Complex Phase Factors (DFT.A1)

- Complex Phase Factors
- N=8 DFT
  - DFT Matrix
  - Exponents of W
  - Common Differences in Exponents of W
  - Complex Phase Factors in Angles
- N=8 IDFT Matrix
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Young Won Lim
11/13/2010

Please send corrections (or suggestions) to youngwlim@hotmail.com.

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DFT

Discrete Fourier Transform

$$X[k] = \sum_{n=0}^{N-1} x[n]\, e^{-j(2\pi/N)kn} \quad \Longleftrightarrow \quad x[n] = \frac{1}{N}\sum_{k=0}^{N-1} X[k]\, e^{+j(2\pi/N)kn}$$

$$W_N \triangleq e^{-j(2\pi/N)}$$

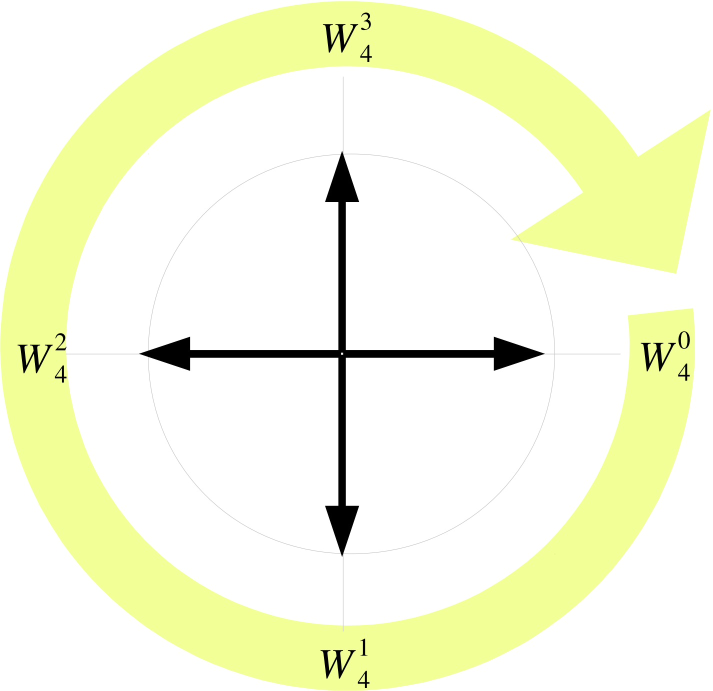
$$W_N^{nk} \triangleq e^{-j(2\pi/N)nk}$$

$$X[k] = \sum_{n=0}^{N-1} x[n]\, W_N^{kn} \quad \Longleftrightarrow \quad x[n] = \frac{1}{N}\sum_{k=0}^{N-1} X[k]\, W_N^{-kn}$$

Complex Phase Factor (1)

$$W_4^{k} = e^{-j\left(\frac{2\pi}{4}\right)k}$$

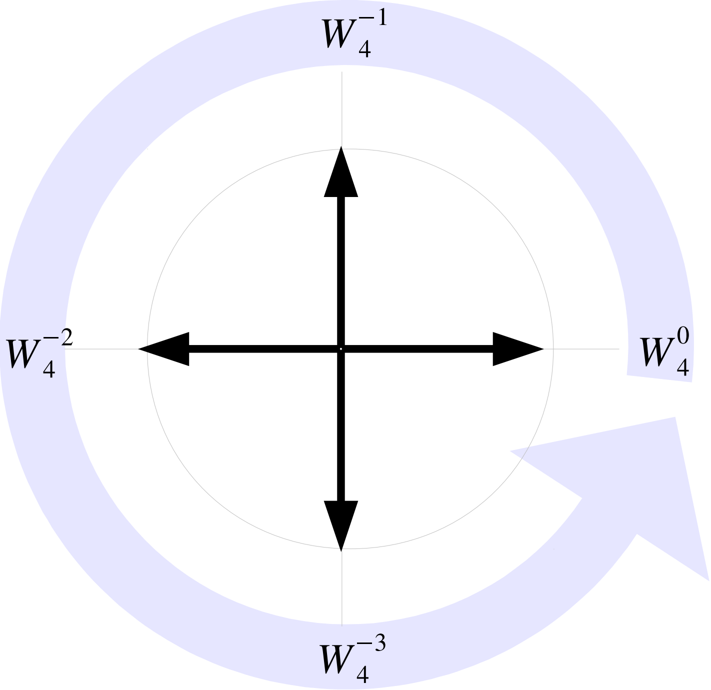
$$W_4^{-k} = e^{+j\left(\frac{2\pi}{4}\right)k}$$



