Vector Functions (1A)

- Vector Functions
- Motion
- Curvature

Young Won Lim
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Vector Valued Functions

Set of points $(x, \ y, \ z)$

Parametric functions $x = f(t) \quad y = g(t) \quad x = h(t)$

$$(x, \ y, \ z) \Rightarrow (f(t), \ g(t), \ h(t))$$

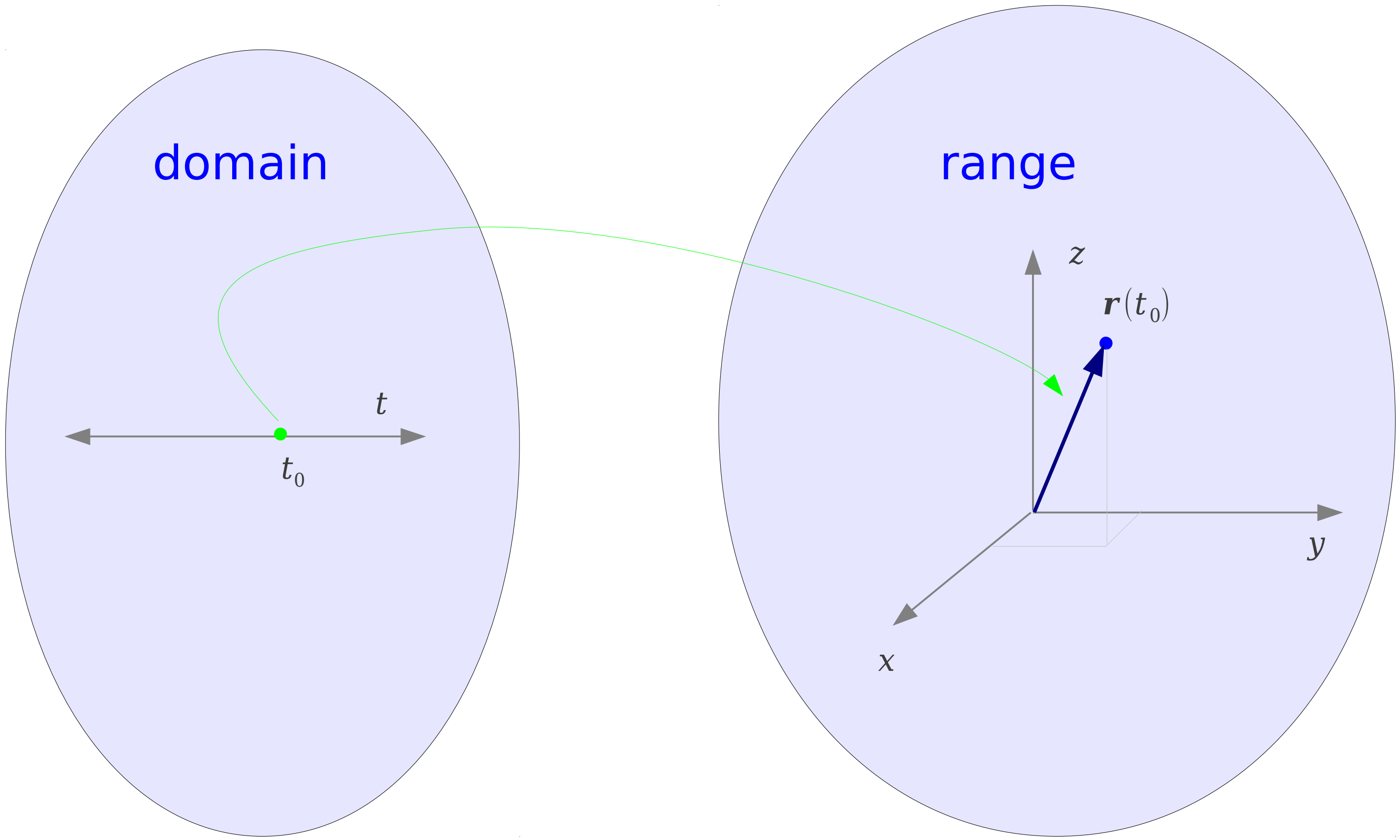
Vector Valued Function $\boldsymbol{r}(t) = \langle f(t), \ g(t), \ h(t) \rangle$

$$= f(t)\boldsymbol{i} + g(t)\boldsymbol{j} + h(t)\boldsymbol{k}$$

Vector Valued Functions (1)

