Spectra (2A)

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Spectral Density Functions

Non-periodic signals

Energy Signal

$$E_x^T = \int_{-T/2}^{+T/2} x^2(t)\, dt$$

Energy Spectral Density

$$\Psi(f) = |X(f)|^2$$

Total Energy

$$\int_{-\infty}^{+\infty} \Psi(f)\, df$$

Periodic signals

Power Signal

$$P_x^T = \frac{1}{T} \int_{-T/2}^{+T/2} x^2(t)\, dt$$

Power Spectral Density

$$G_x(f) = \sum_{n=-\infty}^{+\infty} |c_n|^2\, \delta(f - n f_0)$$

Average Power

$$\int_{-\infty}^{+\infty} G_x(f)\, df$$

Random signals

Power Signal

$$P_x^T = \frac{1}{T} \int_{-T/2}^{+T/2} x^2(t)\, dt$$

Power Spectral Density

$$G_x(f) = \lim_{T \to \infty} \frac{1}{T} |X_T(f)|^2$$

Average Power

$$\int_{-\infty}^{+\infty} G_x(f)\, df$$

Autocorrelation Functions

Energy Signal Autocorrelation

$$R_x(\tau) = \int_{-\infty}^{+\infty} x(t)x(t+\tau)\,dt$$

Power Signal Autocorrelation

$$R_x(\tau) = \lim_{T\to+\infty} \frac{1}{T}\int_{-T/2}^{+T/2} x(t)x(t+\tau)\,dt$$

Random Signal Autocorrelation

$$R_x(\tau) = \boldsymbol{E}\{X(t)\,X(t+\tau)\}$$

Non-periodic signals

$$R_x(\tau) = \int_{-\infty}^{+\infty} x(t)x(t+\tau)\,dt$$

Periodic signals

for a Periodic Signal

$$R_x(\tau) = \frac{1}{T_0}\int_{-T_0/2}^{+T_0/2} x(t)x(t+\tau)\,dt$$

Random signals

for a Ergodic Signal

$$R_x(\tau) = \lim_{T\to+\infty} \frac{1}{T}\int_{-T/2}^{+T/2} X(t)\,X(t+\tau)\,dt$$

Single-Sided Spectrum

$$x(t) = a_0 + \sum_{k=1}^{\infty} \left(a_k \cos(k\omega_0 t) + b_k \sin(k\omega_0 t)\right)$$

$$x(t) = g_0 + \sum_{k=1}^{\infty} g_k \cos(k\omega_0 t + \phi_k)$$

$$a_0 = \frac{1}{T}\int_0^T x(t)\,dt$$

$$a_k = \frac{2}{T}\int_0^T x(t)\cos(k\omega_0 t)\,dt$$

$$b_k = \frac{2}{T}\int_0^T x(t)\sin(k\omega_0 t)\,dt$$

$\boldsymbol{k} = +1, +2, \ldots$

$$g_0 = a_0$$

$$g_k = \sqrt{a_k^2 + b_k^2}$$

$$\phi_k = \tan^{-1}\left(-\frac{b_k}{a_k}\right)$$

$\boldsymbol{k} = +1, +2, \ldots$

$$\cos(\alpha+\beta) = \underline{\cos(\alpha)}\cos(\beta) - \underline{\sin(\alpha)}\sin(\beta)$$

$$g_k \cos(k\omega_0 t + \phi_k) = \underline{g_k \cos(\phi_k)}\cos(k\omega_0 t) - \underline{g_k \sin(\phi_k)}\sin(k\omega_0 t)$$

$$\underline{\qquad a_k}\cos(k\omega_0 t) + \underline{\qquad b_k}\sin(k\omega_0 t)$$

$$a_k = g_k \cos(\phi_k)$$

$$-b_k = g_k \sin(\phi_k)$$

$$a_k^2 + b_k^2 = g_k^2$$

$$-\frac{b_k}{a_k} = \tan(\phi_k)$$

Two-Sided Spectrum

$$x(t) = \sum_{k=-\infty}^{+\infty} C_k \, e^{+jk\omega_0 t}$$

$$C_k = \frac{1}{T}\int_0^T x(t)\, e^{-jk\omega_0 t}\, dt$$

$$k = \ldots, -2, -1, 0, +1, +2, \ldots$$

$$C_k = \begin{cases} a_0 & (k=0) \\ \frac{1}{2}(a_k - jb_k) & (k>0) \\ \frac{1}{2}(a_k + jb_k) & (k<0) \end{cases}$$

$$|C_k| = \begin{cases} a_0 & (k = 0) \\ \frac{1}{2}\sqrt{a_k^2 + b_k^2} & (k \neq 0) \end{cases}$$

$$Arg(C_k) = \begin{cases} \tan^{-1}(-b_k/a_k) & (k > 0) \\ \tan^{-1}(+b_k/a_k) & (k < 0) \end{cases}$$

