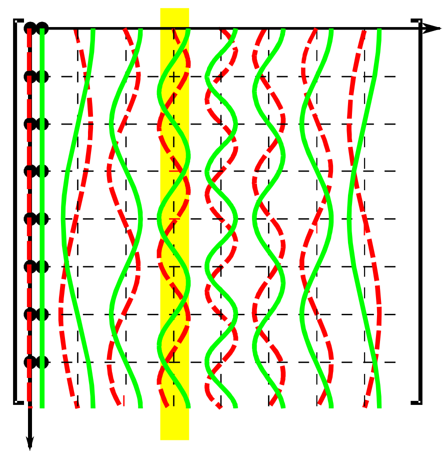
N=8 DFT : X[3] in A·B

$$X[k] = \sum_{n=0}^{7} W_8^{kn}\, x[n] \qquad W_8^{kn} = e^{-j\left(\frac{2\pi}{8}\right)kn}$$

X[3]





$$\boldsymbol{A} \cdot \boldsymbol{B} = \boldsymbol{A} \cdot \boldsymbol{A}^H = N\,\boldsymbol{I}$$

$$\boldsymbol{U} \cdot \boldsymbol{U}^H = \boldsymbol{I}$$

Unitary Matrix

x

When ***x*** *looks like this,* ***X***[3] *is max. (=****N****)*

X[*k*] = 0 *for* $k \neq 3$

$$X[3] = \begin{pmatrix} e^{-j\cdot\frac{\pi}{4}\cdot 0} & e^{-j\cdot\frac{\pi}{4}\cdot 3} & e^{-j\cdot\frac{\pi}{4}\cdot 6} & e^{-j\cdot\frac{\pi}{4}\cdot 1} & e^{-j\cdot\frac{\pi}{4}\cdot 4} & e^{-j\cdot\frac{\pi}{4}\cdot 7} & e^{-j\cdot\frac{\pi}{4}\cdot 2} & e^{-j\cdot\frac{\pi}{4}\cdot 5} \end{pmatrix} \bullet \begin{pmatrix} x[0] & x[1] & x[2] & x[3] & x[4] & x[5] & x[6] & x[7] \end{pmatrix}^T$$

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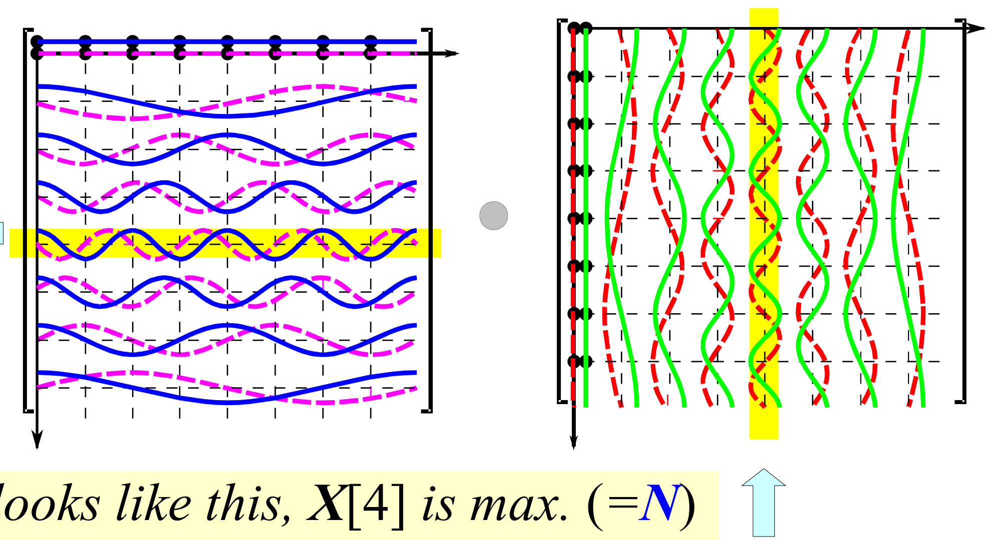
N=8 DFT : X[4] in A·B

$$X[k] = \sum_{n=0}^{7} W_8^{kn}\, x[n] \qquad W_8^{kn} = e^{-j(\frac{2\pi}{8})kn}$$