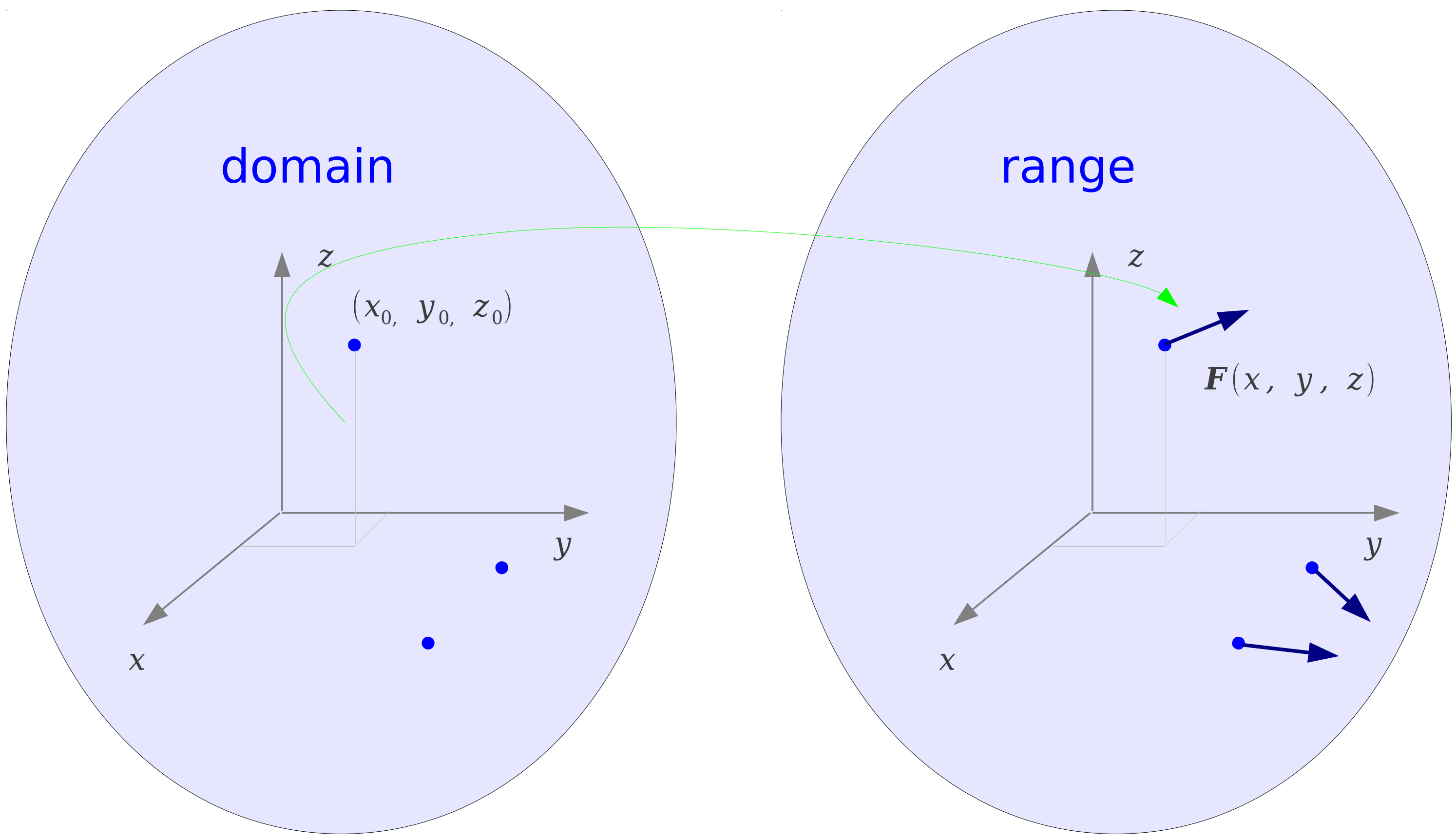
Vector Valued Function $\quad \boldsymbol{r}(t) = \langle f(t),\ g(t),\ h(t) \rangle$

$$= f(t)\boldsymbol{i} + g(t)\boldsymbol{j} + h(t)\boldsymbol{k}$$



Vector Valued Functions (2)