Power Spectrum *Two-Sided*

$$|C_k|^2 + |C_{-k}|^2 = \frac{1}{2} g_k^2 = \frac{1}{2}(a_k^2 + b_k^2)$$

$$x(t) = \sum_{k=-\infty}^{+\infty} C_k \, e^{+jk\omega_0 t}$$

$$C_k = \frac{1}{T}\int_0^T x(t)\, e^{-jk\omega_0 t}\, dt$$

$$k = \ldots, -2, -1, 0, +1, +2, \ldots$$

$$C_k = \begin{cases} a_0 & (k=0) \\ \frac{1}{2} g_k \, e^{+j\phi_k} & (k>0) \\ \frac{1}{2} g_k \, e^{-j\phi_k} & (k<0) \end{cases}$$

$$|C_k| = \begin{cases} a_0 & (k = 0) \\ \frac{1}{2}|g_k| & (k \neq 0) \end{cases}$$

$$Arg(C_k) = \begin{cases} +\phi_k & (k > 0) \\ -\phi_k & (k < 0) \end{cases}$$

Periodogram *One-Sided*

$$2 \cdot |C_k| = g_k = \sqrt{a_k^2 + b_k^2}$$

Power Spectrum (1)

$$x(t) = a_0 + \sum_{n=1}^{\infty} \left(a_k \cos(k\omega_0 t) + b_k \sin(k\omega_0 t) \right)$$

$$a_0 = \frac{1}{T} \int_0^T x(t)\, dt$$

$$a_k = \frac{2}{T} \int_0^T x(t) \cos(k\omega_0 t)\, dt$$

$$b_k = \frac{2}{T} \int_0^T x(t) \sin(k\omega_0 t)\, dt$$

$$k = 1,\ 2,\ \ldots$$

Average Power $P = \frac{1}{T} \int_0^T |x(t)|^2\, dt$

$$P = a_0^2 + \frac{1}{2} \sum_{k=1}^{+\infty} \left(a_k^2 + b_k^2 \right)$$

Power Spectrum

$$P_k = \begin{cases} a_0^2 & (k = 0) \\ \frac{1}{2}\left(a_k^2 + b_k^2 \right) & (k > 0) \end{cases}$$

$$x(t) = \sum_{\mathbf{k}=-\infty}^{+\infty} C_{\mathbf{k}}\, e^{+j\mathbf{k}\omega_0 t}$$

$$C_{\mathbf{k}} = \frac{1}{T} \int_0^T x(t)\, e^{-j\mathbf{k}\omega_0 t}\, dt$$

$$\mathbf{k} = -2,\ -1,\ 0,\ +1,\ +2,\ \ldots$$

$$C_{\mathbf{k}} = \begin{cases} a_0 & (\mathbf{k} = 0) \\ \frac{1}{2}(a_{\mathbf{k}} - jb_{\mathbf{k}}) & (\mathbf{k} > 0) \\ \frac{1}{2}(a_{\mathbf{k}} + jb_{\mathbf{k}}) & (\mathbf{k} < 0) \end{cases}$$

Average Power $P = \frac{1}{T} \int_0^T |x(t)|^2\, dt$

$$P = |C_0|^2 + \sum_{k=1}^{+\infty} \left(|C_k|^2 + |C_{-k}|^2 \right)$$

Power Spectrum

$$P_k = \begin{cases} |C_0|^2 & (k = 0) \\ \left(|C_k|^2 + |C_{-k}|^2 \right) & (k > 0) \end{cases}$$

Power Spectrum (2)

$$x(t) = a_0 + \sum_{n=1}^{\infty} \left(a_n \cos(n\omega_0 t) + b_n \sin(n\omega_0 t)\right)$$

$$x(t) = \sum_{\boldsymbol{k}=-\infty}^{+\infty} C_{\boldsymbol{k}}\, e^{+j\boldsymbol{k}\omega_0 t}$$

