Fourier Integrals (3A)

- Continuous Time Fourier Transform

Please send corrections (or suggestions) to youngwlim@hotmail.com.

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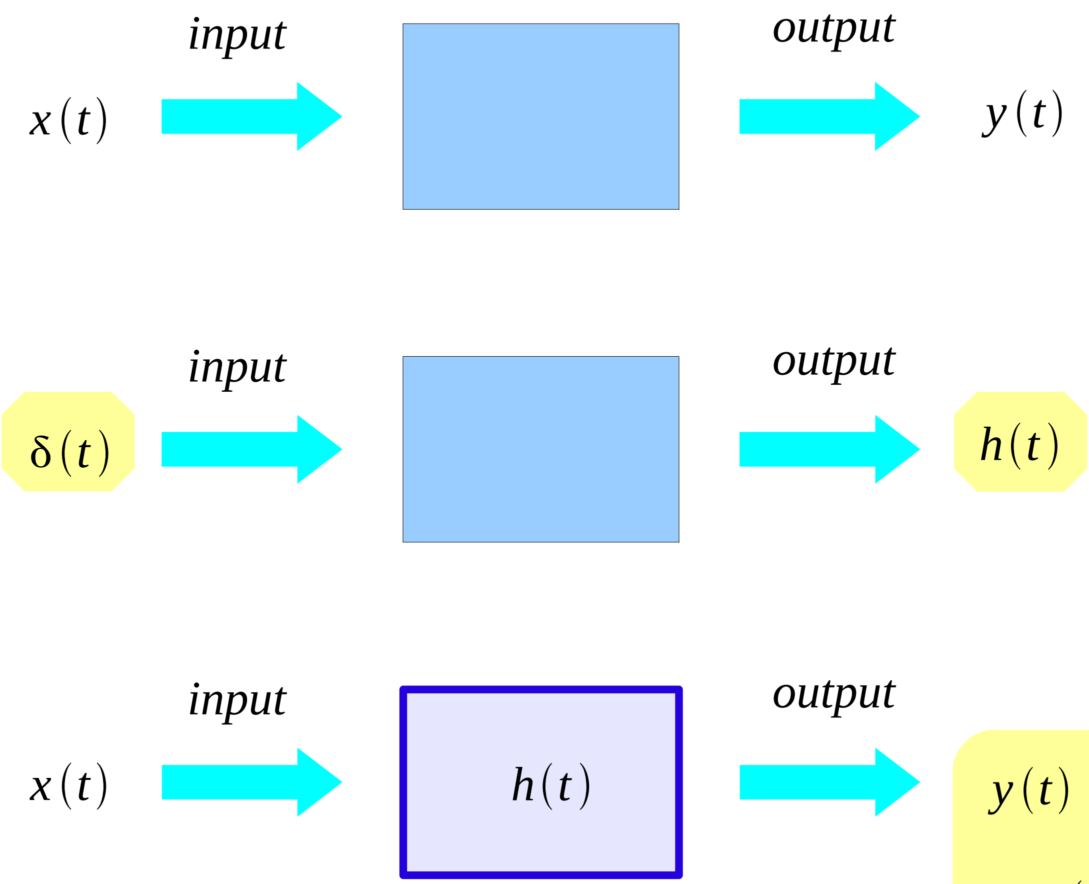
Impulse Response

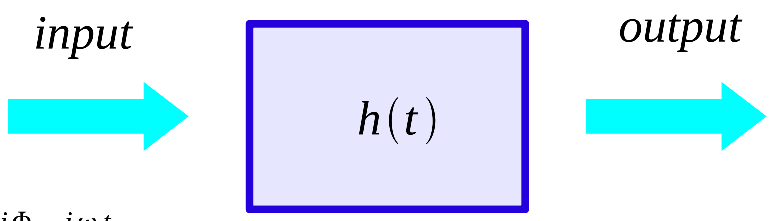


$$
\begin{aligned}
y(t) &\\
&= h(t) * x(t) \\
&= \int_{-\infty}^{+\infty} h(\tau)\, x(t-\tau)\, d\tau
\end{aligned}
$$

Frequency Response

$x(t)$

$= A e^{j\Phi} e^{j\omega t}$



$y(t)$

$= h(t) * x(t)$

$= \int_{-\infty}^{+\infty} h(\tau) x(t-\tau)\, d\tau$

$= \int_{-\infty}^{+\infty} h(\tau) A e^{j\Phi} e^{j\omega(t-\tau)}\, d\tau$

$= \int_{-\infty}^{+\infty} h(\tau) A e^{j\Phi} e^{j\omega t} e^{-j\omega\tau}\, d\tau$

$= A e^{j\Phi} e^{j\omega t} \int_{-\infty}^{+\infty} h(\tau) e^{-j\omega\tau}\, d\tau$

$$H(j\omega) = \int_{-\infty}^{+\infty} h(\tau) e^{-jw\tau}\, d\tau$$

$x(t) = A e^{j\Phi} e^{j\omega t}$

$y(t) = H(j\omega) A e^{j\Phi} e^{j\omega t}$

CTFT of a Rect(t/T) function (1)