Vector Field $(x, \ y, \ z)$ $\boldsymbol{F}(x, \ y, \ z)$



Limit of a Vector Function

Vector Valued Function $\quad \boldsymbol{r}(t) = \langle f(t),\ g(t),\ h(t) \rangle$

Limit of a Vector Valued Function

$$\lim_{t \to a} \boldsymbol{r}(t) = \left\langle \lim_{t \to a} f(t),\ \lim_{t \to a} g(t),\ \lim_{t \to a} h(t) \right\rangle$$

Limit of a Vector Valued Function

$\boldsymbol{r}(a)$ is defined

$\lim_{t \to a} \boldsymbol{r}(t)$ exists

$$\boldsymbol{r}(a) = \lim_{t \to a} \boldsymbol{r}(t)$$

Derivative of a Vector Function

Vector Valued Function $\quad \boldsymbol{r}(t) = \langle f(t),\ g(t),\ h(t) \rangle$

Derivative of a Vector Valued Function

$$\boldsymbol{r}'(t) = \lim_{\Delta t \to 0} \frac{\boldsymbol{r}(t + \Delta t) - \boldsymbol{r}(t)}{\Delta t}$$

$$\boldsymbol{r}'(t) = \langle f'(t),\ g'(t),\ h'(t) \rangle$$

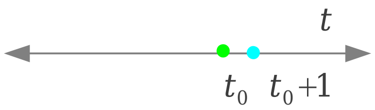
Arc Length (1)

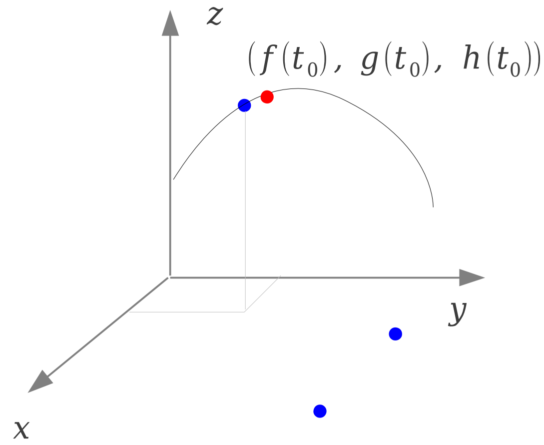
Vector Valued Function $\boldsymbol{r}(t) = \langle f(t),\ g(t),\ h(t) \rangle$

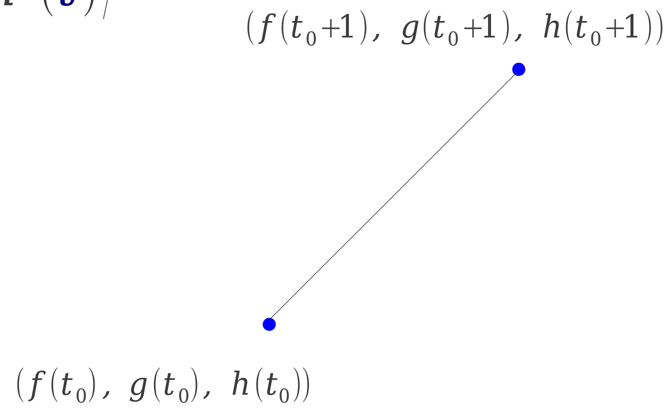
Derivative of a Vector Valued Function

$$\boldsymbol{r}'(t) = \lim_{\Delta t \to 0} \frac{\boldsymbol{r}(t + \Delta t) - \boldsymbol{r}(t)}{\Delta t}$$

$$\boldsymbol{r}'(t) = \langle f'(t),\ g'(t),\ h'(t) \rangle$$







Arc Length (2)

