

CLTI Differential Equation

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Causal LTI Systems (1)

$$a_N \frac{d^N y(t)}{dt^N} + a_{N-1} \frac{d^{N-1} y(t)}{dt^{N-1}} + \dots + a_1 \frac{dy(t)}{dt} + a_0 y(t) = b_M \frac{d^M x(t)}{dt^M} + b_{M-1} \frac{d^{M-1} x(t)}{dt^{M-1}} + \dots + b_1 \frac{dx(t)}{dt} + b_0 x(t)$$

$$\frac{d^N y(t)}{dt^N} + a_1 \frac{d^{N-1} y(t)}{dt^{N-1}} + \dots + a_{N-1} \frac{dy(t)}{dt} + a_N y(t) = b_{N-M} \frac{d^M x(t)}{dt^M} + b_{N-M+1} \frac{d^{M-1} x(t)}{dt^{M-1}} + \dots + b_{N-1} \frac{dx(t)}{dt} + b_N x(t)$$

$$(D^N + a_1 D^{N-1} + \dots + a_{N-1} D + a_N) y(t) = (D^M + b_{N-M+1} D^{M-1} + \dots + b_{N-1} D + b_N) x(t)$$

$$Q(D) y(t) = P(D) x(t)$$

$$M = N$$

$$\frac{d^N y(t)}{dt^N} + a_1 \frac{d^{N-1} y(t)}{dt^{N-1}} + \dots + a_{N-1} \frac{dy(t)}{dt} + a_N y(t) = b_0 \frac{d^M x(t)}{dt^M} + b_1 \frac{d^{M-1} x(t)}{dt^{M-1}} + \dots + b_{N-1} \frac{dx(t)}{dt} + b_N x(t)$$

$$(D^N + a_1 D^{N-1} + \dots + a_{N-1} D + a_N) y(t) = (b_0 D^M + b_1 D^{M-1} + \dots + b_{N-1} D + b_N) x(t)$$

$$Q(D) y(t) = P(D) x(t)$$

Causal LTI Systems (2)

$$a_N \frac{d^N y(t)}{dt^N} + a_{N-1} \frac{d^{N-1} y(t)}{dt^{N-1}} + \dots + a_1 \frac{dy(t)}{dt} + a_0 y(t) = b_M \frac{d^M x(t)}{dt^M} + b_{M-1} \frac{d^{M-1} x(t)}{dt^{M-1}} + \dots + b_1 \frac{dx(t)}{dt} + b_0 x(t)$$

$$\frac{d^N y(t)}{dt^N} + a_1 \frac{d^{N-1} y(t)}{dt^{N-1}} + \dots + a_{N-1} \frac{dy(t)}{dt} + a_N y(t) = b_{N-M} \frac{d^M x(t)}{dt^M} + b_{N-M+1} \frac{d^{M-1} x(t)}{dt^{M-1}} + \dots + b_{N-1} \frac{dx(t)}{dt} + b_N x(t)$$

$$(D^N + a_1 D^{N-1} + \dots + a_{N-1} D + a_N) y(t) = (D^M + b_{N-M+1} D^{M-1} + \dots + b_{N-1} D + b_N) x(t)$$

$$Q(D) y(t) = P(D) x(t)$$

- Zero Input Response
- Zero State Response (Convolution with $h(t)$)

- Natural Response (Homogeneous Solution)
- Forced Response (Particular Solution)

Zero Input Response $y_0(t)$ – (1)

$$\frac{d^N y(t)}{dt^N} + a_1 \frac{d^{N-1} y(t)}{dt^{N-1}} + \dots + a_{N-1} \frac{dy(t)}{dt} + a_N y(t) = b_{N-M} \frac{d^M x(t)}{dt^M} + b_{N-M+1} \frac{d^{M-1} x(t)}{dt^{M-1}} + \dots + b_{N-1} \frac{dx(t)}{dt} + b_N x(t)$$

$$\boxed{(D^N + a_1 D^{N-1} + \dots + a_{N-1} D + a_N)} \cdot y(t) = \boxed{(b_{N-M} D^M + b_{N-M+1} D^{M-1} + \dots + b_{N-1} D + b_N)} \cdot x(t)$$

$$Q(D) \cdot y(t) = P(D) \cdot x(t)$$

$$Q(D)y_0(t) = 0$$



$$(D^N + a_1 D^{N-1} + \dots + a_{N-1} D + a_N)y_0(t) = 0$$

Linear combination of $y_0(t)$ and its derivatives = 0

if and only if

$$y_0(t) = ce^{\lambda t}$$

$$\dot{y}_0(t) = c\lambda e^{\lambda t}$$

$$\ddot{y}_0(t) = c\lambda^2 e^{\lambda t}$$

...

