DFT Frequency (9A)

- Each Row of the DFT Matrix
-

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7/6/10

Please send corrections (or suggestions) to youngwlim@hotmail.com.

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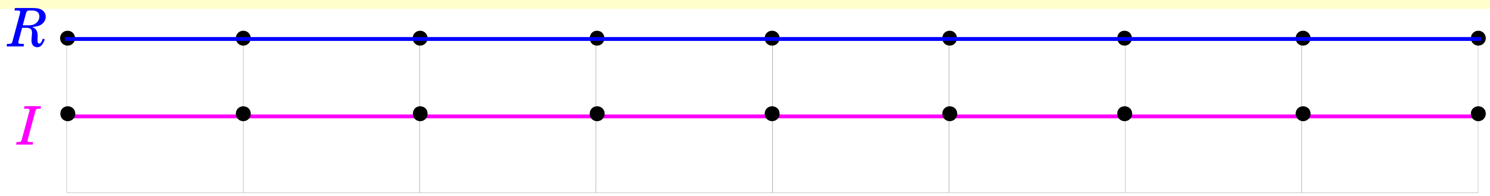
N=8 DFT : The 1st Row of the DFT Matrix

$$\left(e^{-j\cdot\frac{\pi}{4}\cdot 0} \quad e^{-j\cdot\frac{\pi}{4}\cdot 0} \quad e^{-j\cdot\frac{\pi}{4}\cdot 0} \quad e^{-j\cdot\frac{\pi}{4}\cdot 0} \quad e^{-j\cdot\frac{\pi}{4}\cdot 0} \quad e^{-j\cdot\frac{\pi}{4}\cdot 0} \quad e^{-j\cdot\frac{\pi}{4}\cdot 0} \quad e^{-j\cdot\frac{\pi}{4}\cdot 0} \right)$$



0 cycle

$$W_8^{kn} = e^{-j\left(\frac{2\pi}{8}\right)kn} \qquad k = 0, \quad n = 0, 1, \ldots, 7$$

R ➡ *sampled values of* $\cos(-\omega t) = \cos(\omega t)$

I ➡ *sampled values of* $\sin(-\omega t) = -\sin(\omega t)$

$$\omega t = 2\pi f t$$
$$2\pi \cdot \left(\tfrac{0}{8}\right) \cdot f_s \cdot t$$

$\boldsymbol{X}[0]$ *measures how much of the above signal component is present in* $\boldsymbol{x}$.

Sampling Time τ

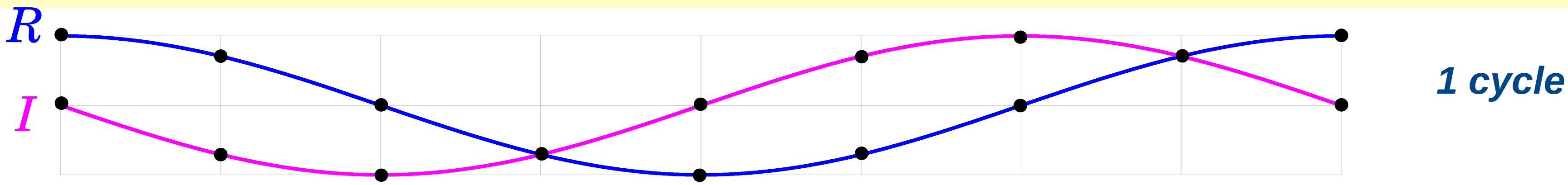
Sampling Frequency $f_s = \frac{1}{\tau}$

Sequence Time Length $T = N\tau$

Zero Frequency

N=8 DFT : The 2nd Row of the DFT Matrix

$$\begin{pmatrix} e^{-j\cdot\frac{\pi}{4}\cdot 0} & e^{-j\cdot\frac{\pi}{4}\cdot 1} & e^{-j\cdot\frac{\pi}{4}\cdot 2} & e^{-j\cdot\frac{\pi}{4}\cdot 3} & e^{-j\cdot\frac{\pi}{4}\cdot 4} & e^{-j\cdot\frac{\pi}{4}\cdot 5} & e^{-j\cdot\frac{\pi}{4}\cdot 6} & e^{-j\cdot\frac{\pi}{4}\cdot 7} \end{pmatrix}$$



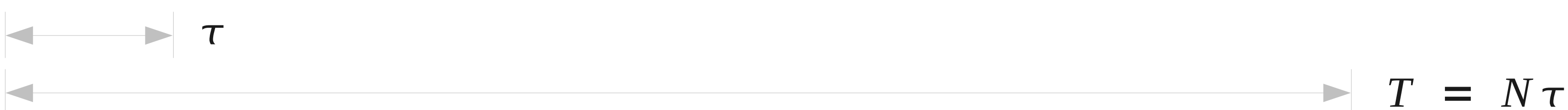
$$W_8^{kn} = e^{-j\left(\frac{2\pi}{8}\right)kn} \qquad k = 1, \quad n = 0, 1, \ldots, 7$$

R ➡ *sampled values of* $\cos(-\omega t) = \cos(\omega t)$

I ➡ *sampled values of* $\sin(-\omega t) = -\sin(\omega t)$

$$\omega t = 2\pi f t$$
$$2\pi\cdot\left(\tfrac{1}{8}\right)\cdot f_s\cdot t$$

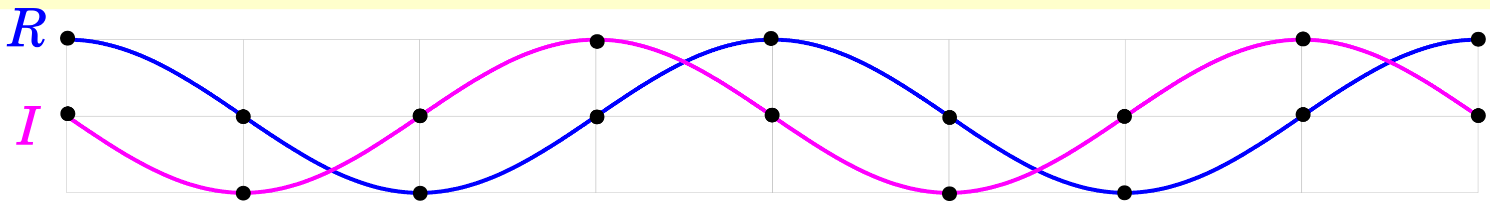
$\boldsymbol{X}[1]$ *measures how much of the above signal component is present in* $\boldsymbol{x}$.