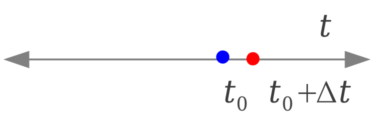
Vector Valued Function

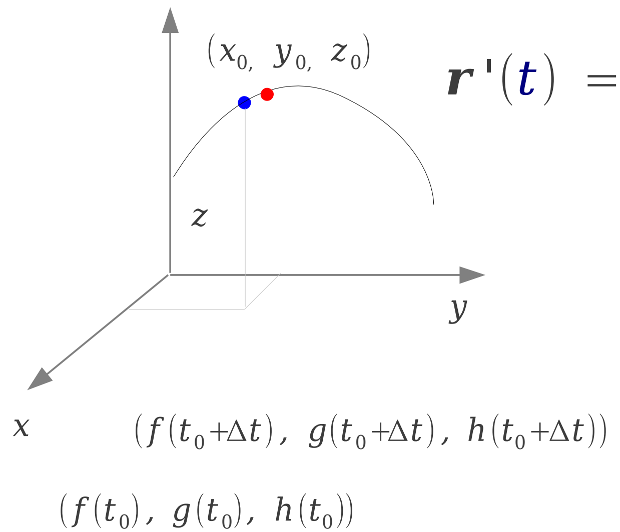
$$\boldsymbol{r}(t) = \langle f(t),\ g(t),\ h(t) \rangle$$

Derivative of a Vector Valued Function

$$\boldsymbol{r}'(t) = \langle f'(t),\ g'(t),\ h'(t) \rangle$$

$$\boldsymbol{r}'(t) = \lim_{\Delta t \to 0} \frac{\boldsymbol{r}(t + \Delta t) - \boldsymbol{r}(t)}{\Delta t}$$