Average Power $P = \frac{1}{T}\int_0^T |x(t)|^2\, dt$

$$P = a_0^2 + \frac{1}{2}\sum_{k=1}^{+\infty}\left(a_k^2 + b_k^2\right)$$

Average Power $P = \frac{1}{T}\int_0^T |x(t)|^2\, dt$

$$P = |C_0|^2 + \sum_{k=1}^{+\infty}\left(|C_k|^2 + |C_{-k}|^2\right)$$

$$|C_k|^2 + |C_{-k}|^2 = \frac{1}{2}\left(a_k^2 + b_k^2\right)$$

$$|C_k|^2 = \frac{1}{4}\left(a_k^2 + b_k^2\right) \qquad |C_{-k}|^2 = \frac{1}{4}\left(a_k^2 + b_k^2\right)$$

$$C_{\boldsymbol{k}} = \begin{cases} a_0 & (\boldsymbol{k} = 0) \\ \frac{1}{2}(a_{\boldsymbol{k}} - jb_{\boldsymbol{k}}) & (\boldsymbol{k} > 0) \\ \frac{1}{2}(a_{\boldsymbol{k}} + jb_{\boldsymbol{k}}) & (\boldsymbol{k} < 0) \end{cases}$$

Power Spectrum

$$P_k = \begin{cases} a_0^2 & (k = 0) \\ \frac{1}{2}\left(a_k^2 + b_k^2\right) & (k > 0) \end{cases}$$

Power Spectrum

$$P_k = \begin{cases} |C_0|^2 & (k = 0) \\ \left(|C_k|^2 + |C_{-k}|^2\right) & (k > 0) \end{cases}$$

Periodogram

$$x(t) = a_0 + \sum_{n=1}^{\infty} \left(a_n \cos(n\omega_0 t) + b_n \sin(n\omega_0 t) \right)$$

$$a_0 = \frac{1}{T} \int_0^T x(t)\, dt$$

$$a_n = \frac{2}{T} \int_0^T x(t) \cos(n\omega_0 t)\, dt$$

$$b_n = \frac{2}{T} \int_0^T x(t) \sin(n\omega_0 t)\, dt$$

$$n = 1,\ 2,\ \ldots$$

Positive Frequency Only

$$f = \frac{n}{T} \qquad n = 1,\ 2,\ \ldots$$

Single-sided Spectrum

$$\sqrt{a_n^2 + b_n^2} \qquad \textit{sometimes} \quad (a_n^2 + b_n^2)$$

An estimate of the spectral density

$$\frac{T}{2} a_n = \int_{t_1}^{t_1+T} x(t) \cos(kt)\, dt$$

$$\frac{T}{2} b_n = \int_{t_1}^{t_1+T} x(t) \sin(kt)\, dt$$

T: integer multiple $\frac{2\pi}{k}$

$$T = \frac{2\pi}{k} \cdot n$$

$$k = \frac{2\pi}{T} \cdot n = n\omega_0$$

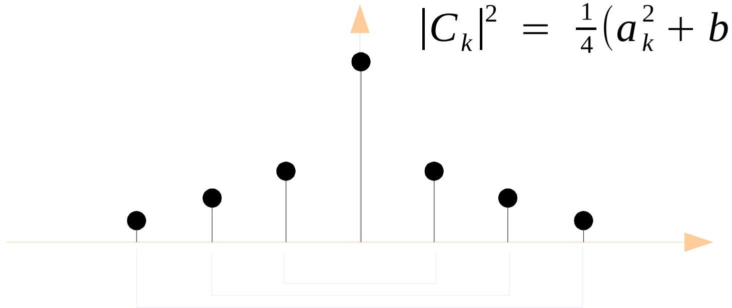
abscissa $\frac{2\pi}{k}$

ordinates $r = \sqrt{a_n^2 + b_n^2}$

Periodogram and Power Spectrum

Power Spectrum Single-Sided

$$|C_k|^2 = \frac{1}{4}(a_k^2 + b_k^2)$$

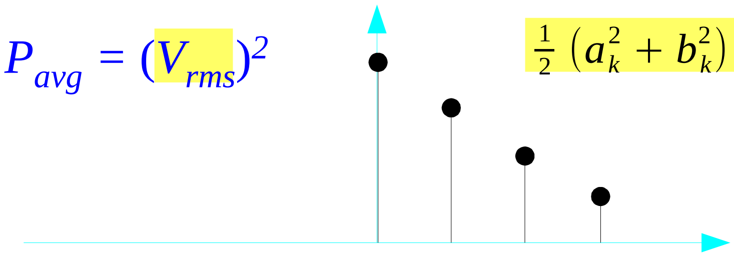


$$(|C_k|^2 + |C_{-k}|^2)$$

Power Spectrum Two-Sided

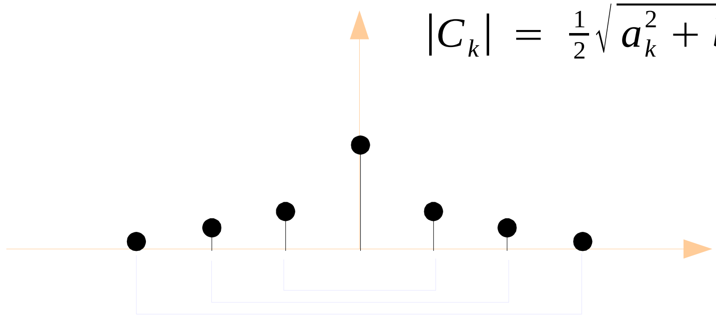
$$P_{avg} = (V_{rms})^2$$

$$\frac{1}{2}\left(a_k^2 + b_k^2\right)$$



$$|C_k|^2 + |C_{-k}|^2 = \frac{1}{2}|g_k|^2 = \frac{1}{2}\left(a_k^2 + b_k^2\right)$$