Sampling Time	τ	*Sampling Frequency*	$f_s = \dfrac{1}{\tau}$
Sequence Time Length	$T = N\tau$	*1st Harmonic Freq*	$f_1 = \dfrac{1}{T} = \dfrac{1}{N\tau} = \dfrac{f_s}{N}$

N=8 DFT : The 3rd Row of the DFT Matrix

$$\begin{pmatrix} e^{-j\cdot\frac{\pi}{4}\cdot 0} & e^{-j\cdot\frac{\pi}{4}\cdot 2} & e^{-j\cdot\frac{\pi}{4}\cdot 4} & e^{-j\cdot\frac{\pi}{4}\cdot 6} & e^{-j\cdot\frac{\pi}{4}\cdot 0} & e^{-j\cdot\frac{\pi}{4}\cdot 2} & e^{-j\cdot\frac{\pi}{4}\cdot 4} & e^{-j\cdot\frac{\pi}{4}\cdot 6} \end{pmatrix}$$



2 cycles

$$W_8^{kn} = e^{-j\left(\frac{2\pi}{8}\right)kn} \qquad k = 2, \quad n = 0, 1, \ldots, 7$$

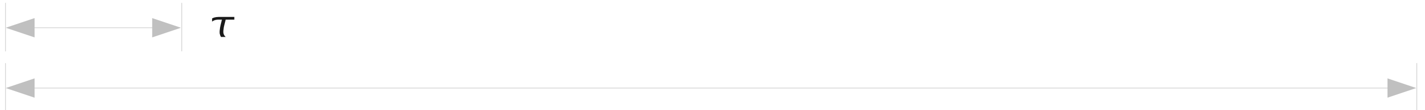
R ➡ *sampled values of* $\cos(-\omega t) = \cos(\omega t)$

I ➡ *sampled values of* $\sin(-\omega t) = -\sin(\omega t)$

$\omega t = 2\pi f t$

$2\pi\cdot\left(\frac{2}{8}\right)\cdot f_s\cdot t$

$\mathbf{X}[2]$ *measures how much of the above signal component is present in* $\mathbf{x}$.



Sampling Time τ

Sampling Frequency $f_s = \frac{1}{\tau}$

Sequence Time Length $T = N\tau$

2nd Harmonic Freq $f_2 = \frac{2}{T} = \frac{2}{N\tau} = \frac{2 f_s}{N}$

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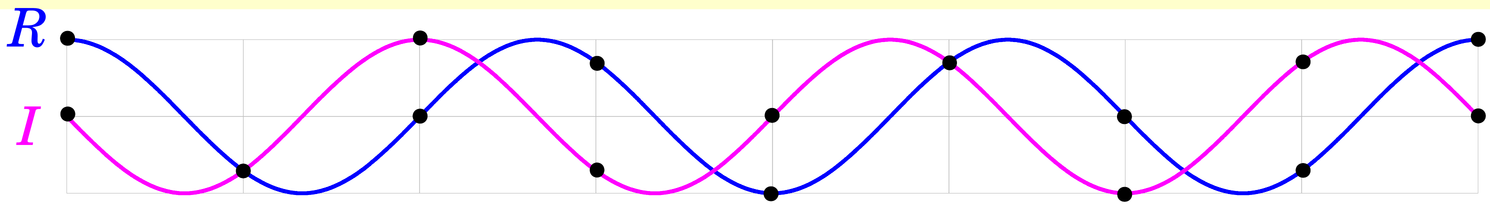
N=8 DFT : The 4th Row of the DFT Matrix

$$\begin{pmatrix} e^{-j\cdot\frac{\pi}{4}\cdot 0} & e^{-j\cdot\frac{\pi}{4}\cdot 3} & e^{-j\cdot\frac{\pi}{4}\cdot 6} & e^{-j\cdot\frac{\pi}{4}\cdot 1} & e^{-j\cdot\frac{\pi}{4}\cdot 4} & e^{-j\cdot\frac{\pi}{4}\cdot 7} & e^{-j\cdot\frac{\pi}{4}\cdot 2} & e^{-j\cdot\frac{\pi}{4}\cdot 5} \end{pmatrix}$$



3 cycles

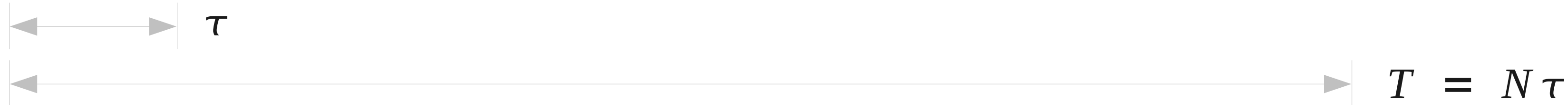
$$W_8^{kn} = e^{-j(\frac{2\pi}{8})kn} \qquad k = 3, \quad n = 0, 1, \ldots, 7$$

R ➡ *sampled values of* $\cos(-\omega t) = \cos(\omega t)$

I ➡ *sampled values of* $\sin(-\omega t) = -\sin(\omega t)$

$$\omega t = 2\pi f t$$
$$2\pi\cdot(\tfrac{3}{8})\cdot f_s\cdot t$$

$\boldsymbol{X}[3]$ *measures how much of the above signal component is present in* $\boldsymbol{x}$.



Sampling Time τ

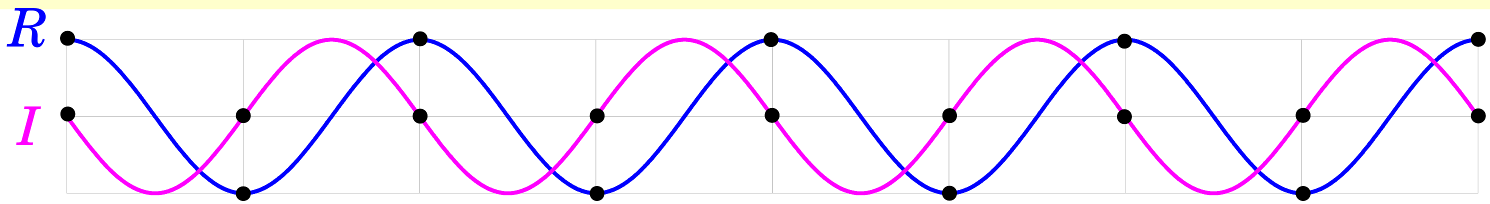
Sampling Frequency $f_s = \dfrac{1}{\tau}$

Sequence Time Length $T = N\tau$

3rd Harmonic Freq $f_3 = \dfrac{3}{T} = \dfrac{3}{N\tau} = \dfrac{3 f_s}{N}$

N=8 DFT : The 5th Row of the DFT Matrix

$$\begin{pmatrix} e^{-j\cdot\frac{\pi}{4}\cdot 0} & e^{-j\cdot\frac{\pi}{4}\cdot 4} & e^{-j\cdot\frac{\pi}{4}\cdot 0} & e^{-j\cdot\frac{\pi}{4}\cdot 4} & e^{-j\cdot\frac{\pi}{4}\cdot 0} & e^{-j\cdot\frac{\pi}{4}\cdot 4} & e^{-j\cdot\frac{\pi}{4}\cdot 0} & e^{-j\cdot\frac{\pi}{4}\cdot 4} \end{pmatrix}$$



4 cycles

$$W_8^{kn} = e^{-j\left(\frac{2\pi}{8}\right)kn} \qquad k = 4, \quad n = 0, 1, \ldots, 7$$

R ➡ *sampled values of* $\cos(-\omega t) = \cos(\omega t)$

I ➡ *sampled values of* $\sin(-\omega t) = -\sin(\omega t)$

$$\omega t = 2\pi f t$$
$$2\pi\cdot\left(\tfrac{4}{8}\right)\cdot f_s\cdot t$$

$X[4]$ *measures how much of the above signal component is present in* x.