$$Q(\lambda) = 0$$



$$\underbrace{(\lambda^N + a_1 \lambda^{N-1} + \dots + a_{N-1} \lambda + a_N)}_{= 0} \underbrace{ce^{\lambda t}}_{\neq 0} = 0$$

Zero Input Response $y_0(t)$ – (2)

$$\frac{d^N y(t)}{dt^N} + a_1 \frac{d^{N-1} y(t)}{dt^{N-1}} + \dots + a_{N-1} \frac{dy(t)}{dt} + a_N y(t) = b_{N-M} \frac{d^M x(t)}{dt^M} + b_{N-M+1} \frac{d^{M-1} x(t)}{dt^{M-1}} + \dots + b_{N-1} \frac{dx(t)}{dt} + b_N x(t)$$

$$\boxed{(D^N + a_1 D^{N-1} + \dots + a_{N-1} D + a_N)} \cdot y(t) = \boxed{(b_{N-M} D^M + b_{N-M+1} D^{M-1} + \dots + b_{N-1} D + b_N)} \cdot x(t)$$

$$Q(D) \cdot y(t) = P(D) \cdot x(t)$$

$$Q(D)y_0(t) = 0 \quad \Rightarrow \quad (D^N + a_1 D^{N-1} + \dots + a_{N-1} D + a_N)y_0(t) = 0$$

$$Q(\lambda) = 0 \quad \Leftrightarrow \quad \underbrace{(\lambda^N + a_1 \lambda^{N-1} + \dots + a_{N-1} \lambda + a_N)}_{= 0} \underbrace{ce^{\lambda t}}_{\neq 0} = 0$$

$$Q(\lambda) = (\lambda^N + a_1 \lambda^{N-1} + \dots + a_{N-1} \lambda + a_N) = 0$$

$$Q(\lambda) = (\lambda - \lambda_1)(\lambda - \lambda_2) \dots (\lambda - \lambda_N) \quad \lambda_i \quad \text{characteristic roots}$$

$$c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t} \dots + c_N e^{\lambda_N t} = y_0(t) \quad e^{\lambda_i t} \quad \text{characteristic modes}$$

ZIR: a linear combination of the characteristic modes of the system

Zero State Response $y(t)$ – (1)

$$\frac{d^N y(t)}{dt^N} + a_1 \frac{d^{N-1} y(t)}{dt^{N-1}} + \dots + a_{N-1} \frac{dy(t)}{dt} + a_N y(t) = b_{N-M} \frac{d^M x(t)}{dt^M} + b_{N-M+1} \frac{d^{M-1} x(t)}{dt^{M-1}} + \dots + b_{N-1} \frac{dx(t)}{dt} + b_N x(t)$$

$$\boxed{(D^N + a_1 D^{N-1} + \dots + a_{N-1} D + a_N)} \cdot y(t) = \boxed{(b_{N-M} D^M + b_{N-M+1} D^{M-1} + \dots + b_{N-1} D + b_N)} \cdot x(t)$$

$$Q(D) \cdot y(t) = P(D) \cdot x(t)$$

All initial conditions are zero

$$y(t) = x(t) * h(t) = \int_{-\infty}^{+\infty} x(\tau) y(t - \tau) d\tau$$

Impulse response $h(t)$

$$y(t) = \int_0^{+t} x(\tau) y(t - \tau) d\tau, \quad t \geq 0$$

Causality

causal system: Response cannot begin before the input

causal input: The input starts at $t=0$ $h(\tau) = 0 \quad \tau < 0$

causal $h(t)$: The causal system's response to a unit impulse cannot begin before $t=0$