$$W_4^1 = W_4^{-3}$$

$$W_4^2 = W_4^{-2}$$

$$W_4^3 = W_4^{-1}$$

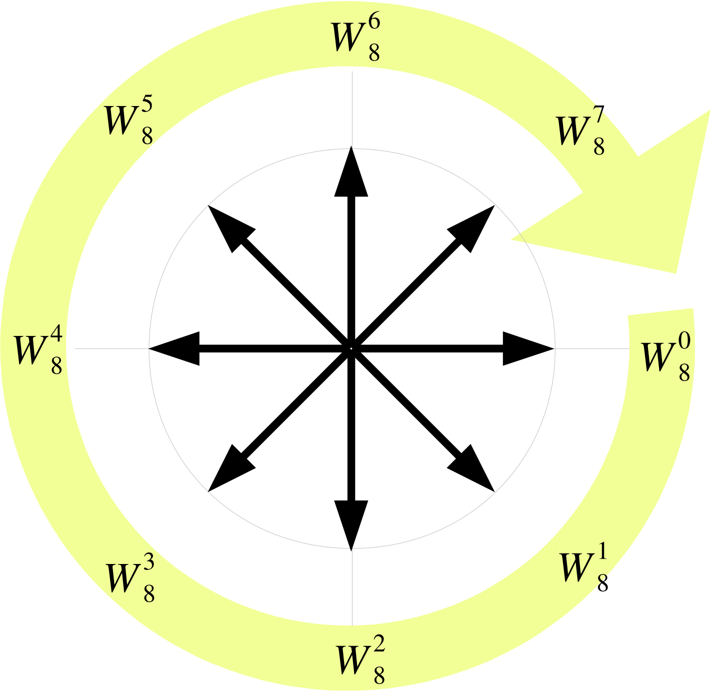


$$W_N^{k \pm N} = W_N^{k}$$

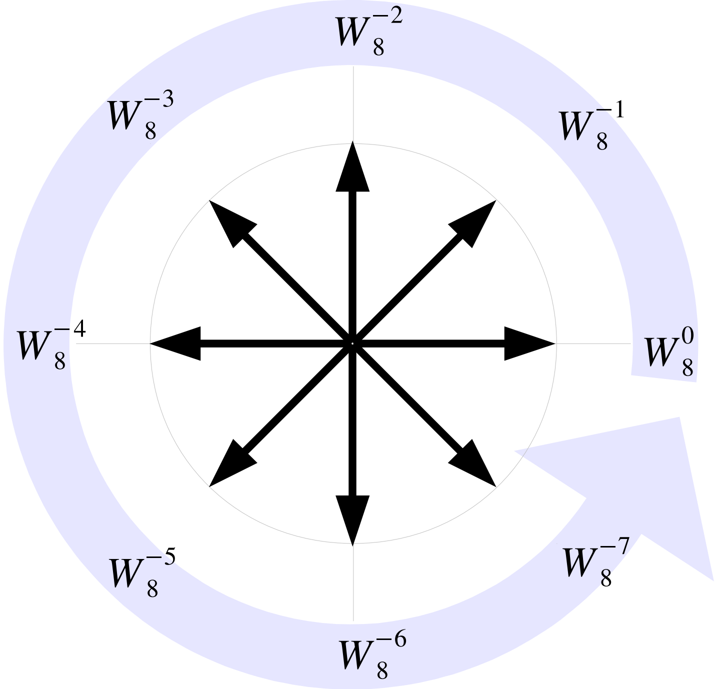
Complex Phase Factor (2)

$$W_8^{k} = e^{-j(\frac{2\pi}{8})k}$$

$$W_8^{-k} = e^{+j(\frac{2\pi}{8})k}$$



$$W_8^1 = W_8^{-7}$$
$$W_8^2 = W_8^{-6}$$
$$W_8^3 = W_8^{-5}$$
$$W_8^4 = W_8^{-4}$$
$$W_8^5 = W_8^{-3}$$
$$W_8^6 = W_8^{-2}$$
$$W_8^7 = W_8^{-1}$$



$$W_N^{k\pm N} = W_N^{k}$$

Complex Phase Factor (3)

$$\boxed{W_N^{k} = e^{-j(\frac{2\pi}{N})k}}$$

$$\boxed{W_N^{-k} = e^{+j(\frac{2\pi}{N})k}}$$

$$\boxed{W_N^{k-N} = W_N^{k}}$$

$$\boxed{W_N^{k+N} = W_N^{k}}$$

$$W_N^{k-N} = e^{-j(\frac{2\pi}{N})(k-N)}$$

$$\frac{W_N^{k-N}}{W_N^{k}} = \frac{e^{-j(\frac{2\pi}{N})(k-N)}}{e^{-j(\frac{2\pi}{N})k}} = e^{j2\pi} = 1$$