$$|C_k| = \frac{1}{2}\sqrt{a_k^2 + b_k^2}$$

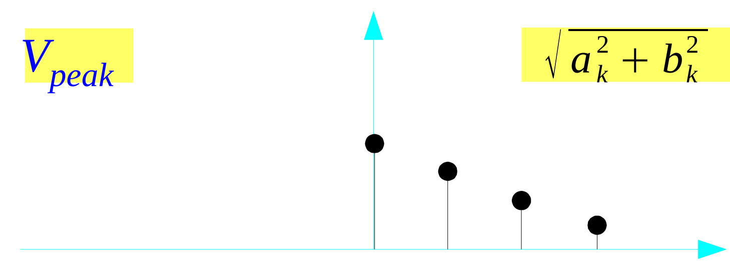


$$(|C_k| + |C_{-k}|)$$

Periodogram

$$V_{peak}$$

$$\sqrt{a_k^2 + b_k^2}$$

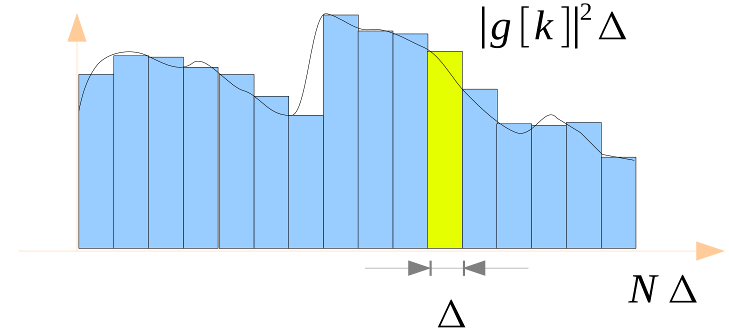


$$|C_k| + |C_{-k}| = |g_k|^2 = \sqrt{a_k^2 + b_k^2}$$

Root Mean Square

Continuous Time

Discrete Time



mean squared amplitude

$$\frac{1}{T}\int_0^T g^2(t)\, dt$$

mean squared amplitude

$$\frac{1}{N\Delta}\sum_{k=0}^{N-1} |g[k]|^2 \Delta = \frac{1}{N}\sum_{k=0}^{N-1} |g[k]|^2$$

root mean squared

$$g_{rms} = \sqrt{\frac{1}{T}\int_0^T g^2(t)\, dt}$$

root mean squared

$$g_{rms} = \frac{1}{N}\sum_{k=0}^{N-1} |g[k]|^2$$

CTFS and CTFT

Continuous Time Fourier <u>Series</u>

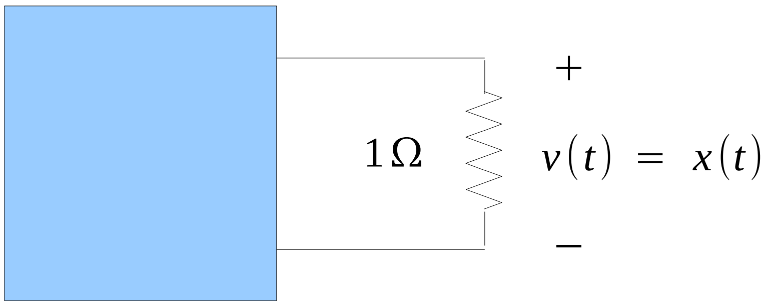
$$C_k = \frac{1}{T}\int_0^T x(t)\, e^{-jk\omega_0 t}\, dt \quad \Leftrightarrow \quad x(t) = \sum_{k=0}^{\infty} C_k\, e^{+jk\omega_0 t}$$

Continuous Time Fourier <u>Transform</u>

$$X(j\omega) = \int_{-\infty}^{+\infty} x(t)\, e^{-j\omega t}\, dt \quad \Leftrightarrow \quad x(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} X(j\omega)\, e^{+j\omega t}\, d\omega$$

$$X(f) = \int_{-\infty}^{+\infty} x(t)\, e^{-j2\pi f t}\, dt \quad \Leftrightarrow \quad x(t) = \int_{-\infty}^{+\infty} X(f)\, e^{+j2\pi f t}\, df$$