$$h(t - \tau) = 0 \quad t - \tau < 0$$

Total Response

$$\frac{d^N y(t)}{dt^N} + a_1 \frac{d^{N-1} y(t)}{dt^{N-1}} + \dots + a_{N-1} \frac{dy(t)}{dt} + a_N y(t) = b_{N-M} \frac{d^M x(t)}{dt^M} + b_{N-M+1} \frac{d^{M-1} x(t)}{dt^{M-1}} + \dots + b_{N-1} \frac{dx(t)}{dt} + b_N x(t)$$

$$\boxed{(D^N + a_1 D^{N-1} + \dots + a_{N-1} D + a_N)} \cdot y(t) = \boxed{(b_{N-M} D^M + b_{N-M+1} D^{M-1} + \dots + b_{N-1} D + b_N)} \cdot x(t)$$

$$Q(D) \cdot y(t) = P(D) \cdot x(t)$$

$$y(t) = \underbrace{\sum_{k=1}^N c_k e^{\lambda_k t}}_{\text{Zero Input Response}} + \underbrace{x(t) * h(t)}_{\text{Zero State Response}}$$

Zero Input Response

Zero State Response

$$y(t) = \underbrace{y_n(t)}_{\text{Natural Response}} + \underbrace{y_\Phi(t)}_{\text{Forced Response}}$$

Natural Response

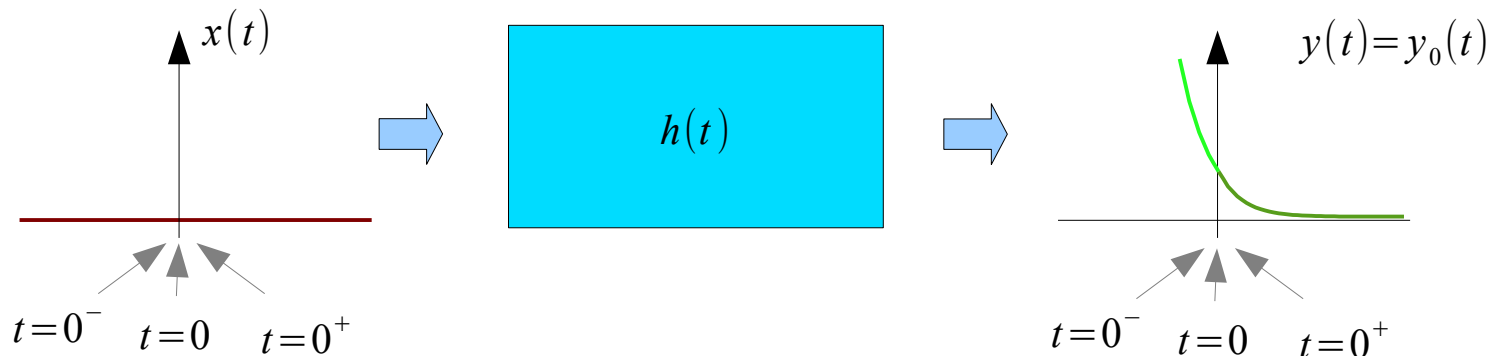
Forced Response

Zero Input Response

$$\frac{d^N y(t)}{dt^N} + a_1 \frac{d^{N-1} y(t)}{dt^{N-1}} + \dots + a_{N-1} \frac{dy(t)}{dt} + a_N y(t) = b_{N-M} \frac{d^M x(t)}{dt^M} + b_{N-M+1} \frac{d^{M-1} x(t)}{dt^{M-1}} + \dots + b_{N-1} \frac{dx(t)}{dt} + b_N x(t)$$

$$\boxed{(D^N + a_1 D^{N-1} + \dots + a_{N-1} D + a_N)} \cdot y(t) = \boxed{(b_{N-M} D^M + b_{N-M+1} D^{M-1} + \dots + b_{N-1} D + b_N)} \cdot x(t)$$

