

# Double Integrals (5A)

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- Double Integral
- Double Integrals in Polar Coordinates
- Green's Theorem

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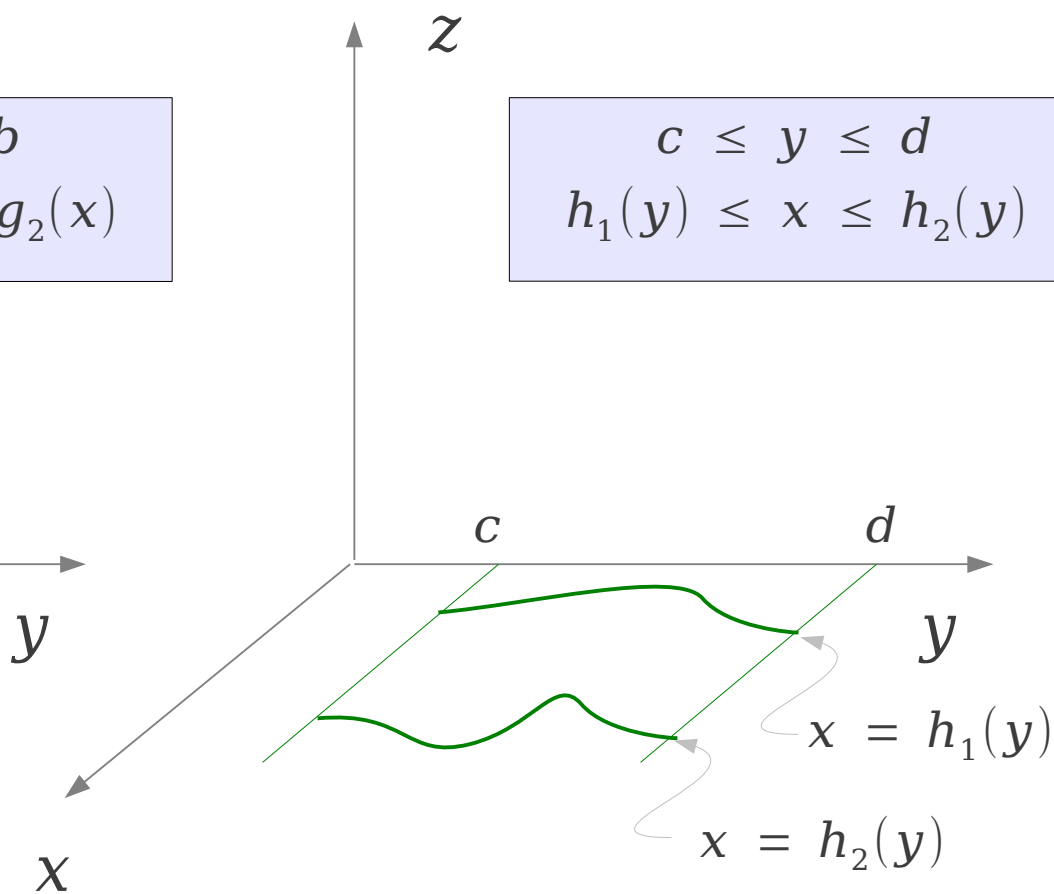
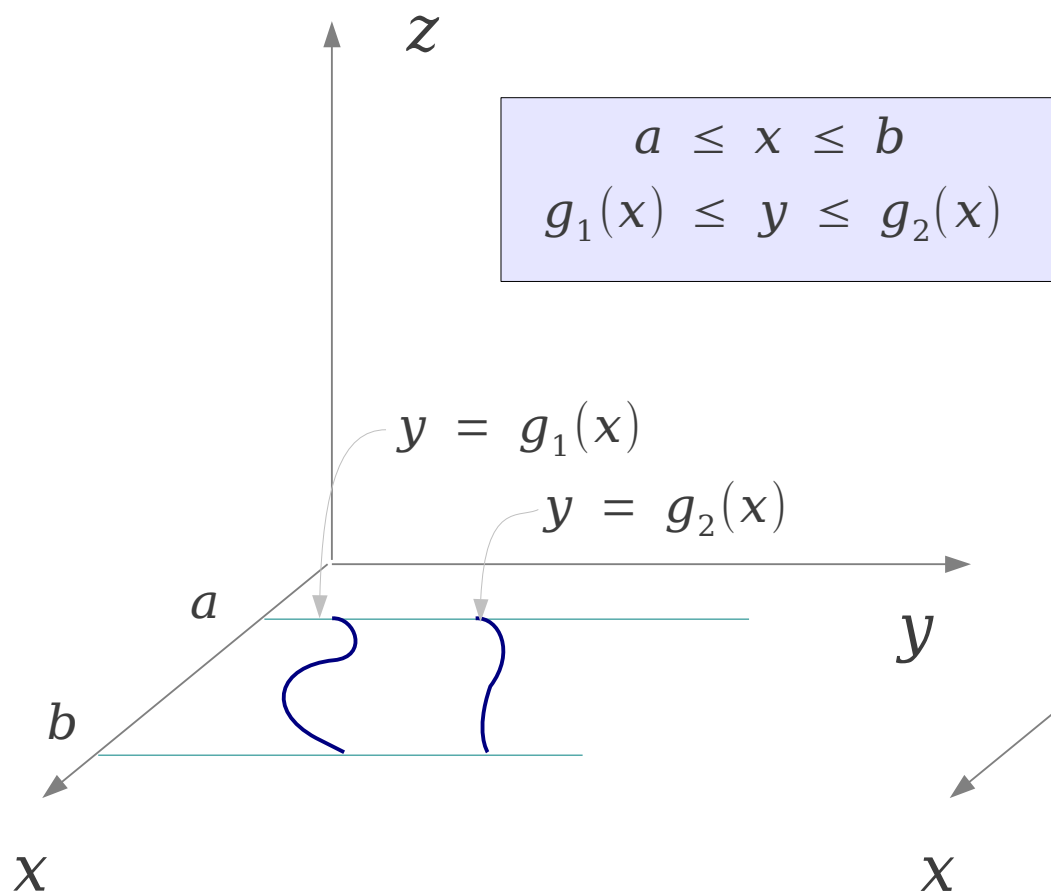
# Area and Volume