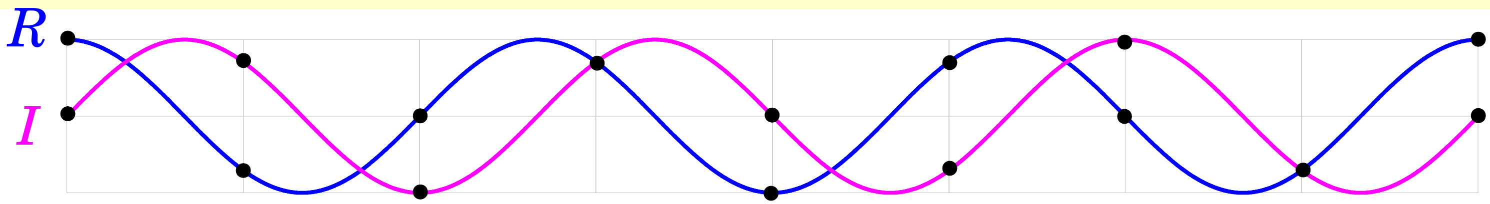
τ

$T = N\tau$

Sampling Time	τ	*Sampling Frequency*	$f_s = \frac{1}{\tau}$
Sequence Time Length	$T = N\tau$	*4th Harmonic Freq*	$f_4 = \frac{4}{T} = \frac{4}{N\tau} = \frac{4 f_s}{N}$

N=8 DFT : The 6th Row of the DFT Matrix

$$\begin{pmatrix} e^{-j\cdot\frac{\pi}{4}\cdot 0} & e^{-j\cdot\frac{\pi}{4}\cdot 5} & e^{-j\cdot\frac{\pi}{4}\cdot 2} & e^{-j\cdot\frac{\pi}{4}\cdot 7} & e^{-j\cdot\frac{\pi}{4}\cdot 4} & e^{-j\cdot\frac{\pi}{4}\cdot 1} & e^{-j\cdot\frac{\pi}{4}\cdot 6} & e^{-j\cdot\frac{\pi}{4}\cdot 3} \end{pmatrix}$$



3 cycles

$$W_8^{kn} = e^{-j(\frac{2\pi}{8})kn} \qquad k = 5, \quad n = 0, 1, \ldots, 7$$

R ➡ *sampled values of* $\cos(\omega' t) = \cos(-(-\omega)t)$

I ➡ *sampled values of* $\sin(\omega' t) = \sin(-(-\omega)t)$

$$-\omega t = -2\pi f t$$

$$2\pi\cdot(\tfrac{-3}{8})\cdot f_s\cdot t$$

$X[5]$ *measures how much of the above signal component is present in* x.

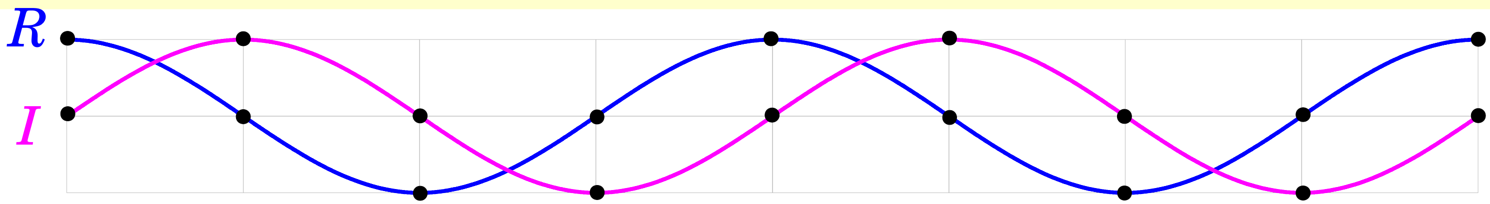
τ

$T = N\tau$

Sampling Time	τ	Sampling Frequency	$f_s = \frac{1}{\tau}$
Sequence Time Length	$T = N\tau$	3rd Harmonic Freq	$f_3 = \frac{3}{T} = \frac{3}{N\tau} = \frac{3 f_s}{N}$

N=8 DFT : The 7th Row of the DFT Matrix

$$\begin{pmatrix} e^{-j\cdot\frac{\pi}{4}\cdot 0} & e^{-j\cdot\frac{\pi}{4}\cdot 6} & e^{-j\cdot\frac{\pi}{4}\cdot 4} & e^{-j\cdot\frac{\pi}{4}\cdot 2} & e^{-j\cdot\frac{\pi}{4}\cdot 0} & e^{-j\cdot\frac{\pi}{4}\cdot 6} & e^{-j\cdot\frac{\pi}{4}\cdot 4} & e^{-j\cdot\frac{\pi}{4}\cdot 2} \end{pmatrix}$$



2 cycles

$$W_8^{kn} = e^{-j\left(\frac{2\pi}{8}\right)kn} \qquad k = 2, \quad n = 0, 1, \ldots, 7$$

R ➡ *sampled values of* $\cos(\omega' t) = \cos(-(-\omega)t)$

I ➡ *sampled values of* $\sin(\omega' t) = \sin(-(-\omega)t)$

$$-\omega t = -2\pi f t$$

$$2\pi\cdot\left(\tfrac{-2}{8}\right)\cdot f_s\cdot t$$

$\boldsymbol{X}[6]$ *measures how much of the above signal component is present in* $\boldsymbol{x}$.