Continuous Periodic Signal



instantaneous power $x^2(t)$

average power $\frac{1}{T}\int_0^T x^2(t)\,dt$

T : $period$

$v(t) = x(t)$ Continuous Periodic

CTFS (Fourier Series)

$$x(t) = \sum_{k=-\infty}^{+\infty} C_k\, e^{+jk\omega_0 t}$$

$$C_k = \frac{1}{T}\int_0^T x(t)\, e^{-jk\omega_0 t}\, dt$$

$$k = \ldots,\ -2,\ -1,\ 0,\ +1,\ +2,\ \ldots$$

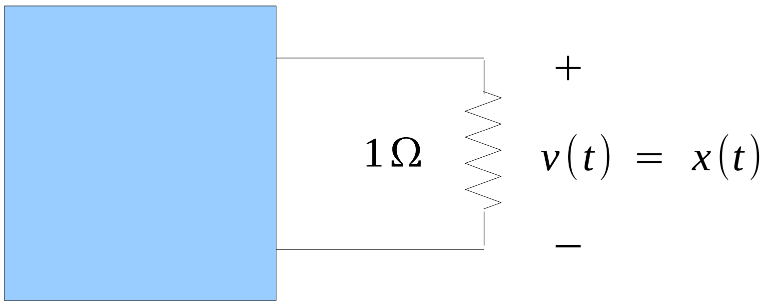
Parseval’s Theorem

$$\frac{1}{T}\int_0^T x^2(t)\,dt = \sum_{n=-\infty}^{+\infty} |C_n|^2$$

average power

sum of power spectrum C_n

Continuous Aperiodic Signal



instantaneous power $x^2(t)$

total energy $\int_{-\infty}^{+\infty} x^2(t)\, dt$

$v(t) = x(t)$ Continuous Aperiodic

CTFT (Fourier Integral)

$$x(t) = \int_{-\infty}^{+\infty} X(f)\, e^{+j2\pi f t}\, df$$

$$X(f) = \int_{-\infty}^{+\infty} x(t)\, e^{-j2\pi f t}\, dt$$

Parseval's Theorem

$$\int_{-\infty}^{+\infty} x^2(t)\, dt = \int_{-\infty}^{\infty} |X(f)|^2\, df$$

total energy

integral of energy spectral density $|X(f)|^2$

Parseval's Theorem – CTFS

Average Power

$$E[x^2(t)] = \frac{1}{T}\int_{-\infty}^{+\infty} x^2(t)\,dt$$

CTFS

$$x(t) = \sum_{k=-\infty}^{+\infty} C_k\, e^{+jk\omega t}$$

$$x^2(t) = x(t)\cdot x^*(t) = \left(\sum_{k=-\infty}^{+\infty} C_k\, e^{+jk\omega t}\right)\left(\sum_{l=-\infty}^{+\infty} C_l^*\, e^{-jl\omega t}\right)$$

$$x^2(t) = \sum_{k=-\infty}^{+\infty}\sum_{l=-\infty}^{+\infty} C_k C_l^*\, e^{+j(k-l)\omega t}$$

$$\frac{1}{T}\int_{-\infty}^{+\infty} x^2(t)\,dt = \sum_{k=-\infty}^{+\infty}\sum_{l=-\infty}^{+\infty} C_k C_l^* \left(\frac{1}{T}\int_{-\infty}^{+\infty} e^{+j(k-l)\omega t}\,dt\right)$$

Average Power

$$\frac{1}{T}\int_{-\infty}^{+\infty} x^2(t)\,dt = \sum_{k=-\infty}^{+\infty} |C_k|^2$$

Power Spectrum

$|C_k|^2$

power associated with individual frequency components

Parseval's Theorem – CTFT

Total Energy

$$\int_{-\infty}^{+\infty} x^2(t)\, dt$$

CTFT

$$x(t) = \int_{-\infty}^{+\infty} X(f)\, e^{+j2\pi ft} df$$

$$x^2(t) = x(t)\cdot x^*(t) = \left(\int_{-\infty}^{+\infty} X(f) e^{+j2\pi ft} df\right)\left(\int_{-\infty}^{+\infty} X^*(\nu) e^{-j2\pi\nu t} d\nu\right)$$

$$x^2(t) = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} |X(f)|^2\, e^{+j2\pi(f-\nu)t} df\, d\nu$$

$$\int_{-\infty}^{+\infty} x^2(t)\, dt = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} |X(f)|^2 \int_{-\infty}^{+\infty} e^{+j2\pi(f-\nu)t} dt\, df\, d\nu$$