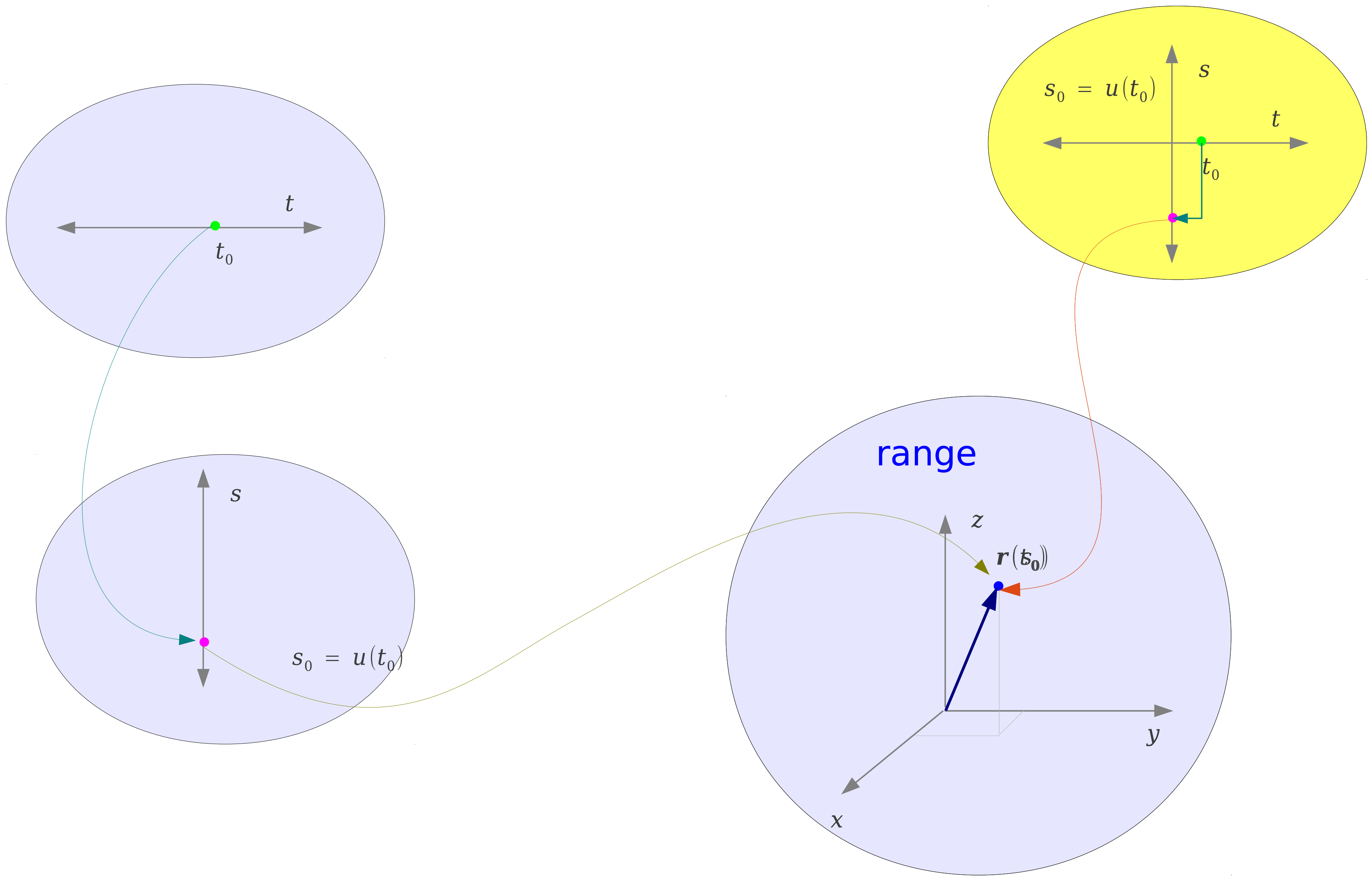


Arc Length

$$s = \int_a^b \sqrt{[f'(t)]^2 + [g'(t)]^2 + [h(t)]^2}\, dt$$

$$= \int_a^b \|\boldsymbol{r}'(t)\|\, dt$$

Composite Function

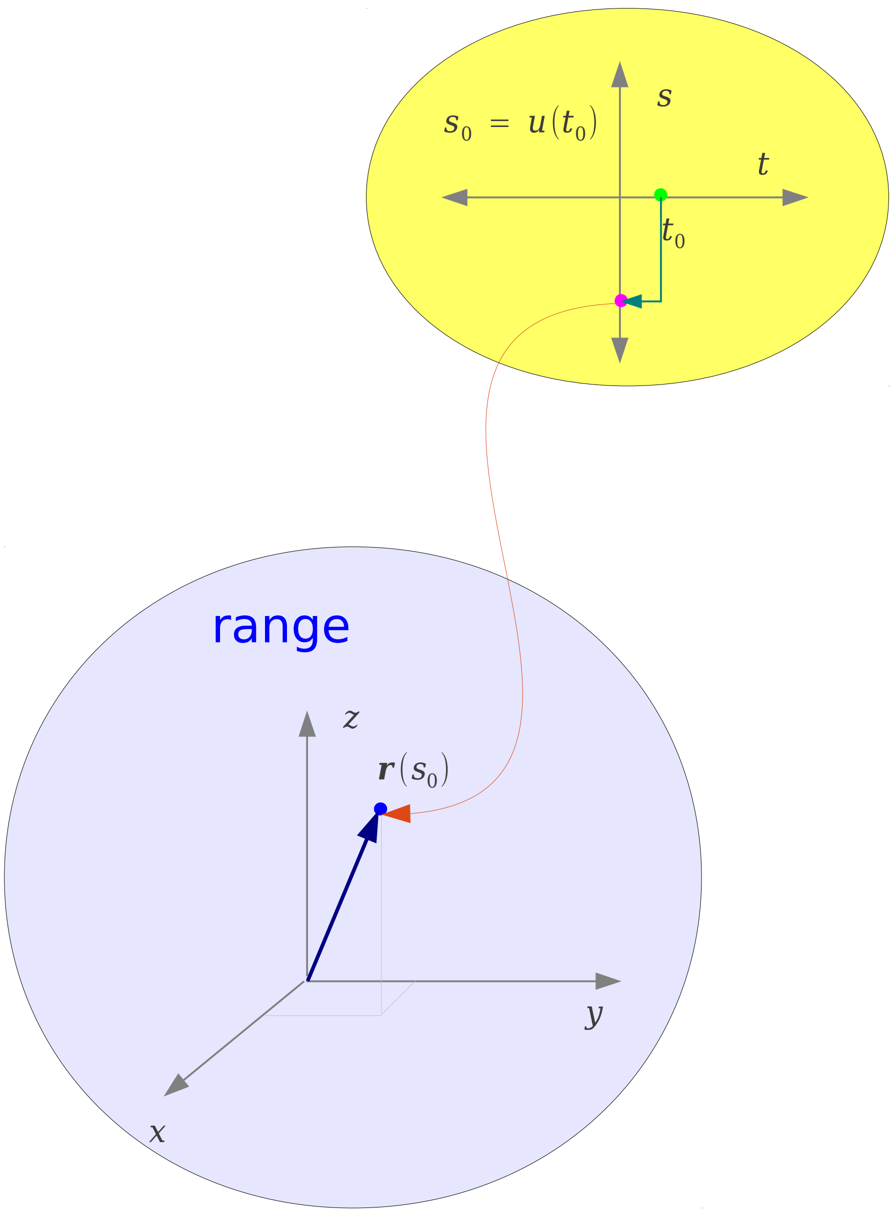


Derivative of a Vector Valued Function

$$s = u(t)$$

$$\frac{ds}{dt} = \frac{du(t)}{dt} \Rightarrow u'(t)$$

$$\frac{d\boldsymbol{r}}{dt} = \frac{d\boldsymbol{r}}{ds}\frac{ds}{dt} = \boldsymbol{r}'(s)\,u'(t)$$



Chain Rule of a Vector Function (2)

Vector Valued Function $\boldsymbol{r}(s) = \langle f(s),\ g(s),\ h(s) \rangle$

Scalar Function $s = u(t)$

$$\boldsymbol{r}(u(t)) = \langle f(u(t)),\ g(u(t)),\ h(u(t)) \rangle$$

Derivative of a Vector Valued Function

$$s = u(t)$$

$$\frac{ds}{dt} = \frac{du(t)}{dt} \Rightarrow u'(t)$$

$$\frac{d\boldsymbol{r}}{dt} = \frac{d\boldsymbol{r}}{ds}\frac{ds}{dt} = \boldsymbol{r}'(s)u'(t)$$

Integration of a Vector Function

Vector Valued Function

$$\mathbf{r}(t) = \langle f(t),\ g(t),\ h(t) \rangle$$

$$= f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k}$$