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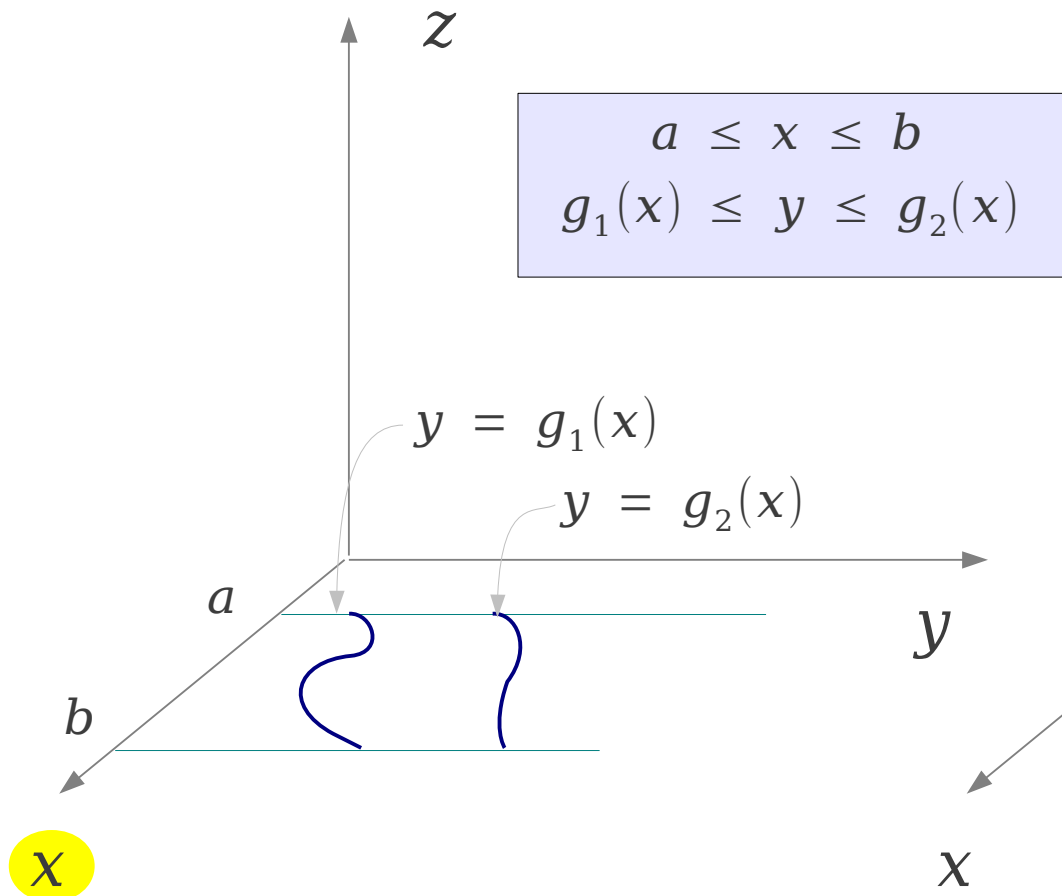
$$A = \iint_R dA$$