Sampling Time τ

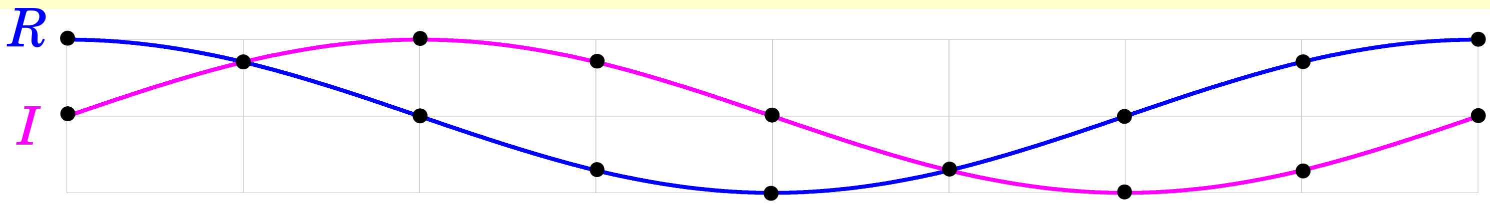
Sampling Frequency $f_s = \frac{1}{\tau}$

Sequence Time Length $T = N\tau$

2nd Harmonic Freq $f_2 = \frac{2}{T} = \frac{2}{N\tau} = \frac{2 f_s}{N}$

N=8 DFT : The 8th Row of the DFT Matrix

$$\begin{pmatrix} e^{-j\cdot\frac{\pi}{4}\cdot 0} & e^{-j\cdot\frac{\pi}{4}\cdot 7} & e^{-j\cdot\frac{\pi}{4}\cdot 6} & e^{-j\cdot\frac{\pi}{4}\cdot 5} & e^{-j\cdot\frac{\pi}{4}\cdot 4} & e^{-j\cdot\frac{\pi}{4}\cdot 3} & e^{-j\cdot\frac{\pi}{4}\cdot 2} & e^{-j\cdot\frac{\pi}{4}\cdot 1} \end{pmatrix}$$



1 cycle

$$W_8^{kn} = e^{-j(\frac{2\pi}{8})kn} \qquad k = 7, \quad n = 0, 1, \dots, 7$$

R ➡ *sampled values of* $\cos(\omega' t) = \cos(-(-\omega)t)$

I ➡ *sampled values of* $\sin(\omega' t) = \sin(-(-\omega)t)$

$$-\omega t = -2\pi f t$$

$$2\pi\cdot(\tfrac{-1}{8})\cdot f_s \cdot t$$

$X[7]$ *measures how much of the above signal component is present in* x.

τ

$T = N\tau$

Sampling Time	τ	Sampling Frequency	$f_s = \frac{1}{\tau}$
Sequence Time Length	$T = N\tau$	1st Harmonic Freq	$f_1 = \frac{1}{T} = \frac{1}{N\tau} = \frac{f_s}{N}$

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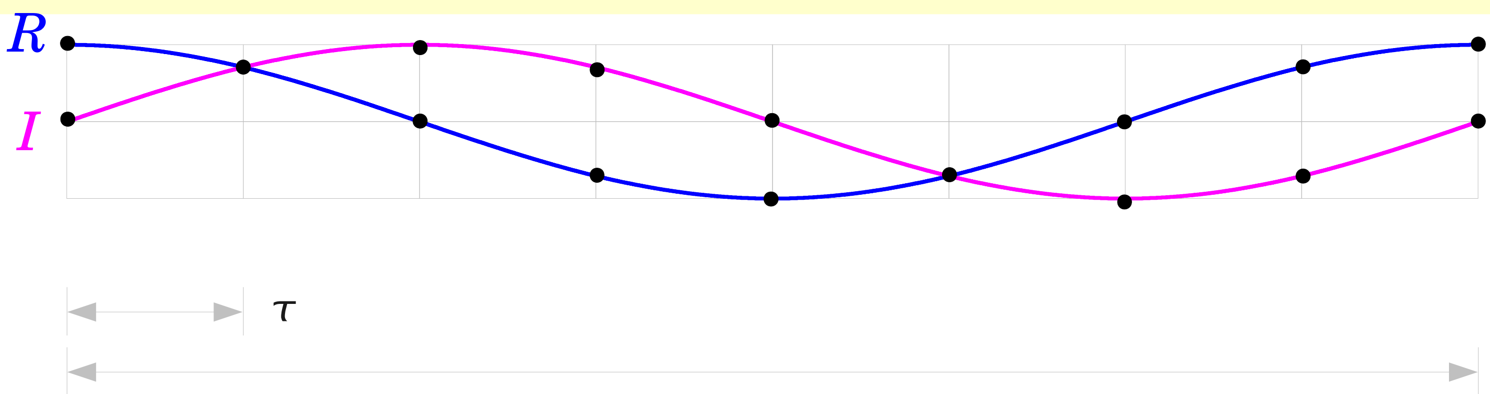
Fundamental Frequency

$$\left(e^{+j\cdot\frac{\pi}{4}\cdot 0} \quad e^{+j\cdot\frac{\pi}{4}\cdot 1} \quad e^{+j\cdot\frac{\pi}{4}\cdot 2} \quad e^{+j\cdot\frac{\pi}{4}\cdot 3} \quad e^{+j\cdot\frac{\pi}{4}\cdot 4} \quad e^{+j\cdot\frac{\pi}{4}\cdot 5} \quad e^{+j\cdot\frac{\pi}{4}\cdot 6} \quad e^{+j\cdot\frac{\pi}{4}\cdot 7} \right)$$



1 cycle

$T = N\tau$

Sampling Time τ

Sampling Frequency $f_s = \frac{1}{\tau}$

Sequence Time Length $T = N\tau$

1st Harmonic Freq $f_1 = \frac{1}{T} = \frac{1}{N\tau} = \frac{f_s}{N}$

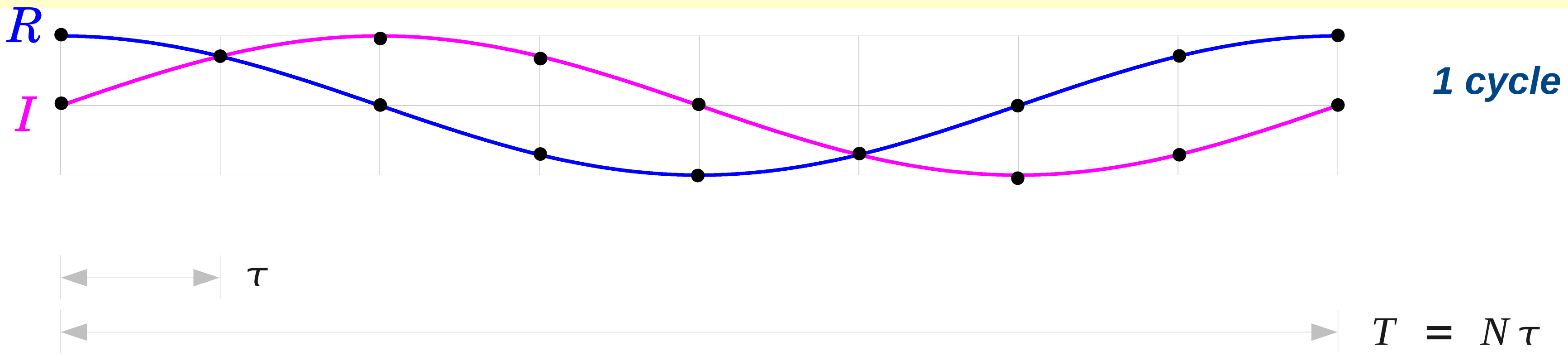
$$f_2 = 2\cdot f_1$$
$$f_3 = 3\cdot f_1$$
$$\cdots$$
$$f_{N-1} = (N-1)\cdot f_1$$