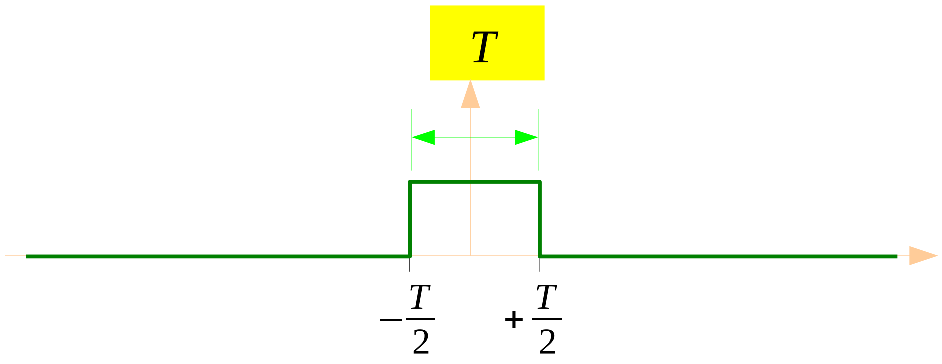
Continuous Time Fourier Transform — Aperiodic Continuous Time Signal

$$X(j\omega) = \int_{-\infty}^{+\infty} x(t)\, e^{-j\omega t}\, dt \quad \Longleftrightarrow \quad x(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} X(j\omega)\, e^{+j\omega t}\, d\omega$$

$$X(j\omega) = \int_{-T/2}^{+T/2} e^{-j\omega t}\, dt$$

$$= \left[\frac{-1}{j\omega} e^{-j\omega t}\right]_{-T/2}^{+T/2} = -\frac{e^{-j\omega T/2} - e^{+j\omega T/2}}{j\omega}$$

$$= \frac{\sin(\omega T/2)}{\omega/2}$$

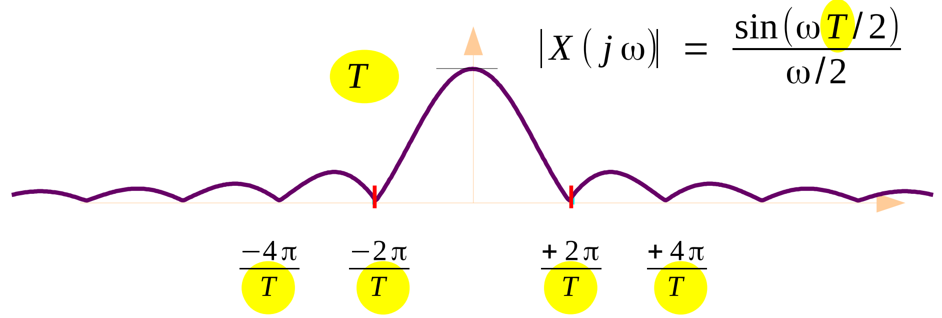


$$X(j0) = \lim_{\omega\to 0}\frac{\sin(\omega T/2)}{\omega/2} = \lim_{\omega\to 0}\frac{T}{2}\frac{\cos(\omega T/2)}{1/2} = T$$

$$\sin(\omega T/2) = 0 \quad \Rightarrow \quad \omega T/2 = \pi n$$

$$\Rightarrow \quad \omega = \frac{2\pi}{T} n$$

$$\Rightarrow \quad \omega = \pm\frac{2\pi}{T}, \pm\frac{4\pi}{T}, \pm\frac{6\pi}{T}, \cdots$$



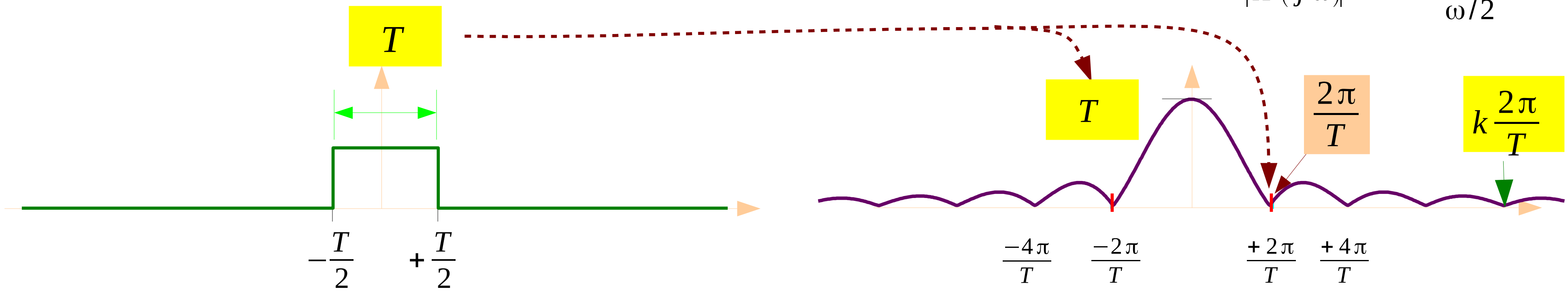
CTFT of a Rect(t/T) function (2)

Continuous Time Fourier Transform   Aperiodic Continuous Time Signal

$$X(j\omega) = \int_{-\infty}^{+\infty} x(t)\, e^{-j\omega t}\, dt \quad \Longleftrightarrow \quad x(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} X(j\omega)\, e^{+j\omega t}\, d\omega$$

$$X(j\omega) = \int_{-T/2}^{+T/2} e^{-j\omega t}\, dt = \frac{\sin(\omega T/2)}{\omega/2}$$

$$|X(j\omega)| = \frac{\sin(\omega T/2)}{\omega/2}$$



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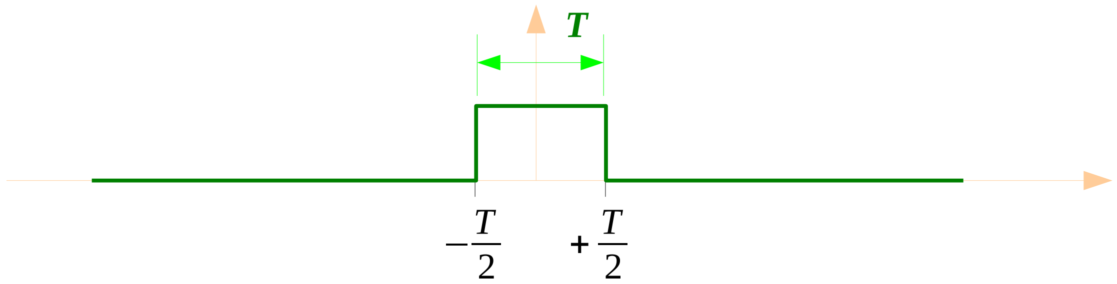
11/4/12