$$Q(D) \cdot y(t) = P(D) \cdot x(t)$$



Input is zero

Only initial conditions
drives the system

$$y_0(0^-) = y_0(0) = y_0(0^+)$$

$$\dot{y}_0(0^-) = \dot{y}_0(0) = \dot{y}_0(0^+)$$

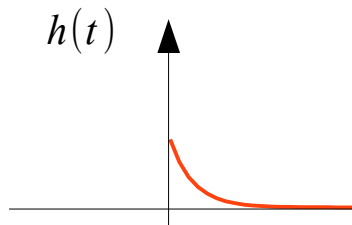
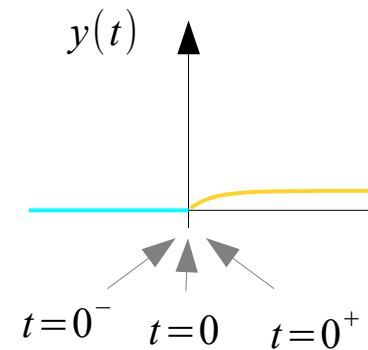
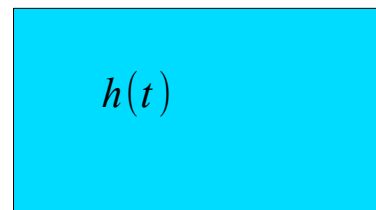
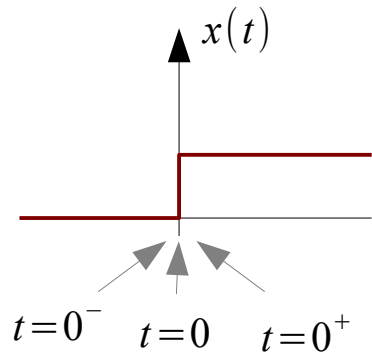
$$\ddot{y}_0(0^-) = \ddot{y}_0(0) = \ddot{y}_0(0^+)$$

Zero State Response

$$\frac{d^N y(t)}{dt^N} + a_1 \frac{d^{N-1} y(t)}{dt^{N-1}} + \dots + a_{N-1} \frac{dy(t)}{dt} + a_N y(t) = b_{N-M} \frac{d^M x(t)}{dt^M} + b_{N-M+1} \frac{d^{M-1} x(t)}{dt^{M-1}} + \dots + b_{N-1} \frac{dx(t)}{dt} + b_N x(t)$$

$$\boxed{(D^N + a_1 D^{N-1} + \dots + a_{N-1} D + a_N)} \cdot y(t) = \boxed{(b_{N-M} D^M + b_{N-M+1} D^{M-1} + \dots + b_{N-1} D + b_N)} \cdot x(t)$$

$$Q(D) \cdot y(t) = P(D) \cdot x(t)$$



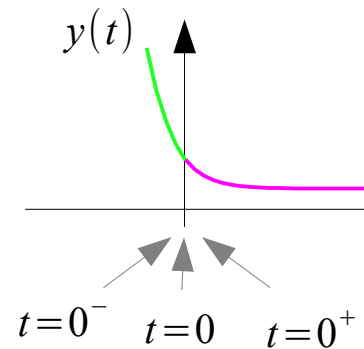
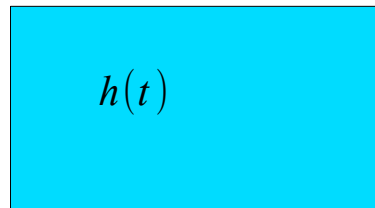
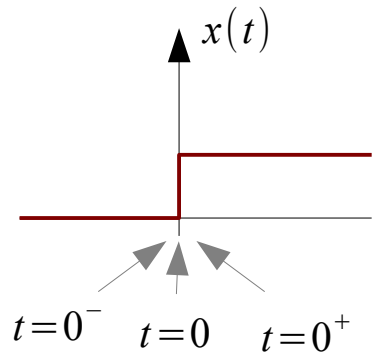
All initial conditions are zero