$$W_N^{k+N} = e^{-j(\frac{2\pi}{N})(k+N)}$$

$$\frac{W_N^{k+N}}{W_N^{k}} = \frac{e^{-j(\frac{2\pi}{N})(k+N)}}{e^{-j(\frac{2\pi}{N})k}} = e^{-j2\pi} = 1$$

$$\boxed{W_N^{kN} = 1}$$

$$\boxed{W_N^{-kN} = 1}$$

$$W_N^{kN} = e^{-j(\frac{2\pi}{N})kN} = e^{-j2\pi k} = 1$$

$$W_N^{kN} = e^{+j(\frac{2\pi}{N})kN} = e^{+j2\pi k} = 1$$

DFT Symmetry

$$X^*[k] = X[N-k]$$

$$\begin{aligned}
X^*[k] &= \sum_{n=0}^{N-1} x[n]\, W_N^{-kn} \\
&= \sum_{n=0}^{N-1} x[n]\, W_N^{nN}\, W_N^{-kn} \\
&= \sum_{n=0}^{N-1} x[n]\, W_N^{n(N-k)} \\
&= X[N-k]
\end{aligned}$$

$$X[k] = \sum_{n=0}^{N-1} x[n]\, W_N^{kn}$$

$$W_N^{nN} = 1$$

$$W_N^{k-N} = W_N^{k}$$

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N=8 DFT Matrix

$$X[k] = \sum_{n=0}^{7} W_8^{kn}\, x[n] \qquad\qquad W_8^{kn} = e^{-j(\frac{2\pi}{8})kn}$$

$$\begin{bmatrix} X[0] \\ X[1] \\ X[2] \\ X[3] \\ X[4] \\ X[5] \\ X[6] \\ X[7] \end{bmatrix} = \begin{bmatrix}
W_8^{0} & W_8^{0} & W_8^{0} & W_8^{0} & W_8^{0} & W_8^{0} & W_8^{0} & W_8^{0} \\
W_8^{0} & W_8^{1} & W_8^{2} & W_8^{3} & W_8^{4} & W_8^{5} & W_8^{6} & W_8^{7} \\
W_8^{0} & W_8^{2} & W_8^{4} & W_8^{6} & W_8^{8} & W_8^{10} & W_8^{12} & W_8^{14} \\
W_8^{0} & W_8^{3} & W_8^{6} & W_8^{9} & W_8^{12} & W_8^{15} & W_8^{18} & W_8^{21} \\
W_8^{0} & W_8^{4} & W_8^{8} & W_8^{12} & W_8^{16} & W_8^{20} & W_8^{24} & W_8^{28} \\
W_8^{0} & W_8^{5} & W_8^{10} & W_8^{15} & W_8^{20} & W_8^{25} & W_8^{30} & W_8^{35} \\
W_8^{0} & W_8^{6} & W_8^{12} & W_8^{18} & W_8^{24} & W_8^{30} & W_8^{36} & W_8^{42} \\
W_8^{0} & W_8^{7} & W_8^{14} & W_8^{21} & W_8^{28} & W_8^{35} & W_8^{42} & W_8^{49}
\end{bmatrix} \begin{bmatrix} x[0] \\ x[1] \\ x[2] \\ x[3] \\ x[4] \\ x[5] \\ x[6] \\ x[7] \end{bmatrix}$$

N=8 DFT Exponents of W

$-n \cdot k \quad mod \quad 8$

N = 8

$$W_N^{nk} = e^{-j(2\pi/N)nk}$$

$$\frac{W_N^{k\pm N}}{W_N^{k}} = \frac{e^{-j(\frac{2\pi}{N})(k\pm N)}}{e^{-j(\frac{2\pi}{N})k}} = e^{\mp j2\pi} = 1$$

$example:$
$-49 \quad mod \quad 8$
$\equiv \; -1 \quad mod \quad 8$

n k	0	1	2	3	4	5	6	7
0	0 0	0 0	0 0	0 0	0 0	0 0	0 0	0 0
1	0 0	-1 -1	-2 -2	-3 -3	-4 -4	-5 -5	-6 -6	-7 -7
2	0 0	-2 -2	-4 -4	-6 -6	-8 0	-10 -2	-12 -4	-14 -6
3	0 0	-3 -3	-6 -6	-9 -1	-12 -4	-15 -7	-18 -2	-21 -5
4	0 0	-4 -4	-8 0	-12 -4	-16 0	-20 -4	-24 0	-28 -4
5	0 0	-5 -5	-10 -2	-15 -7	-20 -4	-25 -1	-30 -6	-35 -3
6	0 0	-6 -6	-12 -4	-18 -2	-24 0	-30 -6	-36 -4	-42 -2
7	0 0	-7 -7	-14 -6	-21 -5	-28 -4	-35 -3	-42 -2	-49 -1

