

# DTFT

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- Discrete Time Fourier Transform

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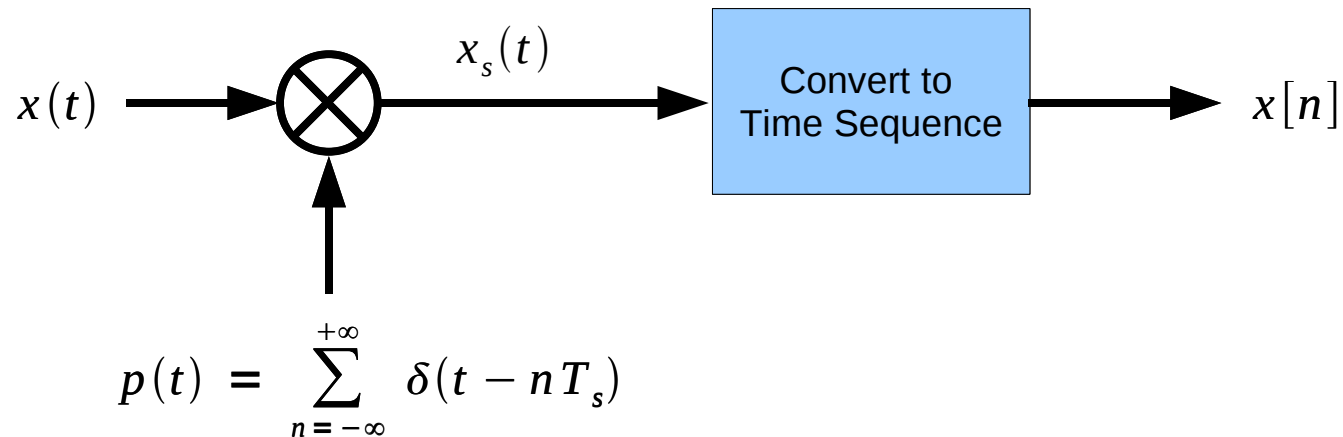
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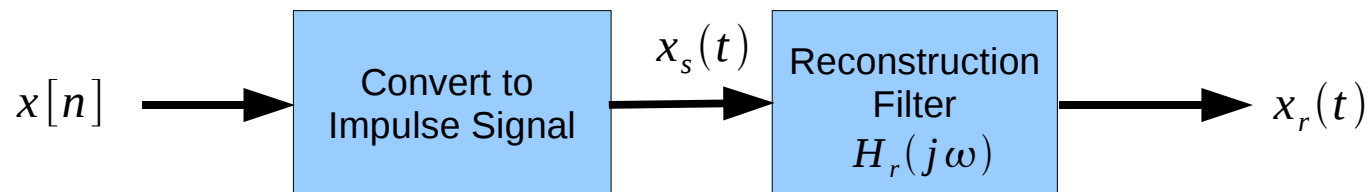
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# Sampling and Reconstruction

## *Ideal Sampling*

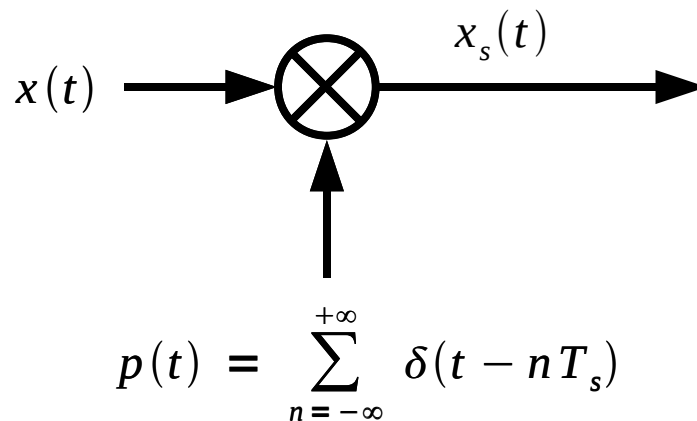


## *Ideal Reconstruction*



# Sampled Signal

## Ideal Sampling



$$\begin{aligned} x_s(t) &= x_c(t) \sum_{n=-\infty}^{+\infty} \delta(t - nT_s) \\ &= \sum_{n=-\infty}^{+\infty} x_c(nT_s) \delta(t - nT_s) \end{aligned}$$