Total Response $y(t)$

$$\frac{d^N y(t)}{dt^N} + a_1 \frac{d^{N-1} y(t)}{dt^{N-1}} + \dots + a_{N-1} \frac{dy(t)}{dt} + a_N y(t) = b_{N-M} \frac{d^M x(t)}{dt^M} + b_{N-M+1} \frac{d^{M-1} x(t)}{dt^{M-1}} + \dots + b_{N-1} \frac{dx(t)}{dt} + b_N x(t)$$

$$\boxed{(D^N + a_1 D^{N-1} + \dots + a_{N-1} D + a_N)} \cdot y(t) = \boxed{(b_{N-M} D^M + b_{N-M+1} D^{M-1} + \dots + b_{N-1} D + b_N)} \cdot x(t)$$

$$Q(D) \cdot y(t) = P(D) \cdot x(t)$$



zero input response
+
zero state response

$$y(t) = y_0(t) \quad t \leq 0^-$$

because the input
has not started yet

$$y(0^-) = y_0(0^-)$$

$$\dot{y}(0^-) = \dot{y}_0(0^-)$$

The total response

$$y(0^-) = y(0^+)$$

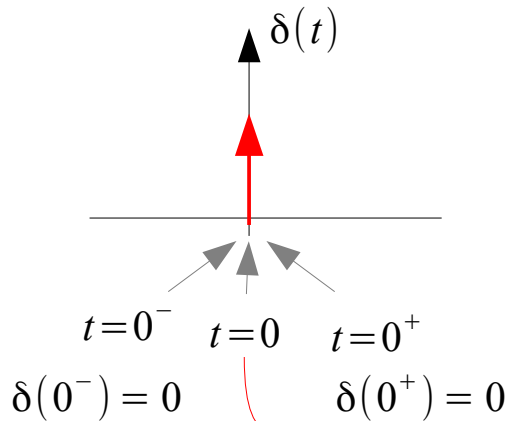
$$\dot{y}(0^-) = \dot{y}(0^+)$$

Impulse Response $h(t)$

$$\frac{d^N y(t)}{dt^N} + a_1 \frac{d^{N-1} y(t)}{dt^{N-1}} + \dots + a_{N-1} \frac{dy(t)}{dt} + a_N y(t) = b_{N-M} \frac{d^M x(t)}{dt^M} + b_{N-M+1} \frac{d^{M-1} x(t)}{dt^{M-1}} + \dots + b_{N-1} \frac{dx(t)}{dt} + b_N x(t)$$

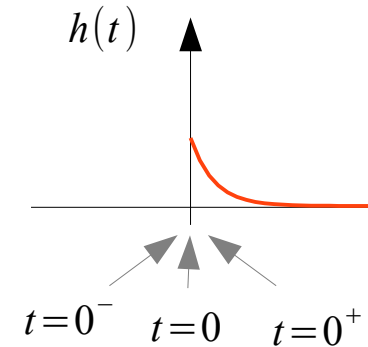
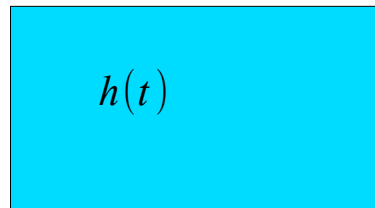
$$\boxed{(D^N + a_1 D^{N-1} + \dots + a_{N-1} D + a_N)} \cdot y(t) = \boxed{(b_{N-M} D^M + b_{N-M+1} D^{M-1} + \dots + b_{N-1} D + b_N)} \cdot x(t)$$

$$Q(D) \cdot y(t) = P(D) \cdot x(t)$$



All init conditions are zero at $t=0^-$

Generates energy storage
 Creates nonzero initial condition at $t=0^+$



$h(t) =$ characteristic mode terms
 $t \geq 0^+$ ($t \neq 0$)