Limit of a Vector Valued Function

$$\int \mathbf{r}(t)\,dt = \left\langle \int f(t)\,dt,\ \int g(t)\,dt,\ \int h(t)\,dt \right\rangle$$

$$= \int f(t)\,dt\,\mathbf{i} + \int g(t)\,dt\,\mathbf{j} + \int h(t)\,dt\,\mathbf{k}$$

Integration of a Vector Function

Vector Valued Function $\quad \boldsymbol{r}(t) = f(t)\boldsymbol{i} + g(t)\boldsymbol{j} + h(t)\boldsymbol{k}$

Displacement $\quad \boldsymbol{v}(t) = \boldsymbol{r}'(t) = f'(t)\boldsymbol{i} + g'(t)\boldsymbol{j} + h'(t)\boldsymbol{k}$

acceleration $\quad \boldsymbol{a}(t) = \boldsymbol{v}'(t) = \boldsymbol{r}''(t) = f''(t)\boldsymbol{i} + g''(t)\boldsymbol{j} + h''(t)\boldsymbol{k}$

Speed

$$\|\boldsymbol{v}(t)\| = \left\|\frac{\boldsymbol{r}(t)}{dt}\right\| = \|f'(t)\boldsymbol{i} + g'(t)\boldsymbol{j} + h'(t)\boldsymbol{k}\|$$

$$= \sqrt{(f'(t))^2 + (g'(t))^2 + (h'(t))^2}$$

$$= \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}$$

Unit Tangent of a Vector Function

Vector Valued Function $\quad \mathbf{r}(t) = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k}$