$$\begin{aligned} x_s(t) &= x_c(t) \sum_{k=-\infty}^{+\infty} \frac{1}{T_s} e^{jk\omega_s t} \\ \omega_s &= \frac{2\pi}{T_s} \end{aligned}$$

# CTFT Frequency Shift Property

## Continuous Time Fourier Transform

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} X(j\omega) e^{+j\omega t} d\omega$$



$$X(j\omega) = \int_{-\infty}^{+\infty} x(t) e^{-j\omega t} dt$$

$$x_c(t)$$



$$X_c(j\omega)$$

$$x_c(t) e^{jk\omega_s t}$$



$$X_c(j(\omega - k\omega_s))$$

$$x_s(t) = x_c(t) \sum_{k=-\infty}^{+\infty} \frac{1}{T_s} e^{jk\omega_s t}$$



$$X_s(j\omega) = \frac{1}{T_s} \sum_{k=-\infty}^{+\infty} X_c(j(\omega - k\omega_s))$$

$$\omega_s = \frac{2\pi}{T_s}$$

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# CTFT Delay Property

## Continuous Time Fourier Transform

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} X(j\omega) e^{+j\omega t} d\omega$$



$$X(j\omega) = \int_{-\infty}^{+\infty} x(t) e^{-j\omega t} dt$$

$$\delta(t - t_d)$$



$$e^{j\omega t_d}$$

$$\delta(t - nT_s)$$



$$e^{-j\omega nT_s}$$

$$x_s(t) = x_c(t) \sum_{n=-\infty}^{+\infty} \delta(t - nT_s)$$



$$X_s(j\omega) = \sum_{n=-\infty}^{+\infty} X_c(nT_s) e^{-j\omega nT_s}$$

$$= \sum_{n=-\infty}^{+\infty} x_c(nT_s) \delta(t - nT_s)$$

$$= \sum_{n=-\infty}^{+\infty} x[n] e^{-j\omega nT_s}$$

# CTFT of a Sampled Signal

## Continuous Time Fourier Transform

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} X(j\omega) e^{+j\omega t} d\omega$$



$$X(j\omega) = \int_{-\infty}^{+\infty} x(t) e^{-j\omega t} dt$$

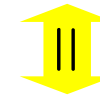
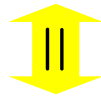
$$x_s(t) = x_c(t) \sum_{k=-\infty}^{+\infty} \frac{1}{T_s} e^{jk\omega_s t}$$



$$X_s(j\omega) = \frac{1}{T_s} \sum_{k=-\infty}^{+\infty} X_c(j(\omega - k\omega_s))$$

$$\omega_s = \frac{2\pi}{T_s}$$

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$$x_s(t) = x_c(t) \sum_{n=-\infty}^{+\infty} \delta(t - nT_s)$$



$$X_s(j\omega) = \sum_{n=-\infty}^{+\infty} X_c(nT_s) e^{-j\omega nT_s}$$

$$= \sum_{n=-\infty}^{+\infty} x_c(nT_s) \delta(t - nT_s)$$

$$= \sum_{n=-\infty}^{+\infty} x[n] e^{-j\omega nT_s}$$

# z-Transform of a Sampled Signal

$$X_s(j\omega) = \sum_{n=-\infty}^{+\infty} X_c(nT_s) e^{-j\omega n T_s}$$

$$= \sum_{n=-\infty}^{+\infty} x[n] e^{-j\omega n T_s}$$



$$X_s(j\omega) = \frac{1}{T_s} \sum_{k=-\infty}^{+\infty} X_c(j(\omega - k\omega_s))$$

$$\omega_s = \frac{2\pi}{T_s}$$

**CTFT** of a sampled signal

$$\sum_{n=-\infty}^{+\infty} x[n] e^{-j\omega n T_s} = \frac{1}{T_s} \sum_{k=-\infty}^{+\infty} X_c(j(\omega - k\omega_s)) = \frac{1}{T_s} \sum_{k=-\infty}^{+\infty} X_c(j(\omega - \frac{2\pi k}{T_s}))$$