Total Energy

$$\int_{-\infty}^{+\infty} x^2(t)\, dt = \int_{-\infty}^{+\infty} |X(f)|^2 df \quad = \frac{1}{2\pi}\int_{-\infty}^{+\infty} |X(\omega)|^2 d\omega$$

Energy Spectral Density

$$|X(f)|^2$$

energy associated with
individual frequency components

Parseval's Theorem – DTFT

Total Energy

$$\int_{-\infty}^{+\infty} x^2(t)\, dt$$

Total energy over all discrete time n

Total energy in fundamental period of frequency

DTFT

$$x[n] = \int_{-0.5}^{+0.5} X(e^{j\hat{f}})\, e^{+j2\pi\hat{f}n} d\hat{f}$$

$$x^2[n] = x[n]\cdot x^*[n] = \left(\int_{-0.5}^{+0.5} X(e^{j\hat{f}})\, e^{+j2\pi\hat{f}n} d\hat{f}\right)\left(\int_{-0.5}^{+0.5} X^*(e^{j\hat{\nu}})\, e^{+j2\pi\hat{\nu}n} d\hat{\nu}\right)$$

$$x^2[n] = \int_{-0.5}^{+0.5}\int_{-0.5}^{+0.5} |X(e^{j\hat{f}})|^2\, e^{+j2\pi(\hat{f}-\hat{\nu})n} df\, d\nu$$

$$\sum_{n=-\infty}^{+\infty} x^2[n]\, dt = \int_{-0.5}^{+0.5}\int_{-0.5}^{+0.5} |X(e^{j\hat{f}})|^2 \sum_{n=-\infty}^{+\infty} e^{+j2\pi(\hat{f}-\hat{\nu})n} df\, d\nu$$

Total Energy

$$\sum_{n=-\infty}^{+\infty} |x[n]|^2 = \int_{-0.5}^{+0.5} |X(\hat{f})|^2 d\hat{f} = \frac{1}{2\pi}\int_{-\pi}^{+\pi} |X(\hat{\omega})|^2 d\hat{\omega}$$

Energy Spectral Density

$$|X(\hat{f})|^2$$

energy associated with individual frequency components

Parseval's Theorem – DFT

Average Power

$$\frac{1}{N}\sum_{n=0}^{N-1} x^2[n]$$

DFT

$$x[n] = \frac{1}{N}\sum_{k=0}^{N-1} X[k]e^{+j\left(\frac{2\pi}{N}\right)kn}$$

$$x^2[n] = x[n]\cdot x^*[n] = \frac{1}{N^2}\left(\sum_{k=0}^{N-1} X[k]e^{+j\left(\frac{2\pi}{N}\right)kn}\right)\left(\sum_{l=0}^{N-1} X^*[l]e^{-j\left(\frac{2\pi}{N}\right)ln}\right)$$

$$x^2[n] = \frac{1}{N^2}\sum_{k=0}^{N-1}\sum_{l=0}^{N-1} X[k]X^*[l]e^{+j\left(\frac{2\pi}{N}\right)(k-l)n}$$

$$\frac{1}{N}\sum_{n=0}^{N-1} x^2[n]\,dt = \frac{1}{N^3}\sum_{k=0}^{N-1}\sum_{l=0}^{N-1} X[k]X^*[l]\sum_{n=0}^{N-1} e^{+j\left(\frac{2\pi}{N}\right)(k-l)n}$$

Average Power

$$\frac{1}{N}\sum_{n=0}^{N-1} |x[n]|^2 = \frac{1}{N^2}\sum_{k=0}^{N-1} |X[k]|^2$$

$$\sum_{n=0}^{N-1} |x[n]|^2 = \frac{1}{N}\sum_{k=0}^{N-1} |X[k]|^2$$

Periodogram

$$\frac{1}{N}|X[k]|^2$$

Average Power of Random Signals

A truncated sample function

$$x_T(t) = x(t) \qquad -\tfrac{T}{2} < t < +\tfrac{T}{2}$$

$$= 0 \qquad otherwise$$

Fourier Transform

$$X_T(\omega) = \int_{-\infty}^{+\infty} x_T(t)\, e^{-j\omega t}\, dt \quad \Longleftrightarrow \quad x_T(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} X_T(\omega)\, e^{+j\omega t}\, d\omega$$

$$X_T(f) = \int_{-\infty}^{+\infty} x_T(t)\, e^{-j2\pi f t}\, dt \quad \Longleftrightarrow \quad x_T(t) = \int_{-\infty}^{+\infty} X_T(f)\, e^{+j2\pi f t}\, df$$