CTFT and CTFS

Continuous Time Fourier Transform — Aperiodic Continuous Time Signal

$$X(j\omega) = \int_{-\infty}^{+\infty} x(t)\, e^{-j\omega t}\, dt \quad \Longleftrightarrow \quad x(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} X(j\omega)\, e^{+j\omega t}\, d\omega$$

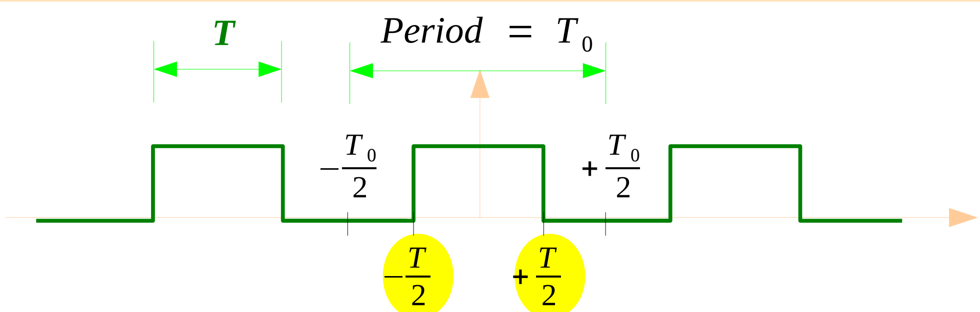
$$X(j\omega) = \frac{\sin(\omega T/2)}{\omega/2}$$



Continuous Time Fourier Series — Periodic Continuous Time Signal

$$C_k = \frac{1}{T}\int_{0}^{T} x(t)\, e^{-jk\omega_0 t}\, dt \quad \Longleftrightarrow \quad x(t) = \sum_{n=0}^{\infty} C_k\, e^{+jk\omega_0 t}$$

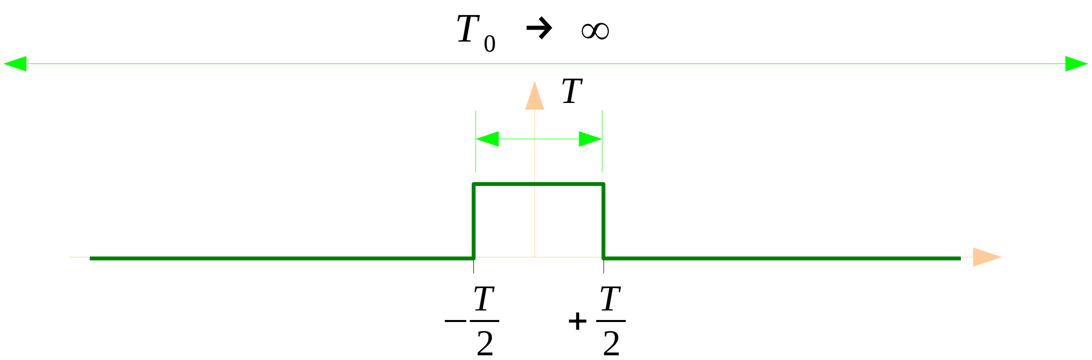
$$C_k = \frac{1}{T_0}\,\frac{\sin(k\omega_0 T/2)}{k\omega_0/2}$$