At $t=0$, at most impulse $A_0 \delta(t)$

Impulse Response $h(t)$

$$\frac{d^N y(t)}{dt^N} + a_1 \frac{d^{N-1} y(t)}{dt^{N-1}} + \dots + a_{N-1} \frac{dy(t)}{dt} + a_N y(t) = b_0 \frac{d^M x(t)}{dt^M} + b_1 \frac{d^{M-1} x(t)}{dt^{M-1}} + \dots + b_{N-1} \frac{dx(t)}{dt} + b_N x(t)$$

$$(D^N + a_1 D^{N-1} + \dots + a_{N-1} D + a_N) y(t) = (b_0 D^M + b_1 D^{M-1} + \dots + b_{N-1} D + b_N) x(t)$$

$$M = N$$

$$Q(D)y(t) = P(D)x(t)$$

If $\delta(t)$ is included in $h(t)$

$$\frac{(D^N + a_1 D^{N-1} + \dots + a_{N-1} D + a_N) h(t)}{\quad \quad \quad \downarrow} = \frac{(b_0 D^M + b_1 D^{M-1} + \dots + b_{N-1} D + b_N) \delta(t)}{\quad \quad \quad \downarrow}$$

The highest order term

$$\delta^{(N+1)}(t)$$



$$\delta^{(N)}(t)$$

contradiction

$h(t)$ cannot contain $\delta^{(i)}(t)$ at all
 $h(t)$ can contain at most $\delta(t)$

Simplified Impulse Matching Method (1)

$$\frac{d^N y(t)}{dt^N} + a_1 \frac{d^{N-1} y(t)}{dt^{N-1}} + \dots + a_{N-1} \frac{dy(t)}{dt} + a_N y(t) = b_0 \frac{d^M x(t)}{dt^M} + b_1 \frac{d^{M-1} x(t)}{dt^{M-1}} + \dots + b_{N-1} \frac{dx(t)}{dt} + b_N x(t)$$

$$(D^N + a_1 D^{N-1} + \dots + a_{N-1} D + a_N) y(t) = (b_0 D^M + b_1 D^{M-1} + \dots + b_{N-1} D + b_N) x(t)$$

$$M = N$$

$$Q(D)y(t) = P(D)x(t)$$

$$h(t) = b_0 \delta(t) + [P(D)y_n(t)]u(t)$$

$y_n(t)$ Linear combination of characteristic modes with the following initial conditions

$$y_n(0) = \dot{y}_n(0) = \ddot{y}_n(0) \dots = y_n^{(N-2)}(0) = 0 \quad y_n^{(N-1)}(0) = 1$$

$$(D^N + a_1 D^{N-1} + \dots + a_{N-1} D + a_N) y_n(t) = \delta(t)$$

$$y_n^{(N)}(t) + a_1 y_n^{(N-1)}(t) + \dots + a_{N-1} y_n^{(1)}(t) + y_n(t) = \delta(t)$$

$$Q(D)y(t) = P(D)x(t)$$

$$P(D) \leftarrow 1$$

$$Q(D)w(t) = x(t)$$

$$Q(D)y_n(t) = \delta(t)$$

$$Q(D)w(t) = x(t)$$

$$Q(D)P(D)w(t) = P(D)x(t)$$

$$Q(D)y(t) = P(D)x(t)$$

Simplified Impulse Matching Method (2)

$$\frac{d^N y(t)}{dt^N} + a_1 \frac{d^{N-1} y(t)}{dt^{N-1}} + \dots + a_{N-1} \frac{dy(t)}{dt} + a_N y(t) = b_0 \frac{d^M x(t)}{dt^M} + b_1 \frac{d^{M-1} x(t)}{dt^{M-1}} + \dots + b_{N-1} \frac{dx(t)}{dt} + b_N x(t)$$

$$(D^N + a_1 D^{N-1} + \dots + a_{N-1} D + a_N) y(t) = (b_0 D^M + b_1 D^{M-1} + \dots + b_{N-1} D + b_N) x(t)$$

$$M = N$$

$$Q(D) y(t) = P(D) x(t)$$

$$(D^N + a_1 D^{N-1} + \dots + a_{N-1} D + a_N) y_n(t) = \delta(t)$$

$$y_n^{(N)}(t) + a_1 y_n^{(N-1)}(t) + \dots + a_{N-1} y_n^{(1)}(t) + y_n(t) = \delta(t)$$