$\boldsymbol{X}[4]$



$\boldsymbol{x}$

$$\boldsymbol{A} \cdot \boldsymbol{B} = \boldsymbol{A} \cdot \boldsymbol{A}^H = N\,\boldsymbol{I}$$

$$\boldsymbol{U} \cdot \boldsymbol{U}^H = \boldsymbol{I}$$

Unitary Matrix

*When **x** looks like this, **X**[4] is max. (=**N**)*

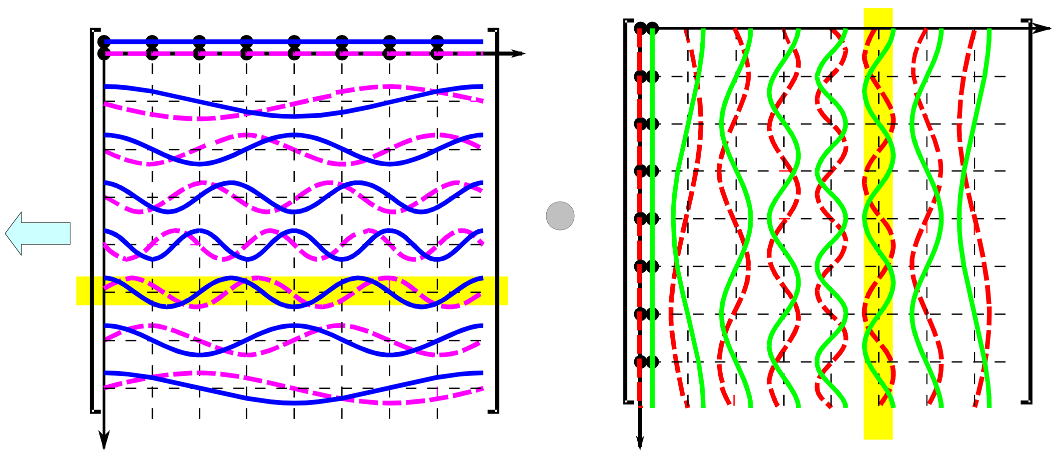
$\boldsymbol{X}[k] = 0$ for $k \neq 4$

$$\boldsymbol{X}[4] = \begin{pmatrix} e^{-j\cdot\frac{\pi}{4}\cdot 0} & e^{-j\cdot\frac{\pi}{4}\cdot 4} & e^{-j\cdot\frac{\pi}{4}\cdot 0} & e^{-j\cdot\frac{\pi}{4}\cdot 4} & e^{-j\cdot\frac{\pi}{4}\cdot 0} & e^{-j\cdot\frac{\pi}{4}\cdot 4} & e^{-j\cdot\frac{\pi}{4}\cdot 0} & e^{-j\cdot\frac{\pi}{4}\cdot 4} \end{pmatrix} \bullet \begin{pmatrix} x[0] & x[1] & x[2] & x[3] & x[4] & x[5] & x[6] & x[7] \end{pmatrix}^T$$

N=8 DFT : X[5] in A·B

$$X[k] = \sum_{n=0}^{7} W_8^{kn}\, x[n] \qquad W_8^{kn} = e^{-j(\frac{2\pi}{8})kn}$$

$\boldsymbol{X}$[5]



$$\boldsymbol{A}\cdot\boldsymbol{B} = \boldsymbol{A}\cdot\boldsymbol{A}^H = N\,\boldsymbol{I}$$

$$\boldsymbol{U}\cdot\boldsymbol{U}^H = \boldsymbol{I}$$

Unitary Matrix

When $\boldsymbol{x}$ *looks like this,* $\boldsymbol{X}$[5] *is max.* (=$\boldsymbol{N}$)