Parseval's Therorem

$$\int_{-\infty}^{+\infty} x_T^2(t)\, dt = \int_{-\infty}^{+\infty} |X_T(f)|^2\, df$$

total energy

Average Power

$$\frac{1}{T}\int_{-\infty}^{+\infty} x_T^2(t)\, dt = \int_{-\infty}^{+\infty} \frac{|X_T(f)|^2}{T}\, dt$$

total energy / T

Power Spectral Density of Random Signals

Average Power

$$\frac{1}{T}\int_{-\infty}^{+\infty} x_T^2(t)\,dt = \int_{-\infty}^{+\infty} \frac{|X_T(f)|^2}{T}\,df \Rightarrow \int_{-\infty}^{+\infty} \hat{S}_{xx}(f)\,df$$

Raw Power Spectral Density

$$\frac{|X_T(f)|^2}{T} = \hat{S}_{xx}(f)$$

Power Spectral Density

$$\lim_{T\to\infty} \frac{\boldsymbol{E}\left[\,|X_T(f)|^2\right]}{T} = S_{xx}(f)$$

$$\lim_{T\to\infty} \frac{1}{T}\int_{-\infty}^{+\infty} x_T^2(t)\,dt = \lim_{T\to\infty} \int_{-\infty}^{+\infty} \frac{|X_T(f)|^2}{T}\,df$$

→ *not converge*

$$E\left[\lim_{T\to\infty} \frac{1}{T}\int_{-\infty}^{+\infty} x_T^2(t)\,dt\right] = E\left[\lim_{T\to\infty} \int_{-\infty}^{+\infty} \frac{|X_T(f)|^2}{T}\,df\right]$$

← *random signal*

$$Var(x(t)) = \sigma_x^2 = \int_{-\infty}^{+\infty} \lim_{T\to\infty} \frac{\boldsymbol{E}\left[\,|X_T(f)|^2\right]}{T}\,df \Rightarrow \int_{-\infty}^{+\infty} S_{xx}(f)\,df$$

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11/6/12

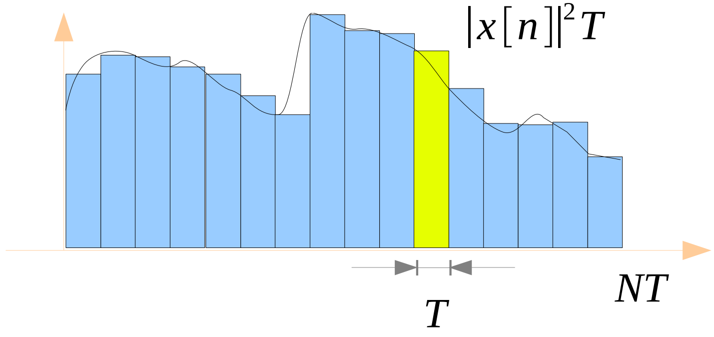
Power and Power Density Spectra (1)

Average Power

$$E[x^2(t)] = \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} x^2(t)\, dt$$

Average Power of N sample of x(t)

$$\frac{1}{N\,T} \int_{0}^{NT} x^2(t)\, dt \approx \frac{1}{N} \sum_{n=0}^{N-1} x^2[n] = \frac{1}{N^2} \sum_{k=0}^{N-1} |X_k|^2$$



$$\sum_{n=0}^{N-1} x^2[n] = \frac{1}{N} \sum_{k=0}^{N-1} |X[k]|^2$$

$$P_{xx}(k) = \frac{1}{N} |X[k]|^2$$

Periodogram:

$$k = 0, 1, \cdots, N-1$$

Average Power

$$\sum_{n=0}^{N-1} x^2[n] = \frac{1}{N} \sum_{k=0}^{N-1} P_{xx}(k)$$

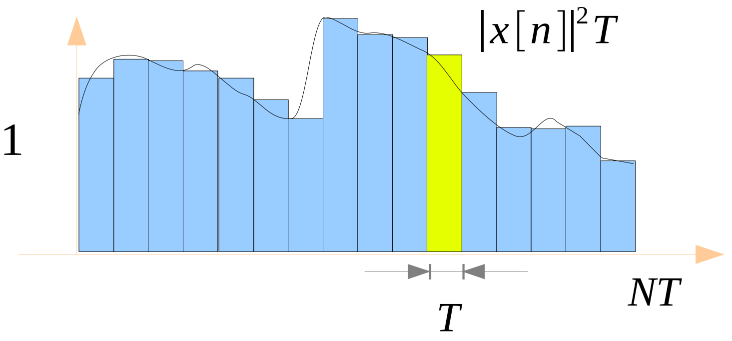
Total Energy

$$T \sum_{n=0}^{N-1} x^2[n] = T \sum_{k=0}^{N-1} P_{xx}(k) \approx \int_{0}^{NT} x^2(t)\, dt$$