$$V = \iint_R f(x, y) dA$$

# Type I and Type II

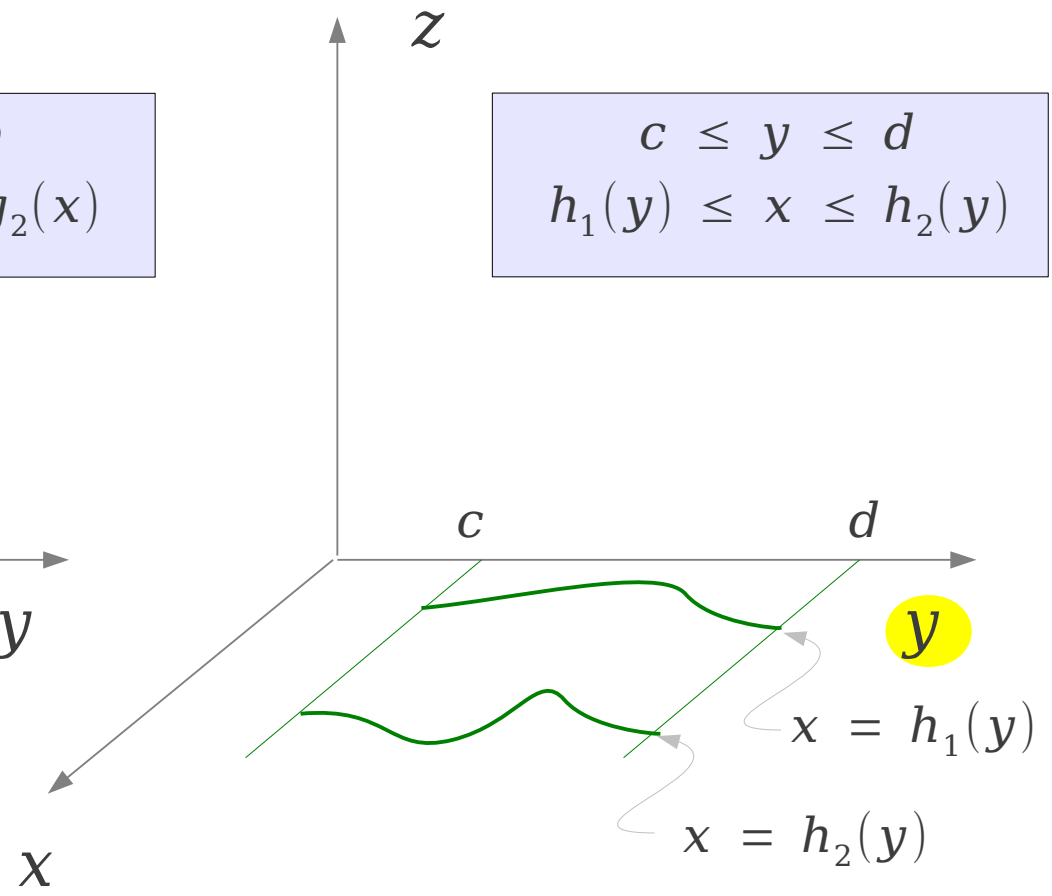


# Fubini's Theorem



$$\begin{aligned} a &\leq x \leq b \\ g_1(x) &\leq y \leq g_2(x) \end{aligned}$$

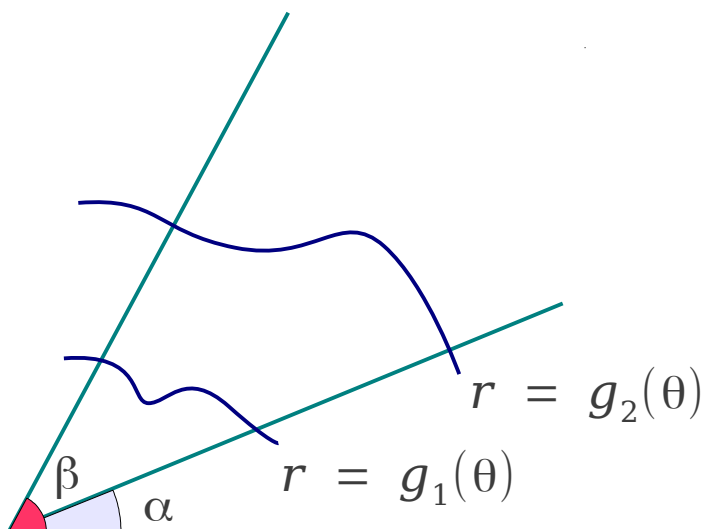
$$\begin{aligned} \iint_R f(x, y) \, dA \\ = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x, y) \, dy \, dx \end{aligned}$$



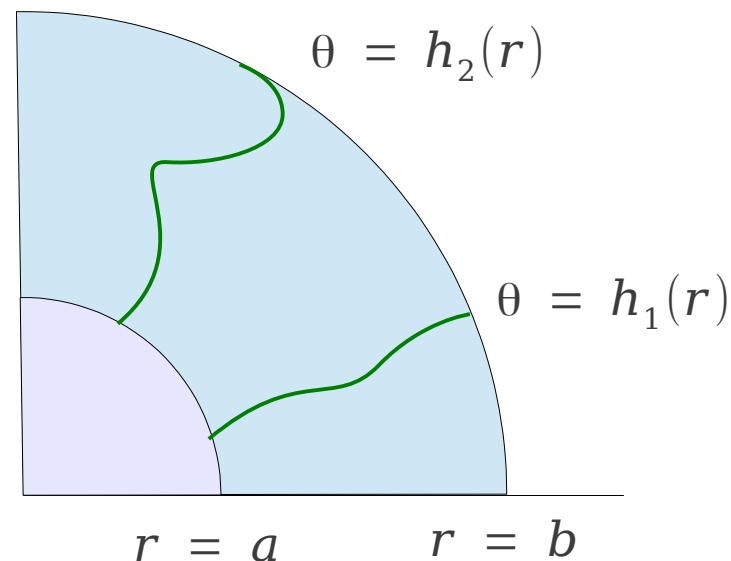
$$\begin{aligned} c &\leq y \leq d \\ h_1(y) &\leq x \leq h_2(y) \end{aligned}$$

$$\begin{aligned} \iint_R f(x, y) \, dA \\ = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x, y) \, dx \, dy \end{aligned}$$