Fundamental Frequency f_o

The Lowest Frequency in a harmonic series. $f_0 = f_1 = \frac{f_s}{N}$

Normalized Frequency

$$\left(e^{+j\cdot\frac{\pi}{4}\cdot 0} \quad e^{+j\cdot\frac{\pi}{4}\cdot 1} \quad e^{+j\cdot\frac{\pi}{4}\cdot 2} \quad e^{+j\cdot\frac{\pi}{4}\cdot 3} \quad e^{+j\cdot\frac{\pi}{4}\cdot 4} \quad e^{+j\cdot\frac{\pi}{4}\cdot 5} \quad e^{+j\cdot\frac{\pi}{4}\cdot 6} \quad e^{+j\cdot\frac{\pi}{4}\cdot 7} \right)$$



Sampling Time τ

Sequence Time Length $T = N\tau$

Sampling Frequency $f_s = \frac{1}{\tau}$ (samples per second)

1st Harmonic Freq $f_1 = \frac{1}{T} = \frac{1}{N\tau} = \frac{f_s}{N}$

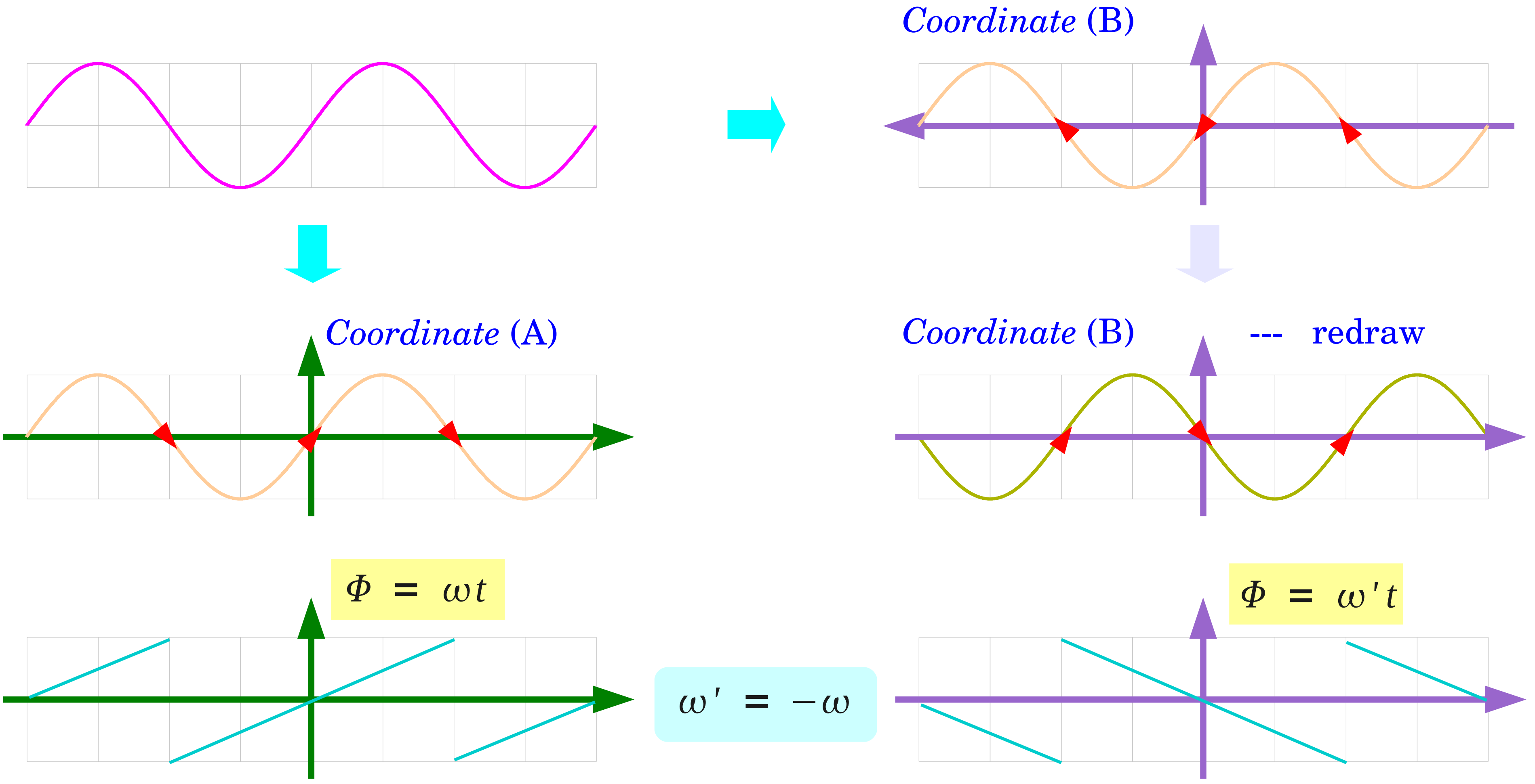
$f_1 = 1\cdot f_1$	$1/N$
$f_2 = 2\cdot f_1$	$2/N$
$f_3 = 3\cdot f_1$	$3/N$
$\dots$	
$f_{N-1} = (N-1)\cdot f_1$	$(N-1)/N$

$n = 0,\ 1,\ 2,\ \dots,\ N-1$

Normalized Frequency (cycles per sample)

$$f_n = \frac{n\cdot f_s}{N} \quad \Rightarrow \quad \frac{f_n}{f_s} = \frac{n}{N}$$

Negative Frequency (1)