N=8 DFT Common Differences in Exponents of W

$-n\cdot k \mod 8$

N = 8

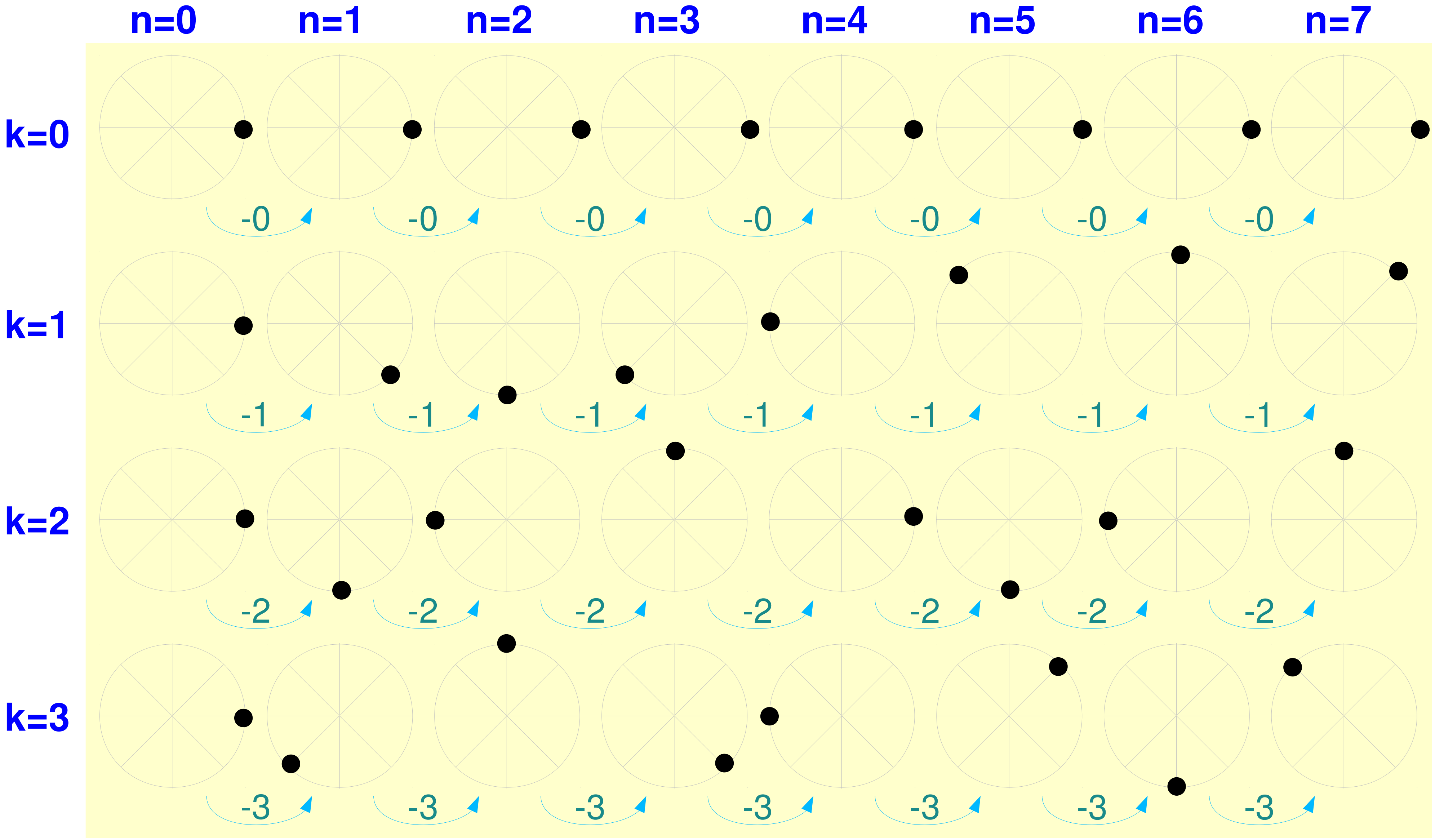
$$W_N^{nk} = e^{-j(2\pi/N)nk}$$

$$\frac{W_N^{k\pm N}}{W_N^{k}} = \frac{e^{-j(\frac{2\pi}{N})(k\pm N)}}{e^{-j(\frac{2\pi}{N})k}}$$