# Type A and Type B



$$\iint_R f(r, \theta) dA \\ = \int_{\alpha}^{\beta} \int_{g_1(\theta)}^{g_2(\theta)} f(r, \theta) r dr d\theta$$



$$\iint_R f(r, \theta) dA \\ = \int_a^b \int_{h_1(r)}^{h_2(r)} f(r, \theta) d\theta r dr$$

# Work using an Arc Length Parameter $s$

$$W = \mathbf{F} \cdot d\mathbf{r}$$

A force field  $\mathbf{F}(x, y) = P(x, y)\mathbf{i} + Q(x, y)\mathbf{j}$

A smooth curve  $C: x = f(t), y = g(t), a \leq t \leq b$

Work done by  $\mathbf{F}$  along  $C$  
$$W = \int_C \mathbf{F}(x, y) \cdot d\mathbf{r}$$
$$= \int_C P(x, y) dx + Q(x, y) dy$$

$$\frac{d\mathbf{r}}{dt} = \frac{d\mathbf{r}}{ds} \frac{ds}{dt}$$

$$d\mathbf{r} = \frac{d\mathbf{r}}{ds} ds \quad d\mathbf{r} = \mathbf{T} ds$$

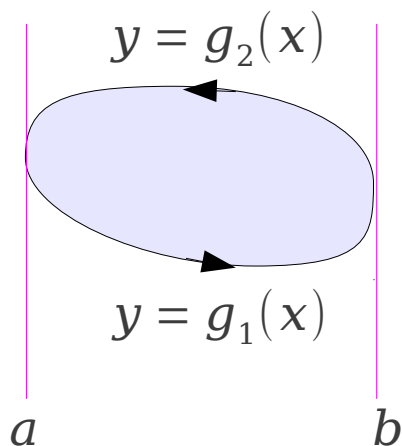
Unit Tangent Vector

$$W = \int_C \mathbf{F} \cdot d\mathbf{r} = \int_C \mathbf{F} \cdot \mathbf{T} ds$$

# Green's Theorem in the Plane (1)

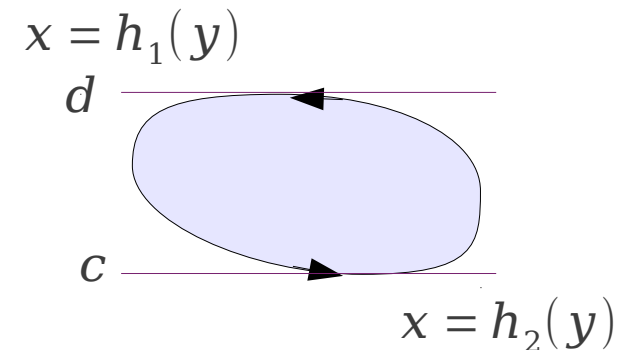
**C**: a piecewise  $c$  simple closed curve  
 bounding a simply connected region **R**

$$\underbrace{\oint_C P dx + Q dy}_{\text{Line Integral}} = \underbrace{\iint_R \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA}_{\text{Double Integral}}$$



$$\iint_R -\frac{\partial P}{\partial y} dA$$

$$\iint_R \frac{\partial Q}{\partial x} dA$$



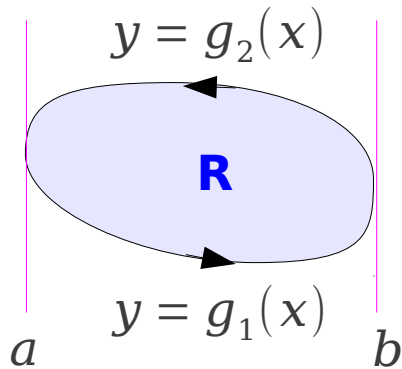


# Green's Theorem in the Plane (2)

**C**: a piecewise c simple closed curve

**R**: a simply connected bounding region

$$\oint_C P dx + Q dy = \iint_R \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA$$