CTFT ← CTFS

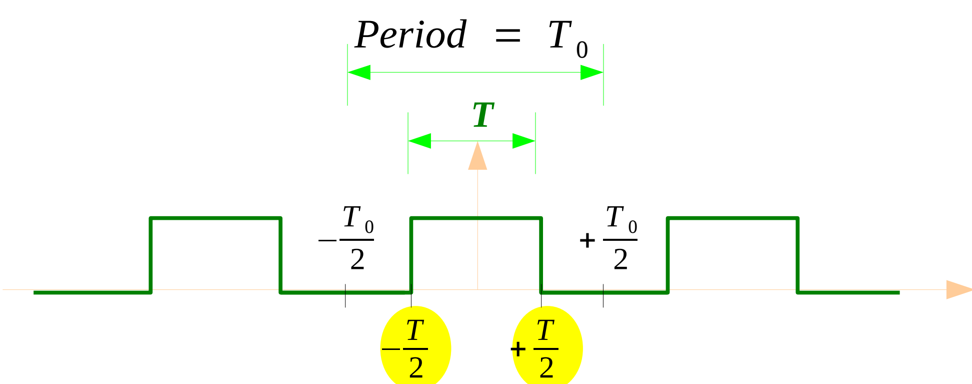
Aperiodic Continuous Time Signal

Continuous Time Fourier Transform



Periodic Continuous Time Signal

Continuous Time Fourier Series



$$x(t)$$

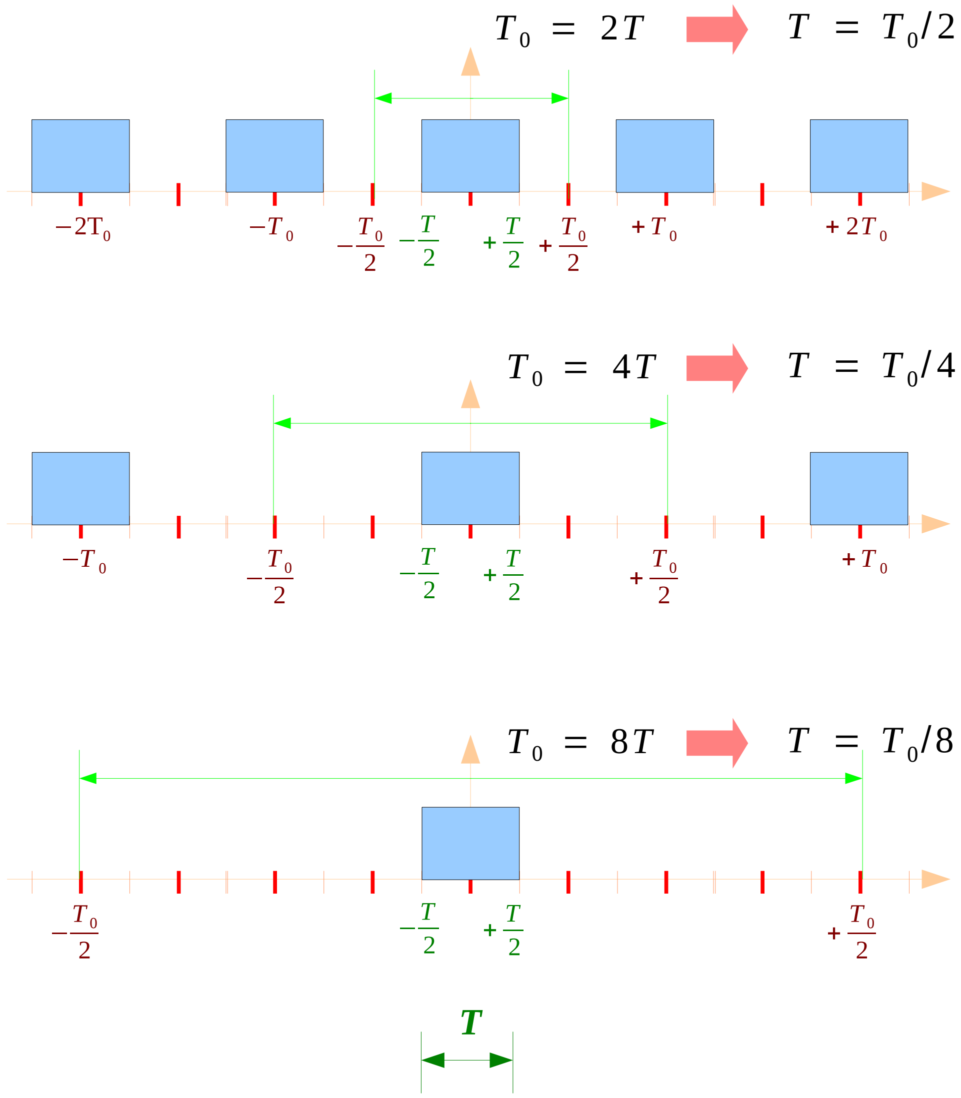
↑

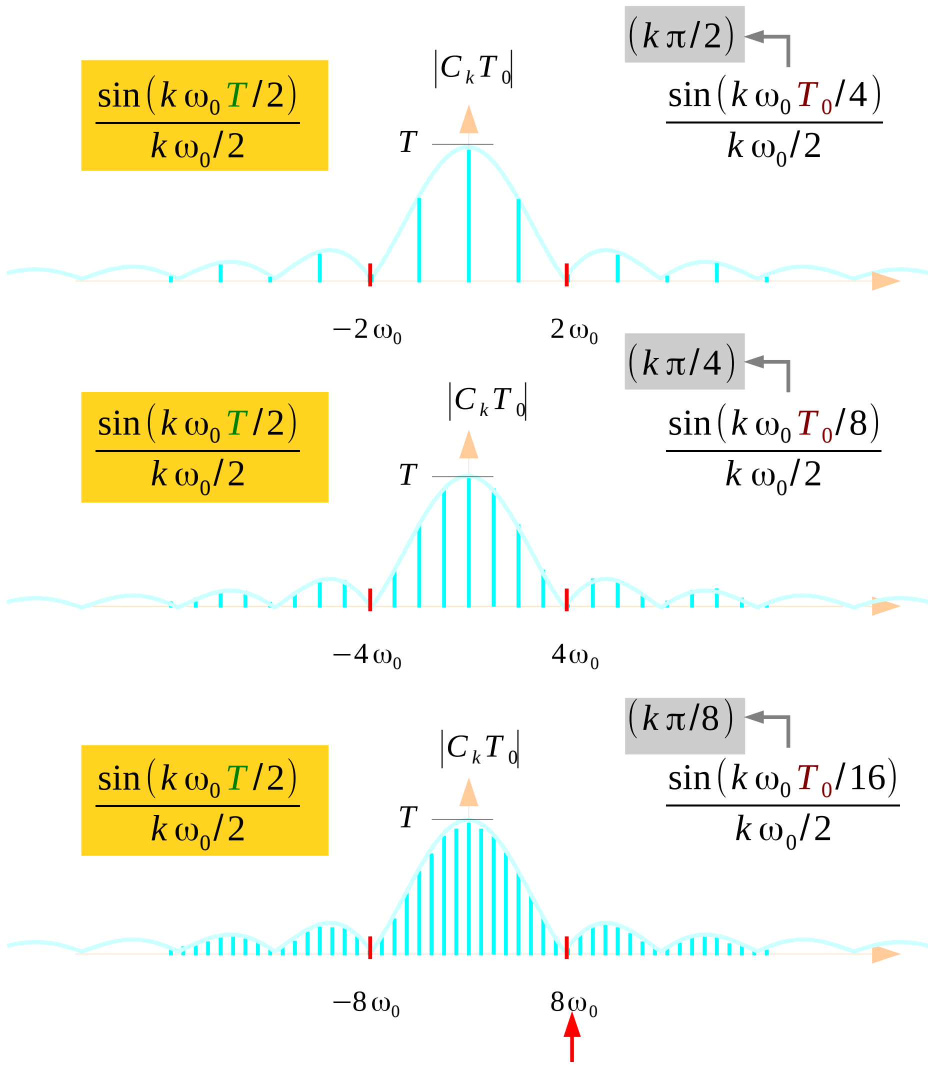
As $T_0 \to \infty$, $x_{T_0}(t) \to x(t)$