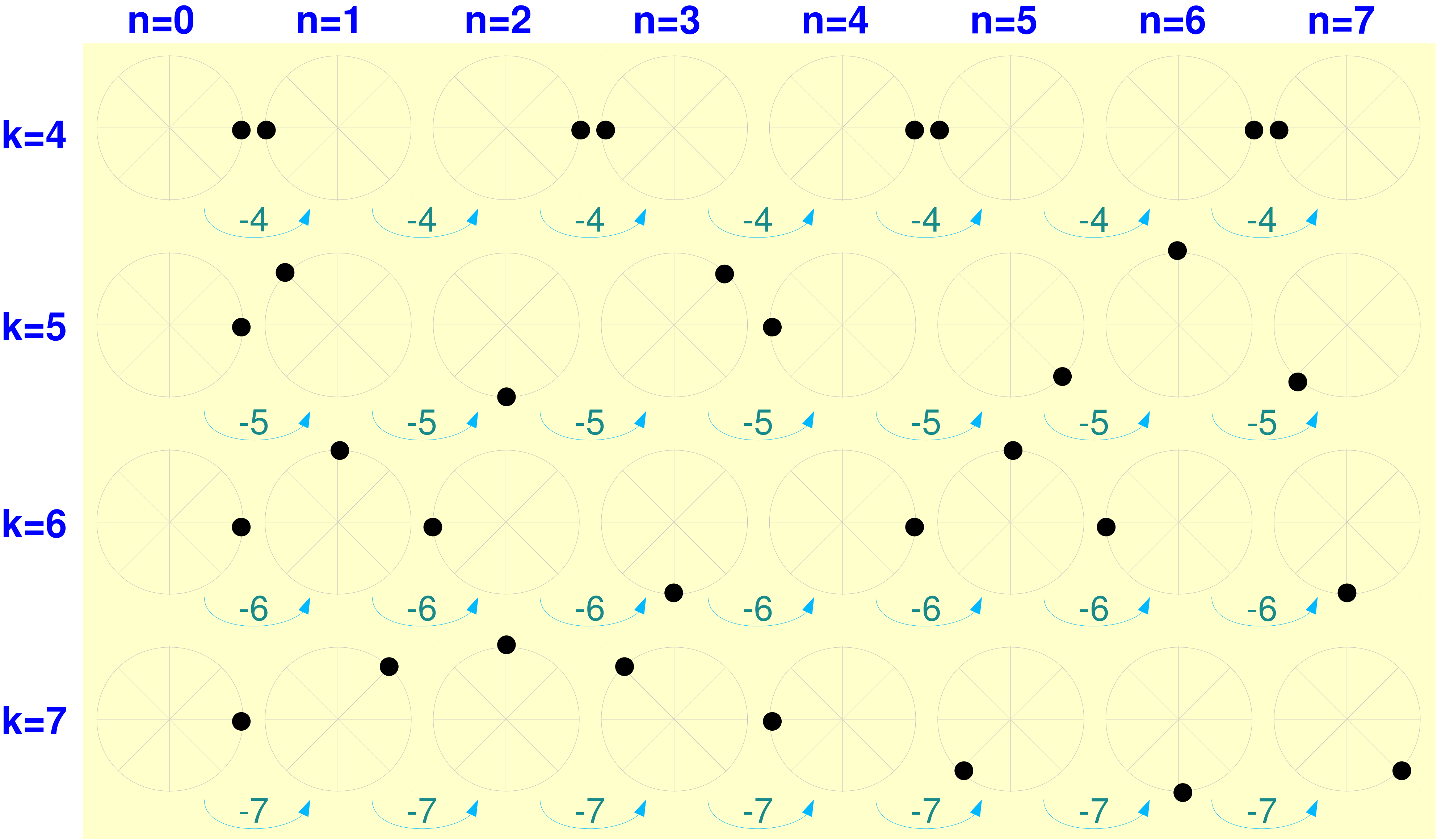
$$= e^{\mp j2\pi} = 1$$

	n=0		n=1		n=2		n=3		n=4		n=5		n=6		n=7
k=0	0	-0	0	-0	0	-0	0	-0	0	-0	0	-0	0	-0	0
k=1	0	-1	-1	-1	-2	-1	-3	-1	-4	-1	-5	-1	-6	-1	-7
k=2	0	-2	-2	-2	-4	-2	-6	-2	0	-2	-2	-2	-4	-2	-6
k=3	0	-3	-3	-3	-6	-3	-1	-3	-4	-3	-7	-3	-2	-3	-5
k=4	0	-4	-4	-4	0	-4	-4	-4	0	-4	-4	-4	0	-4	-4
k=5	0	-5	-5	-5	-2	-5	-7	-5	-4	-5	-1	-5	-6	-5	-3
k=6	0	-6	-6	-6	-4	-6	-2	-6	0	-6	-6	-6	-4	-6	-2
k=7	0	-7	-7	-7	-6	-7	-5	-7	-4	-7	-3	-7	-2	-7	-1