Power and Power Density Spectra (2)

Periodogram:

$$P_{xx}(k) = \frac{1}{N}|X[k]|^2 \qquad k = 0, 1, \cdots, N-1$$



Average Power

$$\sum_{n=0}^{N-1} x^2[n] = \frac{1}{N}\sum_{k=0}^{N-1} P_{xx}(k)$$

$$if \quad P_{xx}(k) \quad is \quad constant \quad P$$

$$\sum_{n=0}^{N-1} x^2[n] = \int_{-0.5}^{+0.5} P_{xx}(\hat{f})\, df \qquad P_{xx}(\hat{f}) = P$$

$$\sum_{n=0}^{N-1} x^2[n] = \int_{-1/2T}^{+1/2T} P_{xx}(f)\, df \qquad P_{xx}(f) = TP$$

$$\sum_{n=0}^{N-1} x^2[n] = \int_{-\pi/T}^{+\pi/T} P_{xx}(\omega)\, d\omega \qquad P_{xx}(\omega) = TP/2\pi$$

Correlation and Power Spectrum (1)

Periodogram:

$$P_{xx}(k) = \frac{1}{N}|X[k]|^2$$

$$k = 0, 1, \cdots, N-1$$

Autocorrelation Function

$$R_{xx}[m] = \frac{1}{N}\sum_{l=0}^{N-1} x[l]x[l+m]$$

$$m = 0, 1, \cdots, N-1$$

Autocorrelation —DFT→ Periodogram

Average Power

$$\frac{1}{N}\sum_{n=0}^{N-1} x^2[n] = R_{xx}[0]$$

$$\sum_{m=0}^{N-1}\left(\frac{1}{N}\sum_{l=0}^{N-1} x[l]x[l+m]\right) e^{-j\left(\frac{2\pi}{N}\right)mk}$$

$$= \sum_{m=0}^{N-1}\left(\frac{1}{N}\sum_{l=0}^{N-1}\left(\frac{1}{N}\sum_{\alpha=0}^{N-1} X_\alpha e^{+j\left(\frac{2\pi}{N}\right)l\alpha}\right)\left(\frac{1}{N}\sum_{\beta=0}^{N-1} X_\beta^* e^{-j\left(\frac{2\pi}{N}\right)(l+m)\beta}\right)\right) e^{-j\left(\frac{2\pi}{N}\right)mk}$$

$$= \frac{1}{N^3}\sum_{\alpha=0}^{N-1}\sum_{\beta=0}^{N-1} X_\alpha X_\beta^* \sum_{m=0}^{N-1}\sum_{l=0}^{N-1}\left(e^{+j\left(\frac{2\pi}{N}\right)l\alpha}\right)\left(e^{-j\left(\frac{2\pi}{N}\right)(l+m)\beta}\right) e^{-j\left(\frac{2\pi}{N}\right)mk}$$

$$= \frac{1}{N^3}\sum_{\alpha=0}^{N-1}\sum_{\beta=0}^{N-1} X_\alpha X_\beta^* \sum_{m=0}^{N-1}\sum_{l=0}^{N-1}\left(e^{+j\left(\frac{2\pi}{N}\right)(l(\alpha-\beta) - m(\beta+k))}\right) \quad = \frac{1}{N}\sum_{\alpha=0}^{N-1}\sum_{\beta=0}^{N-1} X_\alpha X_\beta^*$$

	Periodic Signals	Aperiodic Signals	Random Signals	
Frequency Spacing	$\Delta f = \frac{1}{N \Delta t}$	$\Delta f = \frac{1}{N \Delta t}$	$\sum S \Delta f = \frac{1}{N \Delta t} \sum S$	$\frac{1}{N \Delta t} \sum x^2 \Delta t$
Two Sided	$\frac{1}{N} X(k)$	$\frac{\Delta t}{N} X(k)$	$S(k) = \frac{\Delta t}{N} \lvert X(k) \rvert^2$	$P = \sum_{k=0}^{N-1} S(k) \Delta f$
One Sided				
$k = 0, \frac{N}{2}$	$\frac{1}{N} X(k)$	$\frac{\Delta t}{N} X(k)$	$S_1(k) = 2S(k)$	$P = \sum_{k=0}^{N/2} S_1(k) \Delta f$
$k = 1, \cdots, \frac{N}{2} - 1$	$\frac{2}{N} X(k)$	$\frac{2 \Delta t}{N} X(k)$	$S_1(k) = S(k)$	
Frequency Scale	$k \Delta f$	$k \Delta f$	$k \Delta f$	

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