Complex Functions (1A)

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Cauchy-Riemann Condition

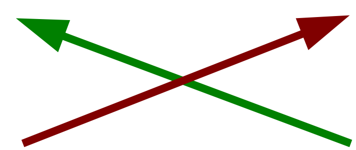
$f(z) = u(x,y) + i\,v(x,y)$: **analytic** in a region

$\Rightarrow$ in that region $\qquad \dfrac{\partial u}{\partial x} = \dfrac{\partial v}{\partial y} \qquad \dfrac{\partial v}{\partial x} = -\dfrac{\partial u}{\partial y}$

$f(z) = u(x,y) + i\,v(x,y)$

$\dfrac{\partial}{\partial x} \qquad \dfrac{\partial}{\partial y}$

$f(z) = u(x,y) + i\,v(x,y)$



$\dfrac{\partial}{\partial x} \qquad \dfrac{\partial}{\partial y}$

Analytic

$$f(z) = u(x,y) + i v(x,y)$$

$u(x,y),\ v(x,y),\ \dfrac{\partial u}{\partial x},\ \dfrac{\partial u}{\partial y},\ \dfrac{\partial v}{\partial x},\ \dfrac{\partial v}{\partial y}$: continuous

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \qquad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$$

$$f(z) = u(x,y) + i v(x,y)$$

: **analytic** at all points inside a region

not necessarily on the boundary

Derivatives

$f(z) = u(x,y) + iv(x,y)$: **analytic** in a region R

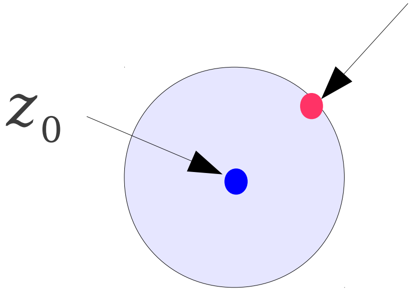
➡ derivatives of all orders at points inside region

$$f'(z_0),\ f''(z_0),\ f^{(3)}(z_0),\ f^{(4)}(z_0),\ f^{(5)}(z_0),\ \cdots$$

➡ Taylor series expansion about any point z_0 inside the region

The power series converges inside the circle about z_0

This circle extends to the nearest singular point



Laplace Equation

$f(z) = u(x,y) + i v(x,y)$: **analytic** in a region R

⇒ $u(x,y)$, $v(x,y)$ satisfy Laplace's equation in the region

harmonic functions

$u(x,y)$, $v(x,y)$ satisfy Laplace's equation in simply connected region

⇒ Real / imaginary part of an analytic function $f(z)$

Cauchy's Theorem

$f(z)$: **analytic** on and inside C ⇒ $\oint_{around\,C} f(z)\,dz = 0$

simple closed curve

a continuously turning tangent

except possibly at a finite number of points

allow a finite number of corners (not smooth)

Cauchy's Integral Formula

$f(z)$: **analytic** on and inside simple close curve C

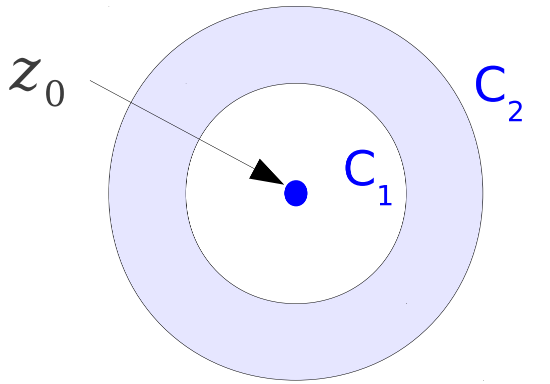
$$f(a) = \frac{1}{2\pi i} \oint \frac{f(z)}{z-a} dz$$