$\boldsymbol{X}[k] = 0 \quad \text{for } k \neq 5$

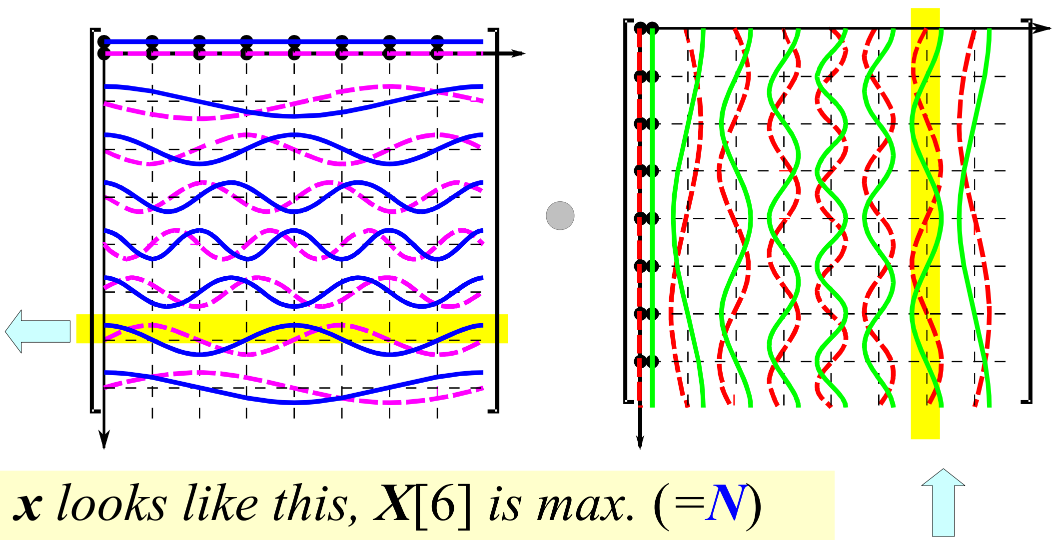
$\boldsymbol{x}$

$$X[5] = \begin{pmatrix} e^{-j\cdot\frac{\pi}{4}\cdot 0} & e^{-j\cdot\frac{\pi}{4}\cdot 5} & e^{-j\cdot\frac{\pi}{4}\cdot 2} & e^{-j\cdot\frac{\pi}{4}\cdot 7} & e^{-j\cdot\frac{\pi}{4}\cdot 4} & e^{-j\cdot\frac{\pi}{4}\cdot 1} & e^{-j\cdot\frac{\pi}{4}\cdot 6} & e^{-j\cdot\frac{\pi}{4}\cdot 3} \end{pmatrix} \bullet \begin{pmatrix} x[0] & x[1] & x[2] & x[3] & x[4] & x[5] & x[6] & x[7] \end{pmatrix}^T$$

N=8 DFT : X[6] in A·B

$$X[k] = \sum_{n=0}^{7} W_8^{kn}\, x[n] \qquad W_8^{kn} = e^{-j(\frac{2\pi}{8})kn}$$

X[6]



x

$$\boldsymbol{A} \cdot \boldsymbol{B} = \boldsymbol{A} \cdot \boldsymbol{A}^H = N\,\boldsymbol{I}$$