**Z-Transform** of a sampled signal

$$X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n} \quad x[n] = x_c(nT_s)$$



$$X(z) \Big|_{z = e^{j\omega T_s}}$$

$$= X(e^{j\omega T_s})$$

*evaluated at*  $z = e^{j\omega T_s}$



# z-Transform and Normalized Frequency

$$X_s(j\omega) = \sum_{n=-\infty}^{+\infty} X_c(nT_s) e^{-j\omega n T_s}$$

$$= \sum_{n=-\infty}^{+\infty} x[n] e^{-j\omega n T_s}$$



$$X_s(j\omega) = \frac{1}{T_s} \sum_{k=-\infty}^{+\infty} X_c(j(\omega - k\omega_s))$$

$$\omega_s = \frac{2\pi}{T_s}$$

$$X(z) \Big|_{z = e^{j\omega T_s}}$$

$$= X(e^{j\omega T_s}) = \sum_{n=-\infty}^{+\infty} x[n] e^{-j\omega n T_s}$$

**z-Transform**



$$\hat{\omega} = \omega T_s$$

**Normalized Frequency**

$$X(z) \Big|_{z = e^{j\hat{\omega}}}$$

$$X(e^{j\hat{\omega}}) = \sum_{n=-\infty}^{+\infty} x[n] e^{-j\hat{\omega} n}$$

**Discrete Time Fourier Transform**

# DTFT and CTFT

$$X_s(j\omega) = \sum_{n=-\infty}^{+\infty} X_c(nT_s) e^{-j\omega n T_s}$$

$$= \sum_{n=-\infty}^{+\infty} x[n] e^{-j\omega n T_s}$$



$$X_s(j\omega) = \frac{1}{T_s} \sum_{k=-\infty}^{+\infty} X_c(j(\omega - k\omega_s))$$

$$\omega_s = \frac{2\pi}{T_s}$$

$$X(e^{j\hat{\omega}}) = \sum_{n=-\infty}^{+\infty} x[n] e^{-j\hat{\omega}n}$$

**DTFT** of a sampled signal

$$X(e^{j\hat{\omega}})$$

$$\hat{\omega} = \omega T_s$$

$$= X(e^{j\omega T_s}) = \frac{1}{T_s} \sum_{k=-\infty}^{+\infty} X_c(j(\omega - k\omega_s))$$

$$= \frac{1}{T_s} \sum_{k=-\infty}^{+\infty} X_c(j(\omega - \frac{2\pi k}{T_s}))$$

**CTFT** of a sampled signal

# CTFS and DTFS

## Continuous Time Fourier Transform

$$X(j\omega) = \int_{-\infty}^{+\infty} x(t) e^{-j\omega t} dt \quad \longleftrightarrow \quad x(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} X(j\omega) e^{+j\omega t} d\omega$$

## Discrete Time Fourier Transform

$$X(e^{j\hat{\omega}}) = \sum_{n=-\infty}^{+\infty} x[n] e^{-j\hat{\omega}n} \quad \longleftrightarrow \quad x[n] = \frac{1}{2\pi} \int_{-\pi}^{+\pi} X(e^{j\hat{\omega}}) e^{+j\hat{\omega}n} d\hat{\omega}$$

## References

- [1] <http://en.wikipedia.org/>
- [2] J.H. McClellan, R.W. Schafer, M.A. Yoder, "Signal Processing First", 2003, Pearson Education