$$\omega_0 = \frac{2\pi}{T_0} \to 0$$

$$x_{T_0}(t) = \sum_{n=-\infty}^{+\infty} x(t - nT_0)$$

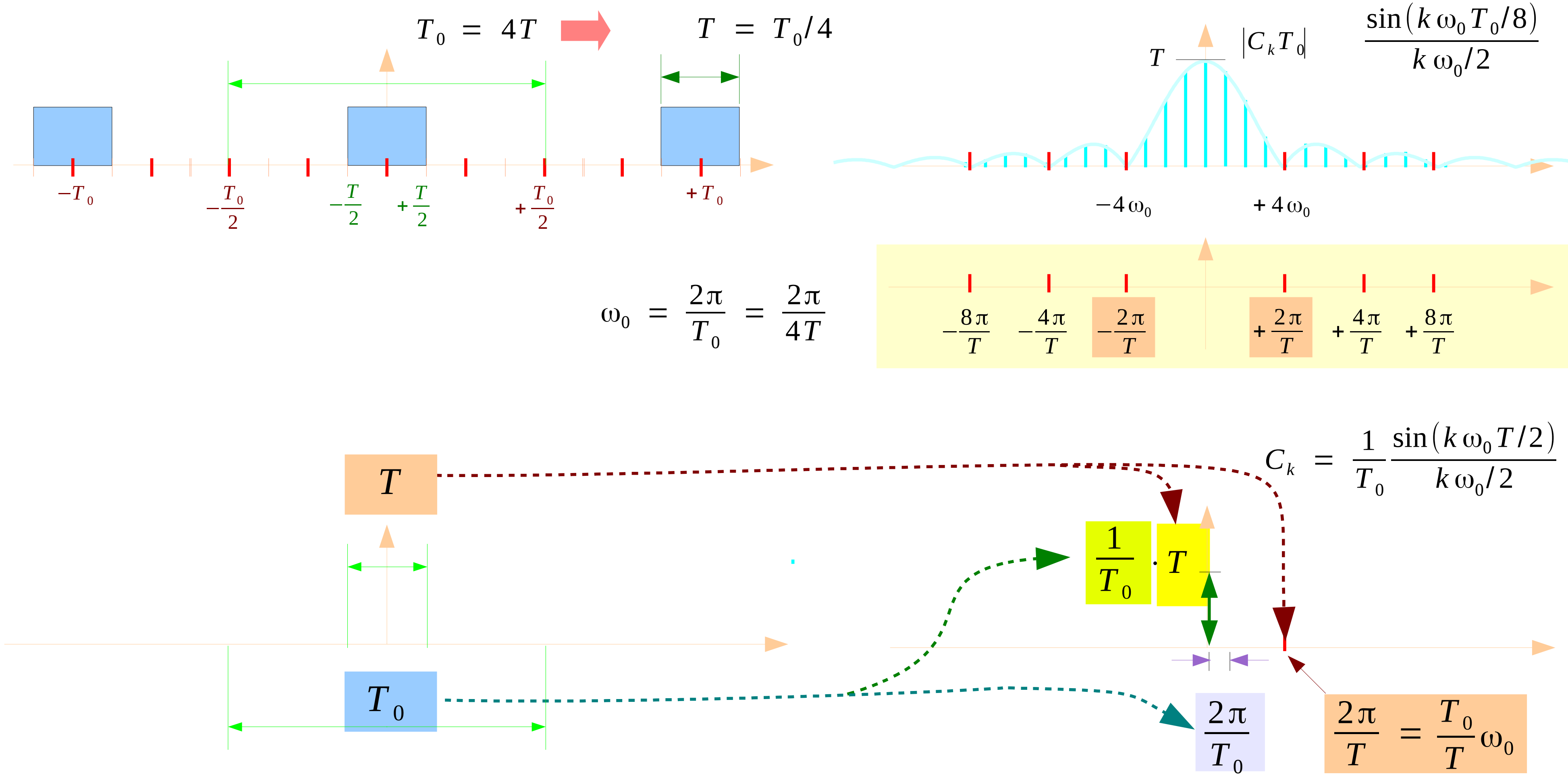
CTFT and CTFS as $T_0 \to \infty$ (1)





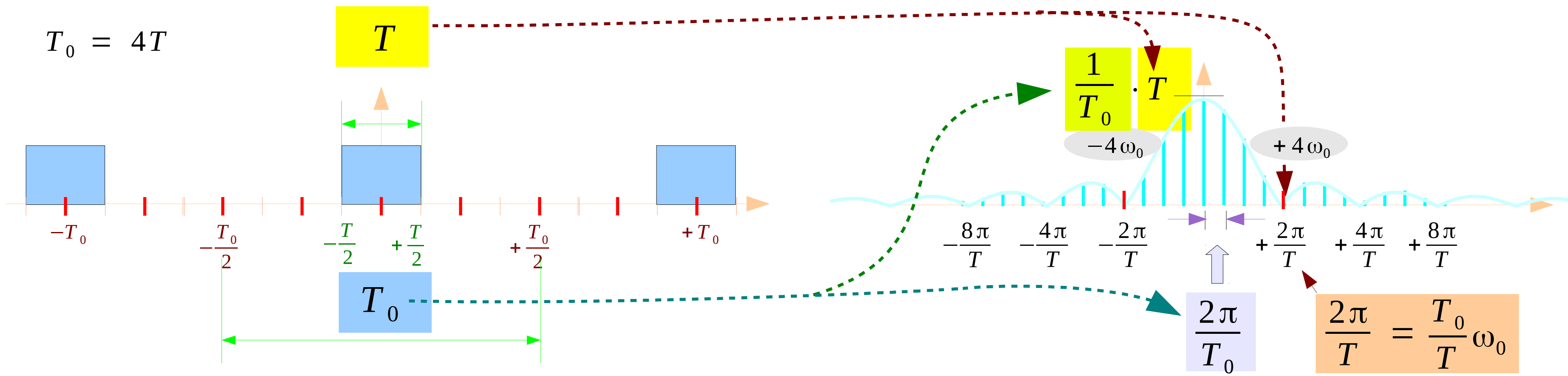
$$\omega_0 = \frac{2\pi}{T_0} = \frac{2\pi}{8T} \Rightarrow \frac{T_0}{T}\omega_0 = \frac{2\pi}{T}$$