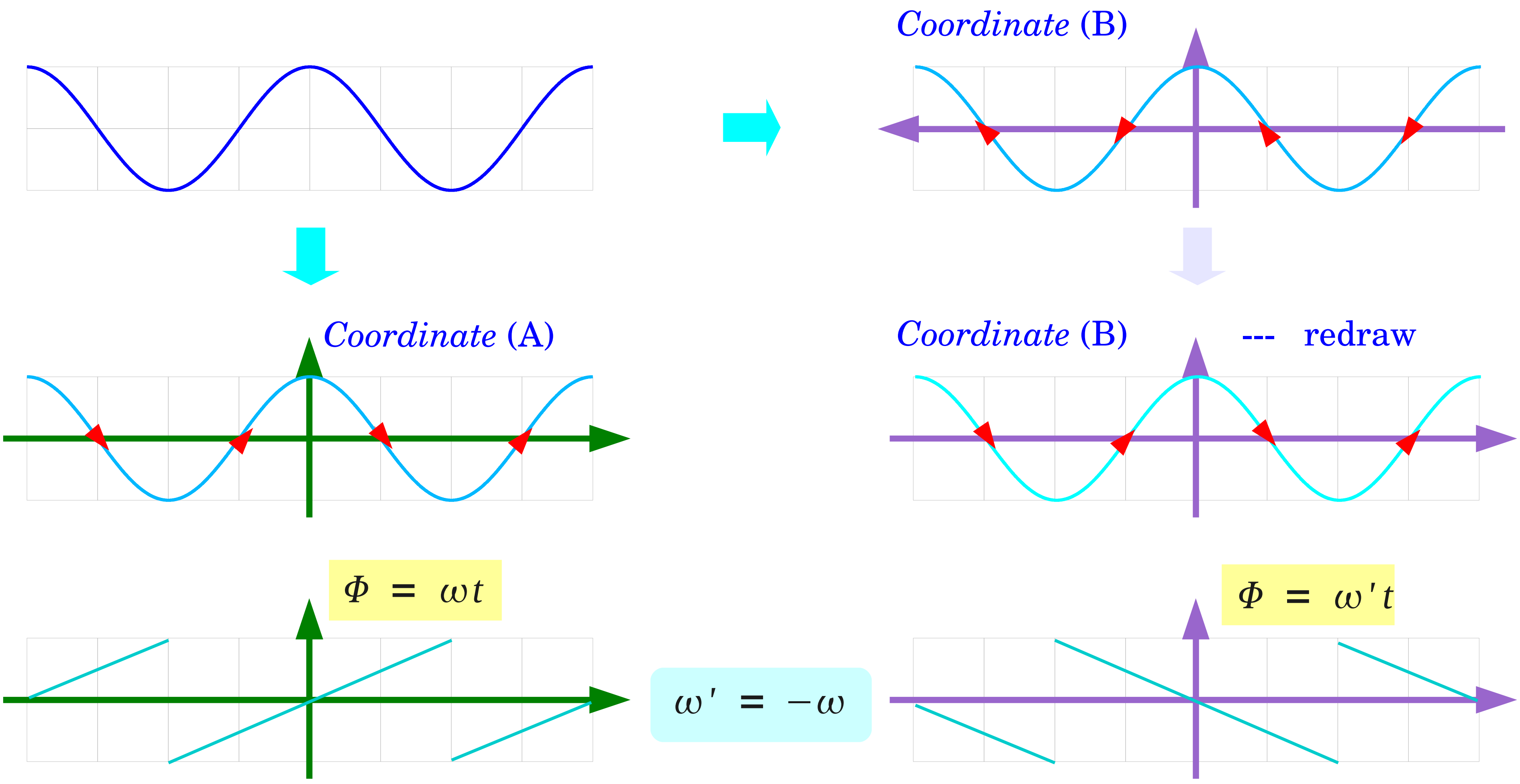
$\Phi = \omega t$

$\Phi = \omega' t$

$\omega' = -\omega$

$\sin(\omega' t) = -\sin(-\omega t)$

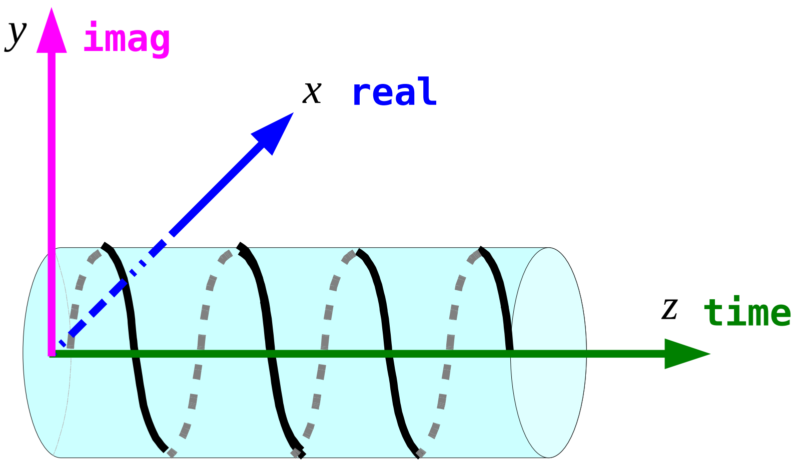
Negative Frequency (2)

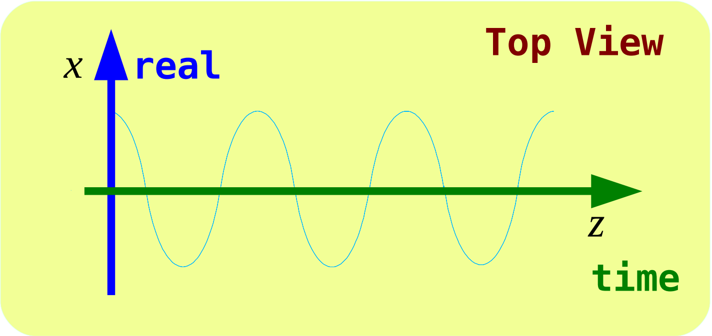


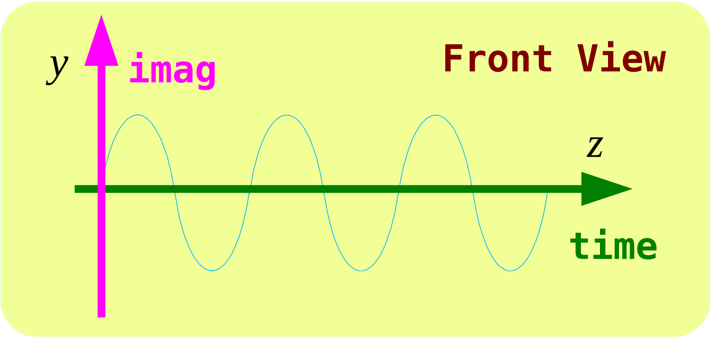
$$\cos(\omega' t) = \cos(-\omega t)$$

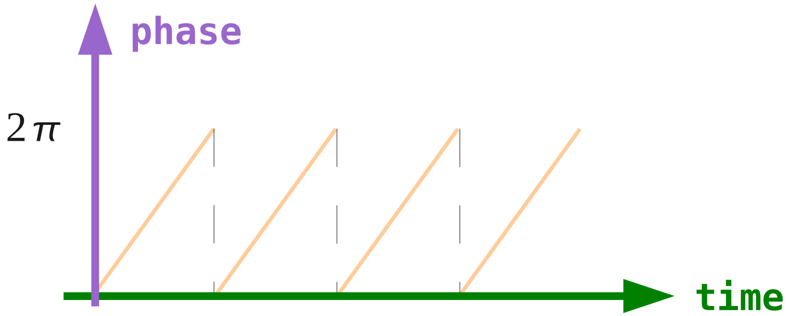
Euler Equation (1)