Displacement $\quad \mathbf{v}(t) = \mathbf{r}'(t) = f'(t)\mathbf{i} + g'(t)\mathbf{j} + h'(t)\mathbf{k}$

Unit Tangent $\quad \mathbf{T}(t) = \dfrac{\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|}$

Arc length $\quad s \quad \Rightarrow \quad \dfrac{d\mathbf{r}}{dt} = \dfrac{d\mathbf{r}}{ds}\dfrac{ds}{dt}$

velocity

- speed
- direction

Unit Tangent $\quad \Rightarrow \quad \dfrac{d\mathbf{r}}{ds} = \dfrac{\dfrac{d\mathbf{r}}{dt}}{\dfrac{ds}{dt}} = \dfrac{\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|} = \mathbf{T}(t)$ direction

speed

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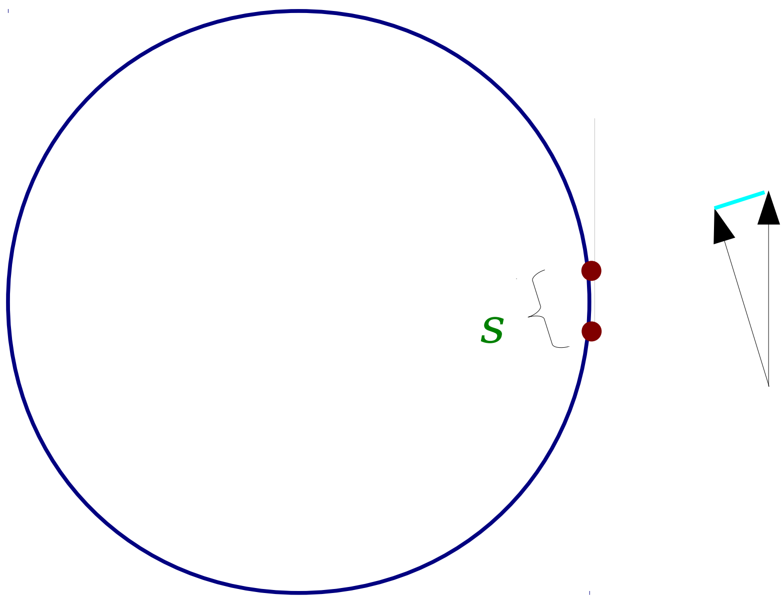
Curvature of a Vector Function (1)

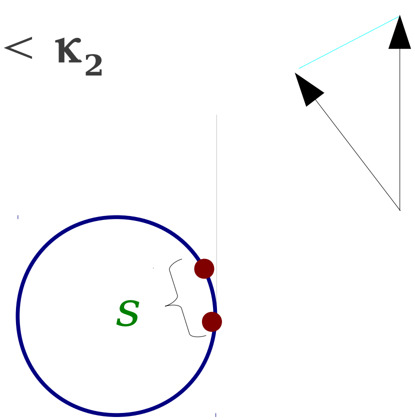
Vector Valued Function $$\boldsymbol{r}(t) = f(t)\boldsymbol{i} + g(t)\boldsymbol{j} + h(t)\boldsymbol{k}$$

Unit Tangent $$\boldsymbol{T}(t) = \frac{\boldsymbol{r}'(t)}{\|\boldsymbol{r}'(t)\|} = \frac{d\boldsymbol{r}}{ds}$$

Curvature $$\kappa = \left\| \frac{d\boldsymbol{T}}{ds} \right\|$$



$$\kappa_1 < \kappa_2$$



Curvature of a Vector Function (2)

Vector Valued Function $\quad \mathbf{r}(t) = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k}$

Unit Tangent $\quad \mathbf{T}(t) = \dfrac{\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|} = \dfrac{d\mathbf{r}}{ds}$

Curvature $\quad \kappa = \left\| \dfrac{d\mathbf{T}}{ds} \right\|$

Arc length $\quad s$

$$\frac{d\mathbf{T}}{dt} = \frac{d\mathbf{T}}{ds}\frac{ds}{dt}$$

$$\frac{d\mathbf{T}}{ds} = \frac{\dfrac{d\mathbf{T}}{dt}}{\dfrac{ds}{dt}} = \frac{\mathbf{T}'(t)}{\|\mathbf{r}'(t)\|} = \kappa(t)$$