CTFT and CTFS as $T_0 \to \infty$ (2)



CTFT of a Rect(t/T) function (3)

$T_0 = 4T$



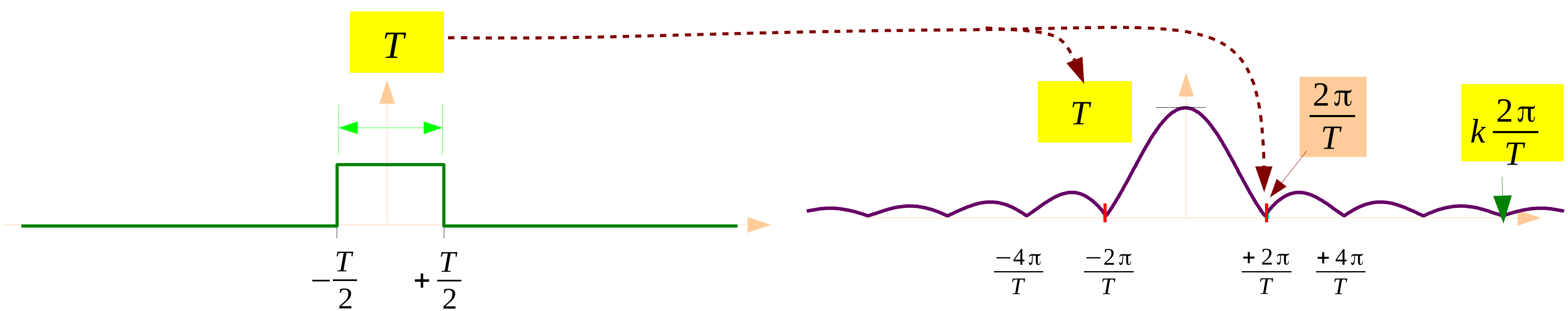
$$C_k T_0 = \frac{\sin(k\omega_0 T/2)}{k\omega_0/2}$$

$$X(j\omega) = \frac{\sin(\omega T/2)}{\omega/2}$$

$$X(j\omega) = \lim_{k\omega_0 \to \omega} \frac{\sin(k\omega_0 T/2)}{k\omega_0/2} = \frac{\sin(\omega T/2)}{\omega/2}$$

$$C_k = \frac{1}{T_0}\frac{\sin(k\omega_0 T/2)}{k\omega_0/2}$$

$$X(j\omega) = \frac{\sin(\omega T/2)}{\omega/2}$$



From CTFS to CTFT

Continuous Time Fourier **Series**

$$C_k = \frac{1}{T}\int_0^T x(t)\, e^{-jk\omega_0 t}\, dt \quad \Longleftrightarrow \quad x(t) = \sum_{n=0}^{\infty} C_k\, e^{+jk\omega_0 t}$$

$$C_k = \frac{1}{T_0}\int_{-T_0/2}^{+T_0/2} x_{T_0}(t)\, e^{-jk\omega_0 t}\, dt \qquad x_{T_0}(t) = \sum_{n=0}^{\infty} C_k\, e^{+jk\omega_0 t} \cdot \frac{2\pi}{2\pi} \cdot \frac{T_0}{T_0}$$

$$C_k T_0 = \int_{-T_0/2}^{+T_0/2} x_{T_0}(t)\, e^{-jk\omega_0 t}\, dt \qquad x_{T_0}(t) = \frac{1}{2\pi}\sum_{k=0}^{\infty} C_k T_0\, e^{+jk\omega_0 t} \cdot \frac{2\pi}{T_0}$$

$T_0 \to \infty \Rightarrow$ $C_k T_0 \to X(j\omega)$ $\quad x_{T_0} \to x(t)$ $\quad \omega_0 = \frac{2\pi}{T_0} \to d\omega$

Continuous Time Fourier **Transform**

$$X(j\omega) = \int_{-\infty}^{+\infty} x(t)\, e^{-j\omega t}\, dt \quad \Longleftrightarrow \quad x(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} X(j\omega)\, e^{+j\omega t}\, d\omega$$

Continuous Time Fourier Transform

$$x(t) = A\delta(t) \iff X(j\omega) = A$$

$$X(j\omega) = \int_{-\infty}^{+\infty} x(t)\, e^{-j\omega t}\, dt$$

$$X(j\omega) = \int_{-\infty}^{+\infty} A\delta(t) e^{-j\omega t} dt$$

$$= \int_{-\infty}^{+\infty} A\delta(t) e^{0} dt$$

$$= A\int_{-\infty}^{+\infty} \delta(t) dt = A$$

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} X(j\omega)\, e^{+j\omega t}\, d\omega$$

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} A e^{+j\omega t} d\omega$$

$$= \frac{A}{2\pi}\int_{-\infty}^{+\infty} e^{+j\omega t} d\omega = A\delta(t)$$

CTFT of Frequency Domain Impulse

Continuous Time Fourier Transform

$$X(j\omega) = 2\pi\,\delta(\omega) \iff x(t) = 1$$

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} X(j\omega)\, e^{+j\omega t} d\omega$$

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} 2\pi\,\delta(\omega) e^{+j\omega t} d\omega$$

$$= \int_{-\infty}^{+\infty} \delta(\omega) e^{0} d\omega = 1$$

$$X(j\omega) = \int_{-\infty}^{+\infty} x(t)\, e^{-j\omega t} dt$$

$$X(j\omega) = \int_{-\infty}^{+\infty} e^{-j\omega t} dt = 2\pi\,\delta(\omega)$$