N=8 DFT Complex Factors in Angles (1)



N=8 DFT Complex Factors in Angles (2)



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  - Common Differences in Exponents of W
  - Complex Phase Factors in Angles

N=8 IDFT

$$x[n] = \frac{1}{N}\sum_{k=0}^{7} W_8^{-kn}\, X[k] \qquad\qquad W_8^{-kn} = e^{+j(\frac{2\pi}{8})kn}$$

$$\begin{bmatrix} x[0] \\ x[1] \\ x[2] \\ x[3] \\ x[4] \\ x[5] \\ x[6] \\ x[7] \end{bmatrix} = \frac{1}{N} \begin{bmatrix} W_8^{0} & W_8^{0} & W_8^{0} & W_8^{0} & W_8^{0} & W_8^{0} & W_8^{0} & W_8^{0} \\ W_8^{0} & W_8^{-1} & W_8^{-2} & W_8^{-3} & W_8^{-4} & W_8^{-5} & W_8^{-6} & W_8^{-7} \\ W_8^{0} & W_8^{-2} & W_8^{-4} & W_8^{-6} & W_8^{-8} & W_8^{-10} & W_8^{-12} & W_8^{-14} \\ W_8^{0} & W_8^{-3} & W_8^{-6} & W_8^{-9} & W_8^{-12} & W_8^{-15} & W_8^{-18} & W_8^{-21} \\ W_8^{0} & W_8^{-4} & W_8^{-8} & W_8^{-12} & W_8^{-16} & W_8^{-20} & W_8^{-24} & W_8^{-28} \\ W_8^{0} & W_8^{-5} & W_8^{-10} & W_8^{-15} & W_8^{-20} & W_8^{-25} & W_8^{-30} & W_8^{-35} \\ W_8^{0} & W_8^{-6} & W_8^{-12} & W_8^{-18} & W_8^{-24} & W_8^{-30} & W_8^{-36} & W_8^{-42} \\ W_8^{0} & W_8^{-7} & W_8^{-14} & W_8^{-21} & W_8^{-28} & W_8^{-35} & W_8^{-42} & W_8^{-49} \end{bmatrix} \begin{bmatrix} X[0] \\ X[1] \\ X[2] \\ X[3] \\ X[4] \\ X[5] \\ X[6] \\ X[7] \end{bmatrix}$$

N=8 IDFT Exponents of W

$$+n\cdot k \ mod \ \ 8$$

N = 8

$$W_N^{-nk} =$$

$$e^{+j(2\pi/N)nk}$$

$$\frac{W_N^{-k\pm N}}{W_N^{-k}} = \frac{e^{+j(\frac{2\pi}{N})(k\pm N)}}{e^{+j(\frac{2\pi}{N})k}}$$

$$= e^{\pm j2\pi} = 1$$

example:
$49 \ mod \ 8$
$\equiv 1 \ mod \ 8$

k n	0	1	2	3	4	5	6	7
0	0 0	0 0	0 0	0 0	0 0	0 0	0 0	0 0
1	0 0	1 1	2 2	3 3	4 4	5 5	6 6	7 7
2	0 0	2 2	4 4	6 6	8 0	10 2	12 4	14 6
3	0 0	3 3	6 6	9 1	12 4	15 7	18 2	21 5
4	0 0	4 4	8 0	12 4	16 0	20 4	24 0	28 4
5	0 0	5 5	10 2	15 7	20 4	25 1	30 6	35 3
6	0 0	6 6	12 4	18 2	24 0	30 6	36 4	42 2
7	0 0	7 7	14 6	21 5	28 4	35 3	42 2	49 1