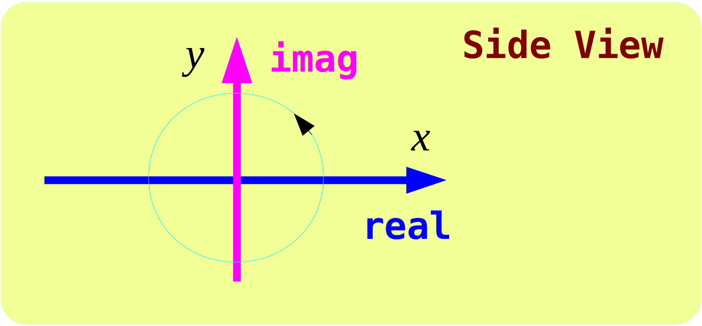
$$e^{+j\omega t} = \cos\omega t + j\sin\omega t$$



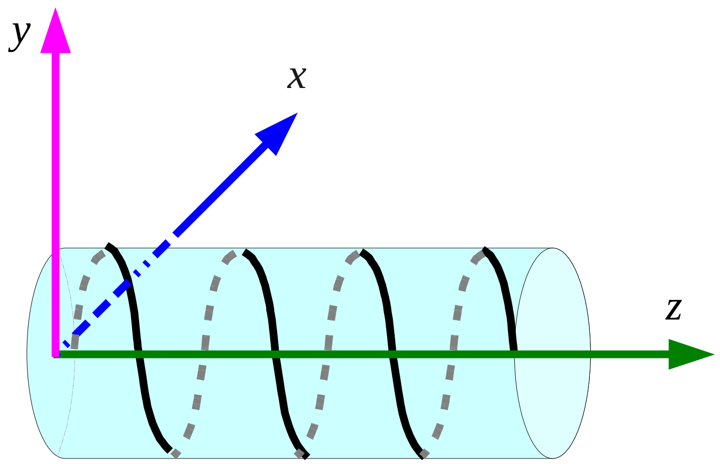


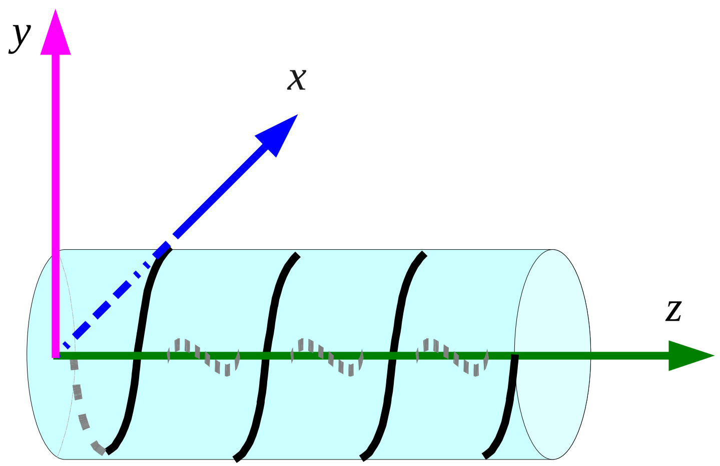




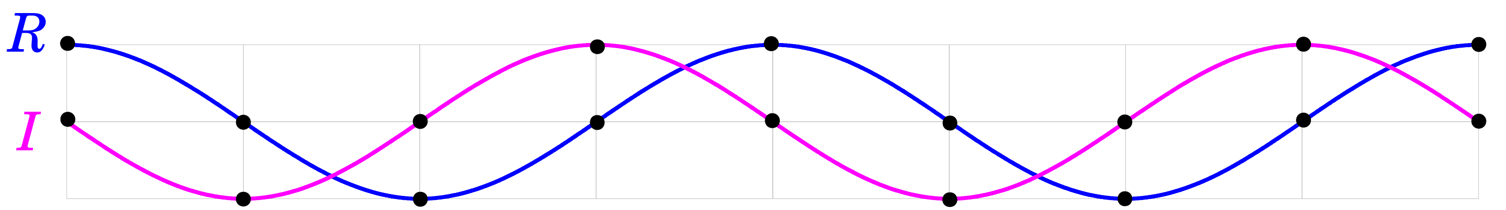


Negative Frequency (3)





Negative Frequency



$$W_8^{kn} = e^{-j\left(\frac{2\pi}{8}\right)kn} \qquad k = 2, \quad n = 0, 1, \dots, 7$$

R ➡ *sampled values of* $\cos(-\omega t) = \cos(\omega t)$

I ➡ *sampled values of* $\sin(-\omega t) = -\sin(\omega t)$

$$\omega t = 2\pi f t$$
$$2\pi \cdot \left(\tfrac{2}{8}\right) \cdot f_s \cdot t$$

$\boldsymbol{X}[2]$ *measures how much of the above signal component is present in* $\boldsymbol{x}$.

τ

$T = N\tau$

Sampling Time τ

Sampling Frequency $f_s = \frac{1}{\tau}$

Sequence Time Length $T = N\tau$

2nd Harmonic Freq $f_2 = \frac{2}{T} = \frac{2}{N\tau} = \frac{2 f_s}{N}$

N=8 DFT : DFT Matrix in + or – Frequencies

$$\omega_0 = 2\pi \cdot \frac{f_s}{N}$$

0th row:	*samples of*	$\cos(\ \ 0\omega_0)t + j\cdot\sin(\ \ 0\omega_0)t$	(0 cycle)
1th row:	*samples of*	$\cos(-1\omega_0)t + j\cdot\sin(-1\omega_0)t$	(1 cycle)
2th row:	*samples of*	$\cos(-2\omega_0)t + j\cdot\sin(-2\omega_0)t$	(2 cycles)
3th row:	*samples of*	$\cos(-3\omega_0)t + j\cdot\sin(-3\omega_0)t$	(3 cycles)
4th row:	*samples of*	$\cos(-4\omega_0)t + j\cdot\sin(-4\omega_0)t$	(4 cycles)
5th row:	*samples of*	$\cos(-5\omega_0)t + j\cdot\sin(-5\omega_0)t$	(5 cycles)
6th row:	*samples of*	$\cos(-6\omega_0)t + j\cdot\sin(-6\omega_0)t$	(6 cycles)
7th row:	*samples of*	$\cos(-7\omega_0)t + j\cdot\sin(-7\omega_0)t$	(7 cycles)

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0th row:	*samples of*	$\cos(\ \ 0\omega_0)t + j\cdot\sin(\ \ 0\omega_0)t$	(0 cycle)
1th row:	*samples of*	$\cos(+7\omega_0)t + j\cdot\sin(+7\omega_0)t$	(7 cycles)
2th row:	*samples of*	$\cos(+6\omega_0)t + j\cdot\sin(+6\omega_0)t$	(6 cycles)
3th row:	*samples of*	$\cos(+5\omega_0)t + j\cdot\sin(+5\omega_0)t$	(5 cycles)
4th row:	*samples of*	$\cos(+4\omega_0)t + j\cdot\sin(+4\omega_0)t$	(4 cycles)
5th row:	*samples of*	$\cos(+3\omega_0)t + j\cdot\sin(+3\omega_0)t$	(3 cycles)
6th row:	*samples of*	$\cos(+2\omega_0)t + j\cdot\sin(+2\omega_0)t$	(2 cycles)
7th row:	*samples of*	$\cos(+1\omega_0)t + j\cdot\sin(+1\omega_0)t$	(1 cycles)