CTFS of Impulse Train

Continuous Time Fourier Series

$$C_n = \frac{1}{T}\int_0^T x(t)\, e^{-jn\omega_0 t}\, dt$$

$$x(t) = \sum_{n=0}^{\infty} C_n\, e^{+jn\omega_0 t}$$

$$p(t) = \sum_{n=-\infty}^{+\infty} \delta(t - nT_s)$$

$$C_n = \frac{1}{T_s}\int_{-T_s/2}^{+T_s/2} \delta(t) e^{-jn\omega_s t}\, dt$$

$$= \frac{1}{T_s}\int_{-T_s/2}^{+T_s/2} \delta(t)\, dt = \frac{1}{T_s}$$

$$p(t) = \sum_{n=-\infty}^{+\infty} C_n e^{jn\omega_s t}$$

$$= \sum_{n=-\infty}^{+\infty} \frac{1}{T_s} e^{jn\omega_s t}$$

$$p(t) = \sum_{n=-\infty}^{+\infty} \delta(t - nT_s) = \sum_{n=-\infty}^{+\infty} \frac{1}{T_s} e^{jn\omega_s t}$$

CTFT of Impulse Train

Continuous Time Fourier Transform

$$p(t) = \sum_{n=-\infty}^{+\infty} \delta(t - nT_s)$$

$$P(j\omega) = \int_{-\infty}^{+\infty} \sum_{n=-\infty}^{+\infty} \delta(t - nT_s) e^{-jn\omega t} dt = \int_{-\infty}^{+\infty} \sum_{n=-\infty}^{+\infty} \frac{1}{T_s} e^{jn\omega_s t} e^{-jn\omega t} dt$$

$$= \sum_{n=-\infty}^{+\infty} \frac{1}{T_s} \int_{-\infty}^{+\infty} e^{-j(\omega - n\omega_s)t} dt$$

$$= \sum_{n=-\infty}^{+\infty} \left(\frac{2\pi}{T_s}\right) \delta(\omega - n\omega_s)$$

$$X(j\omega) = \int_{-\infty}^{+\infty} x(t)\, e^{-j\omega t} dt$$

$$p(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \sum_{n=-\infty}^{+\infty} \left(\frac{2\pi}{T_s}\right) \delta(\omega - n\omega_s)\, e^{+j\omega t} d\omega = \sum_{n=-\infty}^{+\infty} \frac{1}{T_s} \int_{-\infty}^{+\infty} \delta(\omega - n\omega_s)\, e^{+j\omega t} d\omega$$

$$= \sum_{n=-\infty}^{+\infty} \frac{1}{T_s} e^{jn\omega_s t} = \sum_{n=-\infty}^{+\infty} \delta(t - nT_s)$$

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} X(j\omega)\, e^{+j\omega t} d\omega$$

Other Convention

Continuous Time Fourier Transform {unitary, angular frequency}

$$X(j\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} x(t)\, e^{-j\omega t}\, dt \quad \Leftrightarrow$$

$$x(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} X(j\omega)\, e^{+j\omega t}\, d\omega$$

Continuous Time Fourier Transform {non-unitary, angular frequency}

$$X(j\omega) = 1 \cdot \int_{-\infty}^{+\infty} x(t)\, e^{-j\omega t}\, dt \quad \Leftrightarrow$$

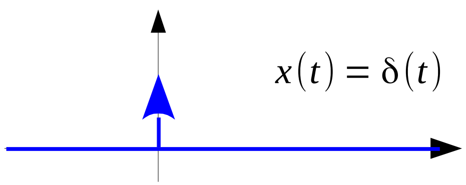
$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} X(j\omega)\, e^{+j\omega t}\, d\omega$$

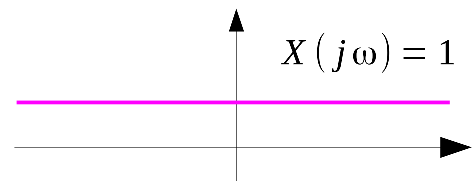
CTFT of Impulse

$$X(j\omega) = \int_{-\infty}^{+\infty} x(t)\, e^{-j\omega t}\, dt \quad \Leftrightarrow \quad x(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} X(j\omega)\, e^{+j\omega t}\, d\omega$$

$$x(t) = A\delta(t)$$

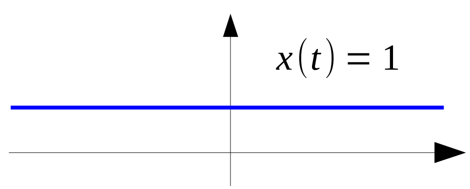
$$X(jw) = \int_{-\infty}^{+\infty} \delta(t)\, e^{-j\omega t}\, dt = \int_{-\infty}^{+\infty} \delta(t)\, e^{-j\omega 0}\, dt = \int_{-\infty}^{+\infty} \delta(t)\, dt = 1$$

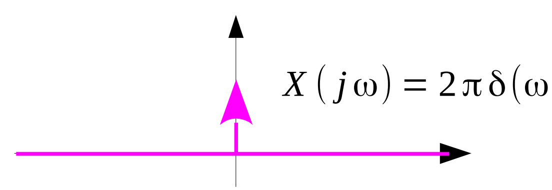




$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} 2\pi\delta(\omega)\, e^{+j\omega t}\, d\omega = \int_{-\infty}^{+\infty} \delta(\omega)\, e^{+j0t}\, d\omega = \int_{-\infty}^{+\infty} \delta(\omega)\, d\omega = 1$$