$$Q(D) w(t) = x(t)$$

$$Q(D) P(D) w(t) = P(D) x(t)$$

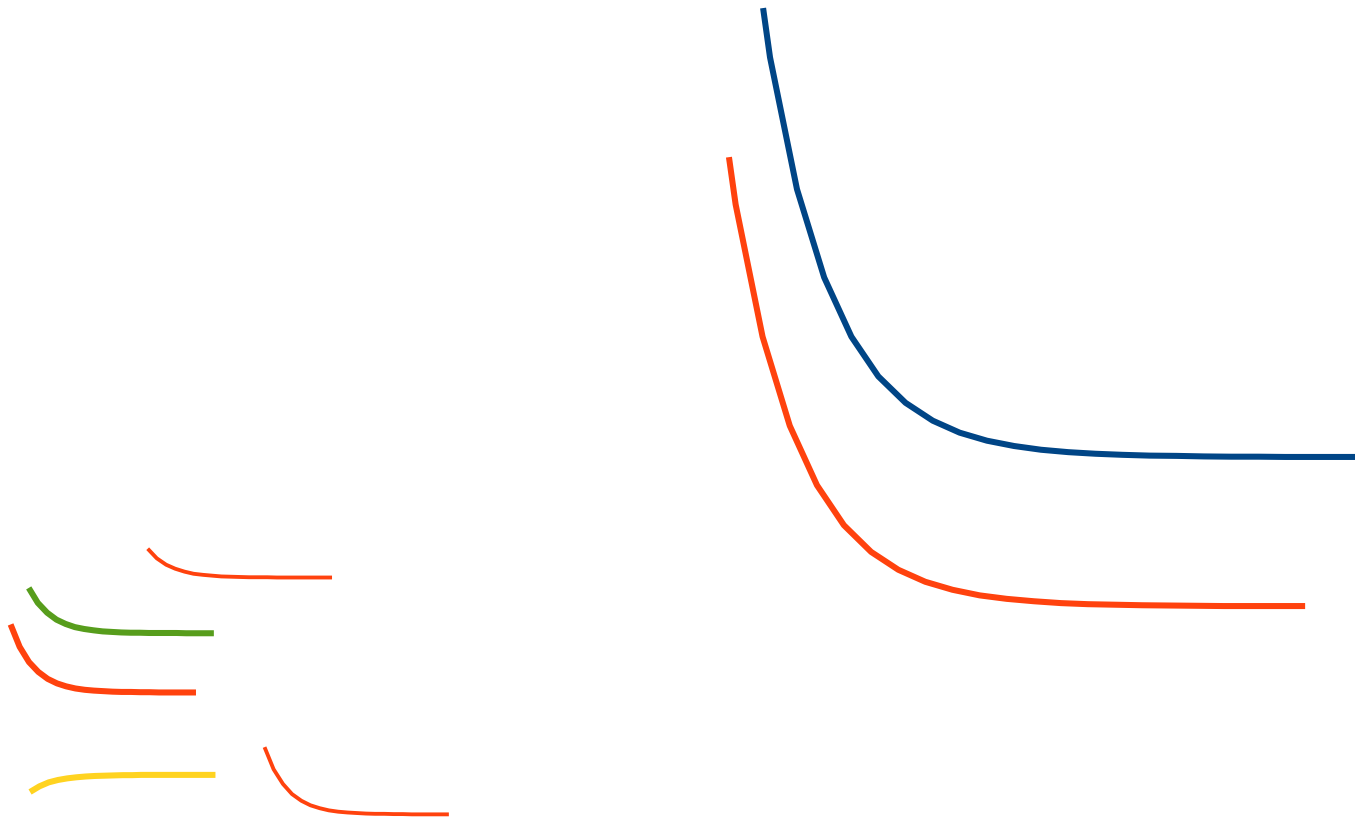
$$Q(D) y(t) = P(D) x(t)$$

$$h(t) = P(D)[y_n(t)u(t)]$$

$$h(t) = b_0 \delta(t) + P(D)y_n(t), \quad t \geq 0$$

$$h(t) = b_0 \delta(t) + [P(D)y_n(t)]u(t)$$

Impulse Response $h(t)$



References

- [1] <http://en.wikipedia.org/>
- [2] J.H. McClellan, et al., Signal Processing First, Pearson Prentice Hall, 2003
- [3] B.P. Lathi, Linear Systems and Signals (2nd Ed)