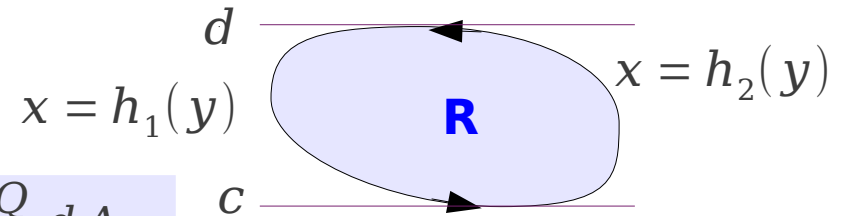
$$\iint_R -\frac{\partial P}{\partial y} dA$$

$$= -\int_a^b \int_{g_1(x)}^{g_2(x)} \frac{\partial P}{\partial y} dy dx$$

$$= -\int_a^b [P(x, g_2(x)) - P(x, g_1(x))] dx$$

$$= \int_a^b P(x, g_1(x)) dx - \int_a^b P(x, g_2(x)) dx$$

$$= \oint_C P dx$$



$$\iint_R \frac{\partial Q}{\partial x} dA$$

$$= -\int_c^d \int_{h_1(y)}^{h_2(y)} \frac{\partial Q}{\partial x} dx dy$$

$$= -\int_c^d [Q(h_2(y), y) - Q(h_1(y), y)] dy$$

$$= \int_c^d Q(h_1(y), y) dy - \int_c^d Q(h_2(y), y) dy$$

$$= \oint_C Q dy$$

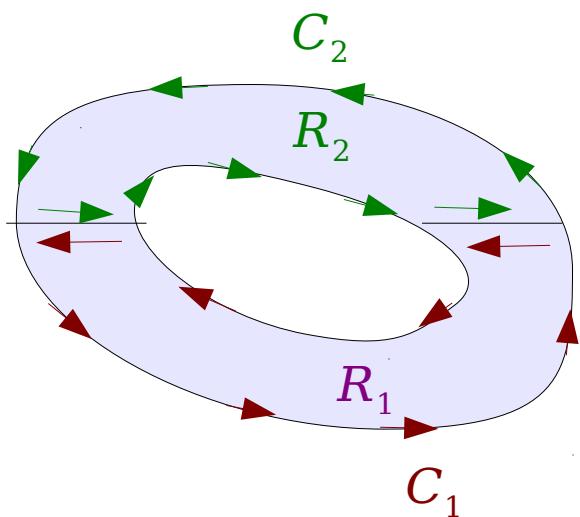
# Region with Holes

**C**: a piecewise c simple closed curve  
 bounding a simply connected region **R**

$$\underbrace{\oint_C P dx + Q dy}_{\text{Line Integral}} = \underbrace{\iint_R \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA}_{\text{Double Integral}}$$

Line Integral

Double Integral



$$\begin{aligned} & \iint_R \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA \\ &= \iint_{R_1} \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA + \iint_{R_2} \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA \\ &= \oint_{C_1} P dx + Q dy + \oint_{C_2} P dx + Q dy \\ &= \oint_C P dx + Q dy \end{aligned}$$

# 2-Divergence

Flux across rectangle boundary

$$\approx \left( \frac{\partial M}{\partial x} \Delta x \right) \Delta y + \left( \frac{\partial N}{\partial y} \Delta y \right) \Delta x = \left( \frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} \right) \Delta x \Delta y$$

Flux density =  $\left( \frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} \right)$       Divergence of  $\mathbf{F}$       Flux Density

## References

- [1] <http://en.wikipedia.org/>
- [2] <http://planetmath.org/>
- [3] M.L. Boas, “Mathematical Methods in the Physical Sciences”
- [4] D.G. Zill, “Advanced Engineering Mathematics”