N=8 DFT : DFT Matrix in Both Frequencies

$$\omega_0 = 2\pi \cdot \frac{f_s}{N}$$

$$\begin{bmatrix}
\text{0th row: } \textit{samples of } \cos(0\omega_0)t + j\cdot\sin(0\omega_0)t \\
\text{1th row: } \textit{samples of } \cos(-1\omega_0)t + j\cdot\sin(-1\omega_0)t \\
\text{2th row: } \textit{samples of } \cos(-2\omega_0)t + j\cdot\sin(-2\omega_0)t \\
\text{3th row: } \textit{samples of } \cos(-3\omega_0)t + j\cdot\sin(-3\omega_0)t \\
\text{4th row: } \textit{samples of } \cos(-4\omega_0)t + j\cdot\sin(-4\omega_0)t \\
\text{5th row: } \textit{samples of } \cos(-5\omega_0)t + j\cdot\sin(-5\omega_0)t \\
\text{6th row: } \textit{samples of } \cos(-6\omega_0)t + j\cdot\sin(-6\omega_0)t \\
\text{7th row: } \textit{samples of } \cos(-7\omega_0)t + j\cdot\sin(-7\omega_0)t
\end{bmatrix}
\begin{matrix}
\text{(0 cycle)} \\ \text{(1 cycle)} \\ \text{(2 cycles)} \\ \text{(3 cycles)} \\ \text{(4 cycles)} \\ \text{(5 cycles)} \\ \text{(6 cycles)} \\ \text{(7 cycles)}
\end{matrix}$$

$$= \begin{bmatrix}
\text{0th row: } \textit{samples of } \cos(0\omega_0)t + j\cdot\sin(0\omega_0)t \\
\text{1th row: } \textit{samples of } \cos(-1\omega_0)t + j\cdot\sin(-1\omega_0)t \\
\text{2th row: } \textit{samples of } \cos(-2\omega_0)t + j\cdot\sin(-2\omega_0)t \\
\text{3th row: } \textit{samples of } \cos(-3\omega_0)t + j\cdot\sin(-3\omega_0)t \\
\text{4th row: } \textit{samples of } \cos(-4\omega_0)t + j\cdot\sin(-4\omega_0)t \\
\text{5th row: } \textit{samples of } \cos(+3\omega_0)t + j\cdot\sin(+3\omega_0)t \\
\text{6th row: } \textit{samples of } \cos(+2\omega_0)t + j\cdot\sin(+2\omega_0)t \\
\text{7th row: } \textit{samples of } \cos(+1\omega_0)t + j\cdot\sin(+1\omega_0)t
\end{bmatrix}
\begin{matrix}
\text{(0 cycle)} \\ \text{(1 cycle)} \\ \text{(2 cycles)} \\ \text{(3 cycles)} \\ \text{(4 cycles)} \\ \text{(3 cycles)} \\ \text{(2 cycles)} \\ \text{(1 cycles)}
\end{matrix}$$

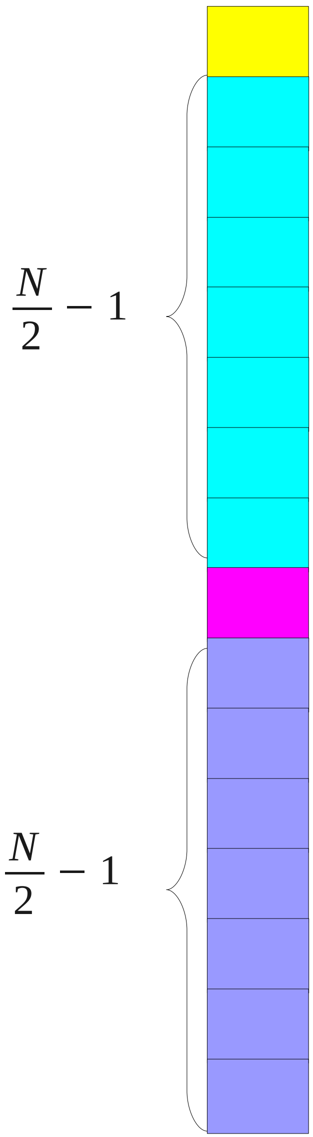
Frequency View of a DFT Matrix

		Normalized Frequency
$f = 0$	$(0\, cycle)$	0
$f = -1 * f_o$	$(1\, cycle)$	$\frac{-1}{N}$
$f = -2 * f_o$	$(2\, cycles)$	$\frac{-2}{N}$
$f = -(\frac{N}{2} - 1) * f_o$	$(\frac{N}{2} - 1\, cycles)$	$\frac{-1}{2} + \frac{1}{N}$
$f = -(\frac{N}{2}) * f_o$	$(\frac{N}{2}\, cycles)$	$\frac{-1}{2}$
$f = +(\frac{N}{2} - 1) * f_o$	$(\frac{N}{2} - 1\, cycles)$	$\frac{1}{2} - \frac{1}{N}$
$f = +2 * f_o$	$(2\, cycles)$	$\frac{2}{N}$
$f = +1 * f_o$	$(1\, cycle)$	$\frac{1}{N}$

$\frac{N}{2} - 1$ (rows $f = -1 * f_o$ through $f = -(\frac{N}{2} - 1) * f_o$)

$\frac{N}{2} - 1$ (rows $f = +(\frac{N}{2} - 1) * f_o$ through $f = +1 * f_o$)

Frequency View of a X[i] Vector



			Normalized Frequency
$X[0]$	$f = 0$	$(0\,cycle)$	0
$X[1]$	$f = +1 * f_o$	$(1\,cycle)$	$\frac{1}{N}$
$X[2]$	$f = +2 * f_o$	$(2\,cycles)$	$\frac{2}{N}$
$X[\frac{N}{2}-1]$	$f = +(\frac{N}{2}-1) * f_o$	$(\frac{N}{2}-1\,cycles)$	$\frac{1}{2}-\frac{1}{N}$
$X[\frac{N}{2}]$	$f = +(\frac{N}{2}) * f_o$	$(\frac{N}{2}\ \ cycles)$	$\frac{1}{2}$
$X[\frac{N}{2}+1]$	$f = -(\frac{N}{2}-1) * f_o$	$(\frac{N}{2}-1\,cycles)$	$\frac{-1}{2}+\frac{1}{N}$
$X[N-2]$	$f = -2 * f_o$	$(2\,cycles)$	$\frac{-2}{N}$
$X[N-1]$	$f = -1 * f_o$	$(1\,cycle)$	$\frac{-1}{N}$

References

[1] http://en.wikipedia.org/
[2] J.H. McClellan, et al., Signal Processing First, Pearson Prentice Hall, 2003
[3] A “graphical interpretation” of the DFT and FFT, by Steve Mann