$$\boldsymbol{U} \cdot \boldsymbol{U}^H = \boldsymbol{I}$$

Unitary Matrix

*When **x** looks like this, **X**[6] is max. (=**N**)*

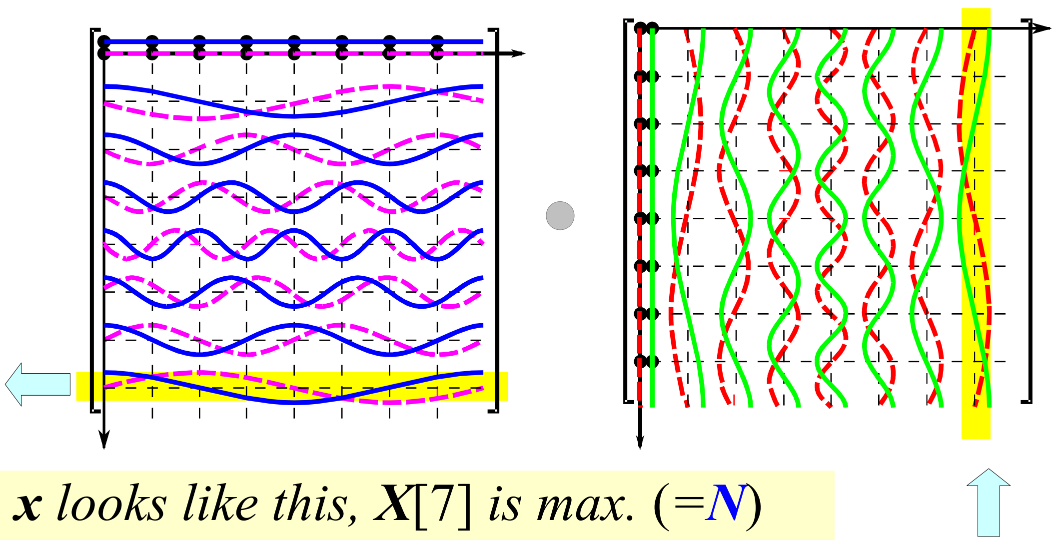
$\boldsymbol{X}[k] = 0$ for $k \neq 6$

$$\boldsymbol{X}[6] = \begin{pmatrix} e^{-j\cdot\frac{\pi}{4}\cdot 0} & e^{-j\cdot\frac{\pi}{4}\cdot 6} & e^{-j\cdot\frac{\pi}{4}\cdot 4} & e^{-j\cdot\frac{\pi}{4}\cdot 2} & e^{-j\cdot\frac{\pi}{4}\cdot 0} & e^{-j\cdot\frac{\pi}{4}\cdot 6} & e^{-j\cdot\frac{\pi}{4}\cdot 4} & e^{-j\cdot\frac{\pi}{4}\cdot 2} \end{pmatrix} \bullet \begin{pmatrix} x[0] & x[1] & x[2] & x[3] & x[4] & x[5] & x[6] & x[7] \end{pmatrix}^T$$

N=8 DFT : X[7] in A·B

$$X[k] = \sum_{n=0}^{7} W_8^{kn}\, x[n] \qquad W_8^{kn} = e^{-j(\frac{2\pi}{8})kn}$$

X[7]



x

$$\boldsymbol{A}\cdot\boldsymbol{B} = \boldsymbol{A}\cdot\boldsymbol{A}^H = N\,\boldsymbol{I}$$

$$\boldsymbol{U}\cdot\boldsymbol{U}^H = \boldsymbol{I}$$

Unitary Matrix

*When **x** looks like this, **X**[7] is max. (=**N**)*

$\boldsymbol{X}[k] = 0$ for $k \neq 7$

$$X[7] = \begin{pmatrix} e^{-j\cdot\frac{\pi}{4}\cdot 0} & e^{-j\cdot\frac{\pi}{4}\cdot 7} & e^{-j\cdot\frac{\pi}{4}\cdot 6} & e^{-j\cdot\frac{\pi}{4}\cdot 5} & e^{-j\cdot\frac{\pi}{4}\cdot 4} & e^{-j\cdot\frac{\pi}{4}\cdot 3} & e^{-j\cdot\frac{\pi}{4}\cdot 2} & e^{-j\cdot\frac{\pi}{4}\cdot 1} \end{pmatrix} \bullet \begin{pmatrix} x[0] & x[1] & x[2] & x[3] & x[4] & x[5] & x[6] & x[7] \end{pmatrix}^T$$

References

[1] http://en.wikipedia.org/
[2] J.H. McClellan, et al., Signal Processing First, Pearson Prentice Hall, 2003
[3] A "graphical interpretation" of the DFT and FFT, by Steve Mann