$$X(jw) = 2\pi\delta(\omega)$$





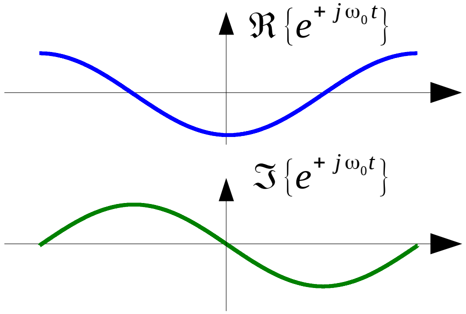
CTFT of Sinusoid

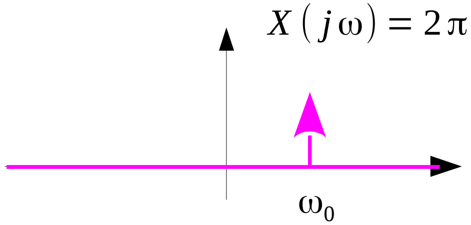
$$X(j\omega) = \int_{-\infty}^{+\infty} x(t)\, e^{-j\omega t}\, dt \quad \Longleftrightarrow \quad x(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} X(j\omega)\, e^{+j\omega t}\, d\omega$$

$$X(jw) = 2\pi\delta(\omega - \omega_0)$$

$$\begin{aligned}
x(t) &= \frac{1}{2\pi}\int_{-\infty}^{+\infty} 2\pi\delta(\omega - \omega_0)\, e^{+j\omega t}\, d\omega \\
&= \int_{-\infty}^{+\infty} \delta(\omega - \omega_0)\, e^{+j\omega_0 t}\, d\omega \\
&= e^{+j\omega_0 t}\int_{-\infty}^{+\infty} \delta(\omega)\, d\omega \\
&= e^{+j\omega_0 t} \\
&= \cos\omega_0 t + j\sin\omega_0 t
\end{aligned}$$

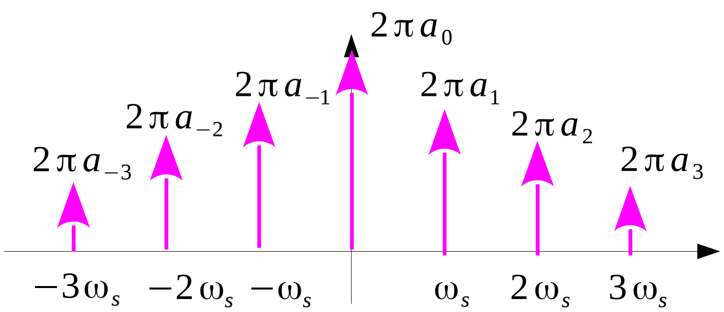
$$\omega_0 = \frac{2\pi}{T_0}$$





CTFT of Periodic Signals

A general formula for the CTFT
of any periodic function
for which a CTFS exists



Period

$$T_s \Rightarrow \omega_s = \frac{2\pi}{T_s}$$

Fourier Series Expansion

$$x(t) = \sum_{k=-\infty}^{+\infty} a_k \, e^{+jk\omega_s t}$$

Fourier Series Coefficients

$$a_k = \frac{1}{T_s} \int_{-T_s/2}^{+T_s/2} x(t) \, e^{-jk\omega_s t} \, dt$$

Fourier Transform

$$X(j\omega) = \sum_{k=-\infty}^{\infty} a_k \int_{-\infty}^{+\infty} e^{jk\omega_s t} \, e^{-j\omega t} \, dt$$

$$= \sum_{k=-\infty}^{\infty} a_k \int_{-\infty}^{+\infty} e^{-j(\omega - k\omega_s)t} \, dt$$

$$= \sum_{k=-\infty}^{\infty} 2\pi \, a_k \, \delta(\omega - k\omega_s)$$

Period Time Signal → CTFS → Sampling in Frequency

CTFT of Impulse Train

A general formula for the CTFT
of any periodic function
for which a CTFS exists

Period

$$T_s \Rightarrow \omega_s = \frac{2\pi}{T_s}$$

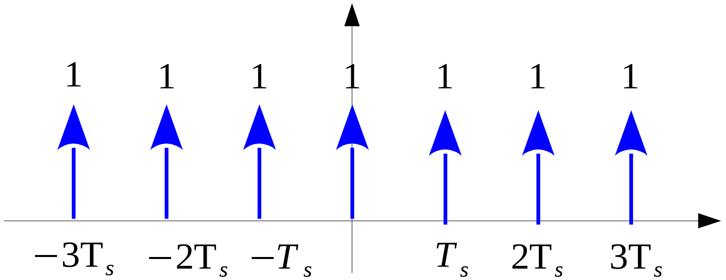
Fourier Series Expansion

$$p(t) = \sum_{k=-\infty}^{+\infty} a_k \, e^{+jk\omega_s t}$$

Fourier Series Coefficients

$$a_k = \frac{1}{T_s} \int_{-T_s/2}^{+T_s/2} x(t)\, e^{-jk\omega_s t}\, dt$$

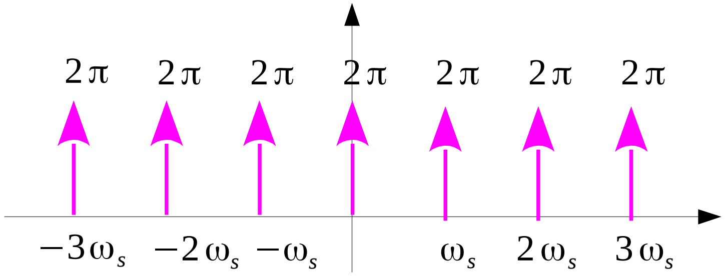
$$= \frac{1}{T_s} \int_{-T_s/2}^{+T_s/2} \delta(t)\, dt = \frac{1}{T_s}$$



Fourier Transform

$$X(j\omega) = \sum_{k=-\infty}^{\infty} 2\pi a_k \delta(\omega - k\omega_s)$$

$$= \sum_{k=-\infty}^{\infty} \left(\frac{2\pi}{T_s}\right) \delta(\omega - k\omega_s)$$



References

[1] http://en.wikipedia.org/
[2] J.H. McClellan, et al., Signal Processing First, Pearson Prentice Hall, 2003