the value of $f(z)$

at a point $z = a$ inside C

$$f(z) = \frac{1}{2\pi i} \oint \frac{f(w)}{w-z} dw$$

Laurent's Theorem



$f(z)$: **analytic** in the region R

between circles C_1, C_2

centered at z_0

$$f(z) = a_0 + a_1(z-z_0) + a_2(z-z_0)^2 + \cdots$$

$$+ \frac{b_1}{(z-z_0)} + \frac{b_2}{(z-z_0)^2} + \cdots$$

Principal part

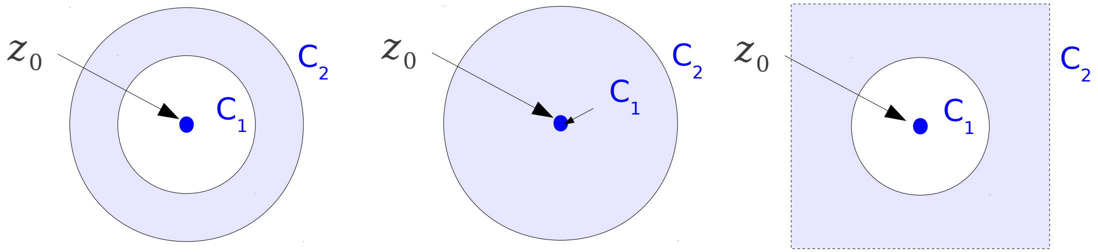
: **convergent** in the region R

Laurent's Theorem - Coefficients

$f(z)$: **analytic** in the region R

between circles C_1, C_2

centered at z_0



$$f(z) = a_0 + a_1(z-z_0) + a_2(z-z_0)^2 + \cdots$$

$$+ \frac{b_1}{(z-z_0)} + \frac{b_2}{(z-z_0)^2} + \cdots$$

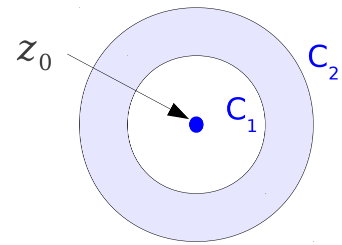
: **convergent** in the region R

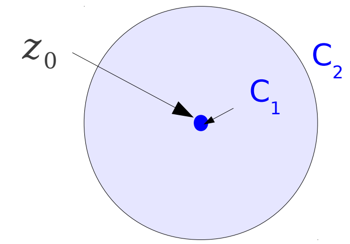
$$a_n = \frac{1}{2\pi i}\oint_C \frac{f(z)\,dz}{(z-z_0)^{n+1}}$$

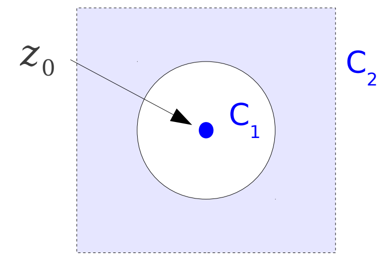
$$b_n = \frac{1}{2\pi i}\oint_C \frac{f(z)\,dz}{(z-z_0)^{-n+1}}$$

Laurent's Theorem – Some Points

$f(z)$: **analytic** in the region R
between circles C_1, C_2
centered at z_0







$$f(z) = a_0 + a_1(z-z_0) + a_2(z-z_0)^2 + \cdots$$

regular point z_0

$$f(z) = a_0 + a_1(z-z_0) + a_2(z-z_0)^2 + \cdots + \frac{b_1}{(z-z_0)} + \frac{b_2}{(z-z_0)^2} + \cdots + \frac{b_n}{(z-z_0)^n}$$

pole of order n z_0

$$f(z) = a_0 + a_1(z-z_0) + a_2(z-z_0)^2 + \cdots + \frac{b_1}{(z-z_0)}$$

simple pole z_0

b_1 *residue of* $f(z)$

$$f(z) = a_0 + a_1(z-z_0) + a_2(z-z_0)^2 + \cdots + \frac{b_1}{(z-z_0)} + \frac{b_2}{(z-z_0)^2} + \cdots + \cdots$$

essential singularity z_0

Finding Residues (1)

$$\oint_C f(z)dz = 2\pi i \cdot \sum \textit{the residues of } f(z) \textit{ inside } C$$

The integral around C is in the counterclockwise direction

Methods of Finding Residues

Laurent Series: b_1 $1/(z-z_0)$

Simple Pole : $f(z)\cdot(z-z_0)$

Multiple Pole : $f(z)\cdot(z-z_0)^m$

Finding Residues (2)

$$\oint_C f(z)\,dz = 2\pi i \cdot \sum \textit{the residues of } f(z) \textit{ inside } C$$

The integral around C is in the counterclockwise direction

Laurent Series: b_1 $\quad 1/(z-z_0)$

Simple Pole : $f(z)\cdot(z-z_0)$

Multiple Pole : $f(z)\cdot(z-z_0)^m$

$$R(z_0) = \lim_{z \to z_0} (z-z_0) f(z)$$

$$R(z_0) = \frac{g(z_0)}{h'(z_0)} \quad \Leftarrow \quad f(z) = \frac{g(z)}{h(z)} \qquad g(z_0) \neq 0 \quad h'(z_0) \neq 0 \quad h(z_0) = 0$$

$$R(z_0) = \frac{1}{(m-1)!} \frac{d^{m-1}}{dz^{m-1}} (z-z_0)^m f(z)$$

References

[1] http://en.wikipedia.org/
[2] http://planetmath.org/
[3] M.L. Boas, “Mathematical Methods in the Physical Sciences”