N=8 IDFT Common Differences in Exponents of W

$+n \cdot k \;\; mod \;\; 8$

N = 8

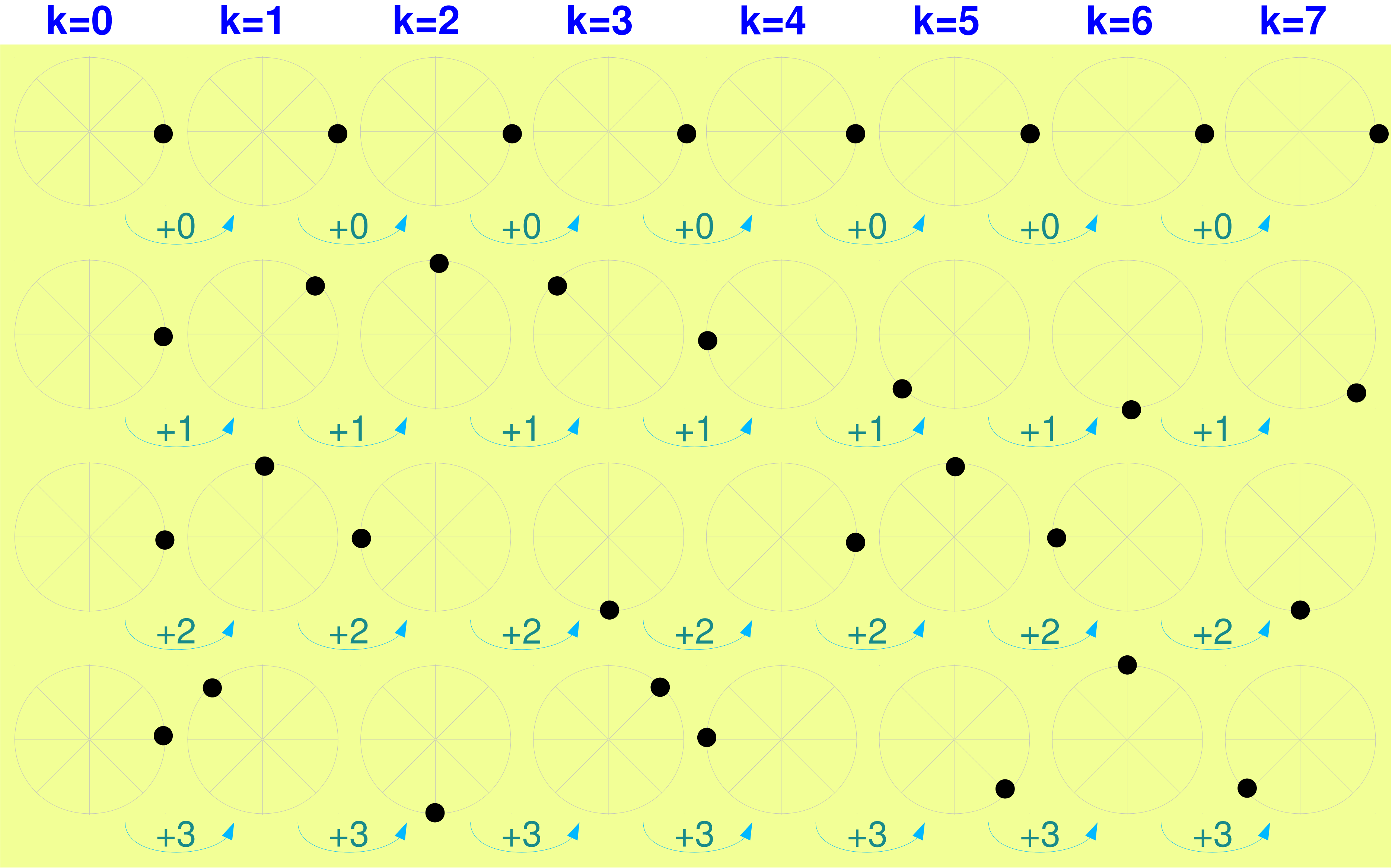
$$W_N^{-nk} = e^{+j(2\pi/N)nk}$$

$$\frac{W_N^{-k \pm N}}{W_N^{-k}} = \frac{e^{+j(\frac{2\pi}{N})(k \pm N)}}{e^{+j(\frac{2\pi}{N})k}}$$