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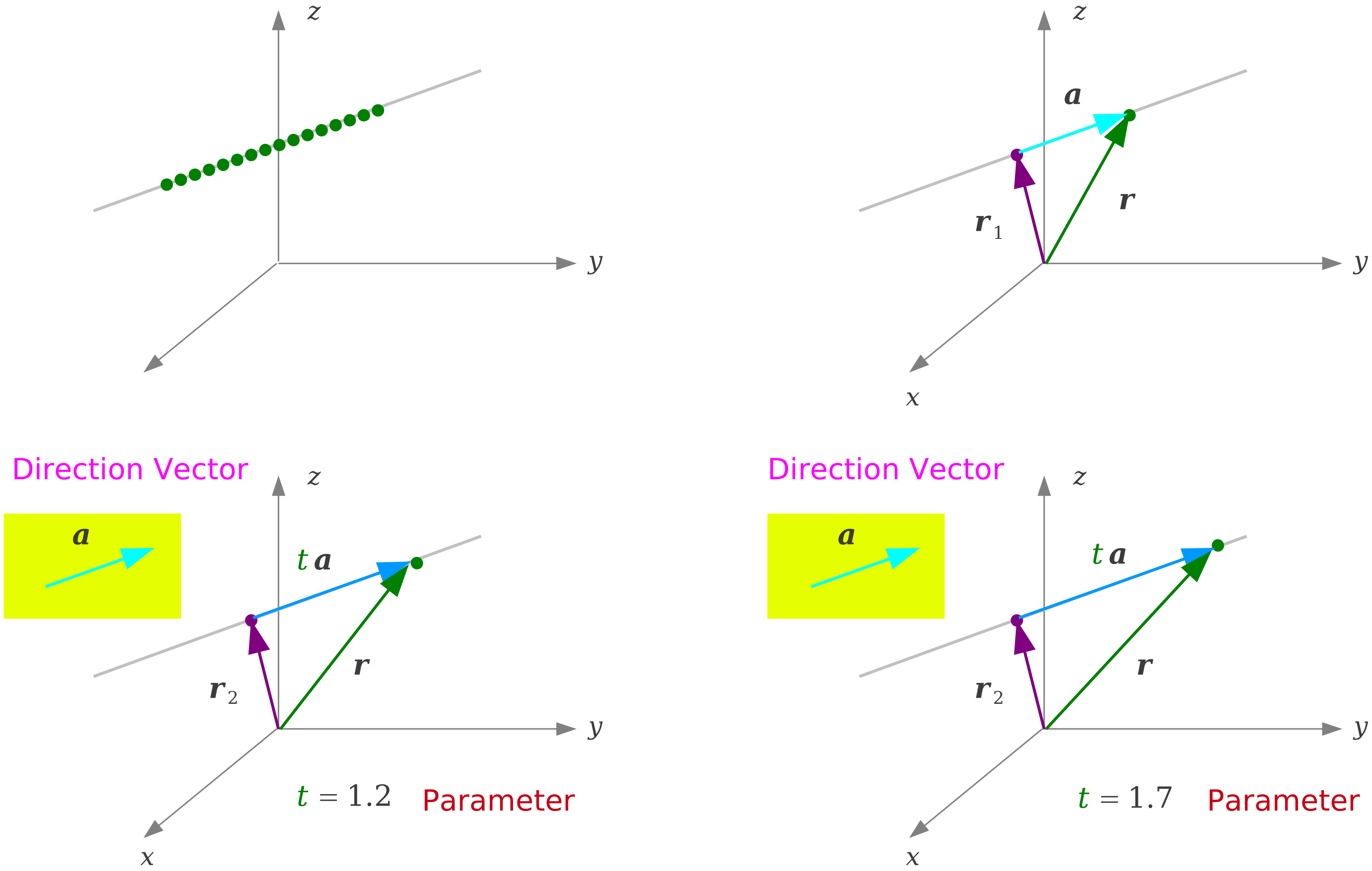
Line Equations (2)



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