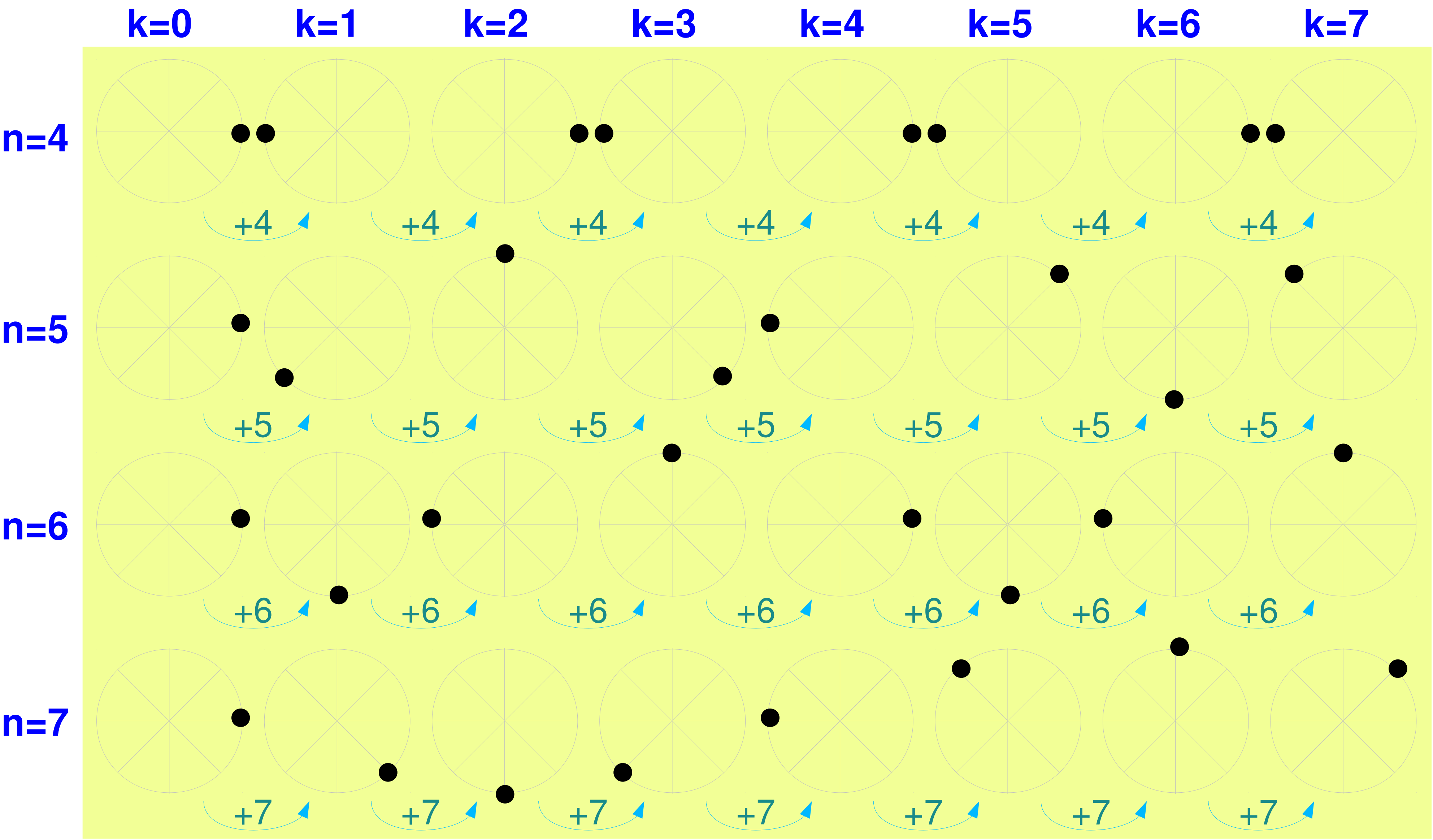
$$= e^{\pm j2\pi} = 1$$

	k=0		k=1		k=2		k=3		k=4		k=5		k=6		k=7
n=0	0	+0	0	+0	0	+0	0	+0	0	+0	0	+0	0	+0	0
n=1	0	+1	+1	+1	+2	+1	+3	+1	+4	+1	+5	+1	+6	+1	+7
n=2	0	+2	+2	+2	+4	+2	+6	+2	0	+2	+2	+2	+4	+2	+6
n=3	0	+3	+3	+3	+6	+3	+1	+3	+4	+3	+7	+3	+2	+3	+5
n=4	0	+4	+4	+4	0	+4	+4	+4	0	+4	+4	+4	0	+4	+4
n=5	0	+5	+5	+5	+2	+5	+7	+5	+4	+5	+1	+5	+6	+5	+3
n=6	0	+6	+6	+6	+4	+6	+2	+6	0	+6	+6	+6	+4	+6	+2
n=7	0	+7	+7	+7	+6	+7	+5	+7	+4	+7	+3	+7	+2	+7	+1

N=8 IDFT Complex Factors in Angles (1)



N=8 IDFT Complex Factors in Angles (2)



References

[1] http://en.wikipedia.org/
[2] J.H. McClellan, et al., Signal Processing First, Pearson Prentice Hall, 2003