DFT Analysis (9A)

- Each Row of the DFT Matrix
-

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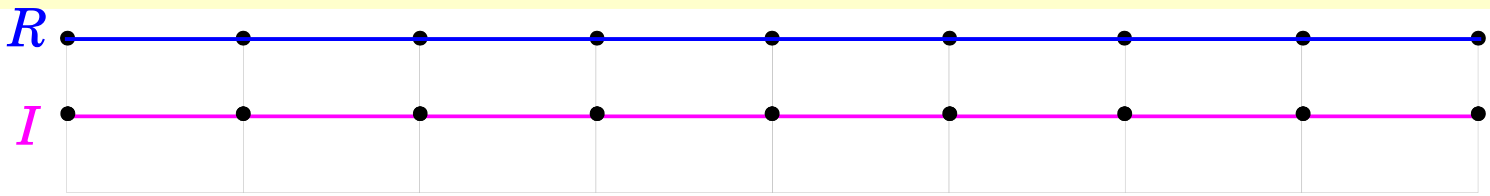
N=8 DFT : The 1st Row of the DFT Matrix

$$\begin{pmatrix} e^{-j\cdot\frac{\pi}{4}\cdot 0} & e^{-j\cdot\frac{\pi}{4}\cdot 0} & e^{-j\cdot\frac{\pi}{4}\cdot 0} & e^{-j\cdot\frac{\pi}{4}\cdot 0} & e^{-j\cdot\frac{\pi}{4}\cdot 0} & e^{-j\cdot\frac{\pi}{4}\cdot 0} & e^{-j\cdot\frac{\pi}{4}\cdot 0} & e^{-j\cdot\frac{\pi}{4}\cdot 0} \end{pmatrix}$$



0 cycle

$$W_8^{kn} = e^{-j\left(\frac{2\pi}{8}\right)kn} \qquad k = 0, \quad n = 0, 1, \ldots, 7$$

R ➡ *sampled values of* $\cos(-\omega t) = \cos(\omega t)$

I ➡ *sampled values of* $\sin(-\omega t) = -\sin(\omega t)$

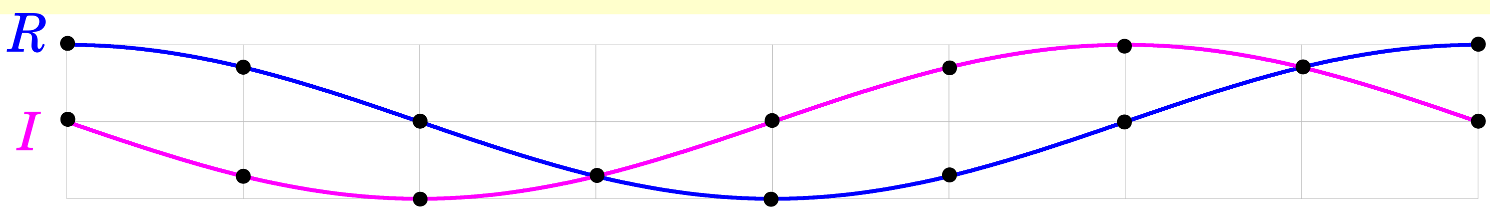
$$\omega t = 2\pi f t$$
$$2\pi\cdot\left(\tfrac{0}{8}\right)\cdot f_s\cdot t$$

$\boldsymbol{X}[0]$ *measures how much of the above signal component is present in* $\boldsymbol{x}$.

Sampling Time	τ	*Sampling Frequency*	$f_s = \frac{1}{\tau}$
Sequence Time Length	$T = N\tau$	*Zero Frequency*	

N=8 DFT : The 2nd Row of the DFT Matrix

$$\left(e^{-j\cdot\frac{\pi}{4}\cdot 0} \quad e^{-j\cdot\frac{\pi}{4}\cdot 1} \quad e^{-j\cdot\frac{\pi}{4}\cdot 2} \quad e^{-j\cdot\frac{\pi}{4}\cdot 3} \quad e^{-j\cdot\frac{\pi}{4}\cdot 4} \quad e^{-j\cdot\frac{\pi}{4}\cdot 5} \quad e^{-j\cdot\frac{\pi}{4}\cdot 6} \quad e^{-j\cdot\frac{\pi}{4}\cdot 7} \right)$$



1 cycle

$$W_8^{kn} = e^{-j\left(\frac{2\pi}{8}\right)kn} \qquad k = 1, \quad n = 0, 1, \ldots, 7$$

R ➡ *sampled values of* $\cos(-\omega t) = \cos(\omega t)$

I ➡ *sampled values of* $\sin(-\omega t) = -\sin(\omega t)$

$$\omega t = 2\pi f t$$
$$2\pi\cdot\left(\tfrac{1}{8}\right)\cdot f_s\cdot t$$

$\boldsymbol{X}[1]$ *measures how much of the above signal component is present in* $\boldsymbol{x}$.

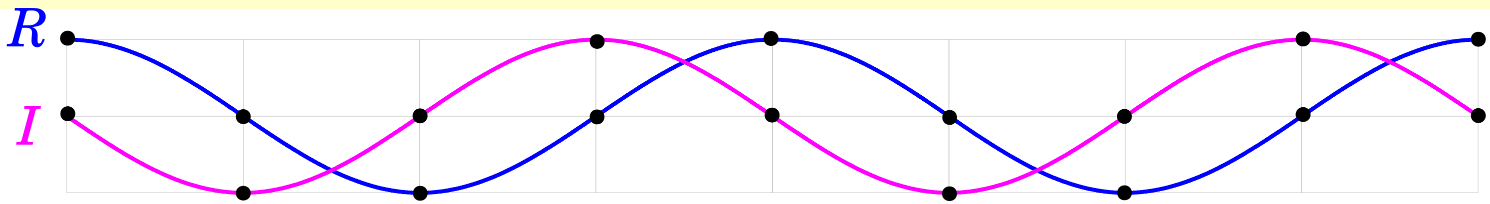
τ

$T = N\tau$

Sampling Time	τ	Sampling Frequency	$f_s = \frac{1}{\tau}$
Sequence Time Length	$T = N\tau$	1st Harmonic Freq	$f_1 = \frac{1}{T} = \frac{1}{N\tau} = \frac{f_s}{N}$

N=8 DFT : The 3rd Row of the DFT Matrix

$$\begin{pmatrix} e^{-j\cdot\frac{\pi}{4}\cdot 0} & e^{-j\cdot\frac{\pi}{4}\cdot 2} & e^{-j\cdot\frac{\pi}{4}\cdot 4} & e^{-j\cdot\frac{\pi}{4}\cdot 6} & e^{-j\cdot\frac{\pi}{4}\cdot 0} & e^{-j\cdot\frac{\pi}{4}\cdot 2} & e^{-j\cdot\frac{\pi}{4}\cdot 4} & e^{-j\cdot\frac{\pi}{4}\cdot 6} \end{pmatrix}$$



2 cycles

$$W_8^{kn} = e^{-j\left(\frac{2\pi}{8}\right)kn} \qquad k = 2, \quad n = 0, 1, \ldots, 7$$

R ➡ *sampled values of* $\cos(-\omega t) = \cos(\omega t)$

I ➡ *sampled values of* $\sin(-\omega t) = -\sin(\omega t)$

$$\omega t = 2\pi f t$$
$$2\pi\cdot\left(\tfrac{2}{8}\right)\cdot f_s\cdot t$$

$\boldsymbol{X}[2]$ *measures how much of the above signal component is present in* $\boldsymbol{x}$.

τ

$T = N\tau$

Sampling Time τ

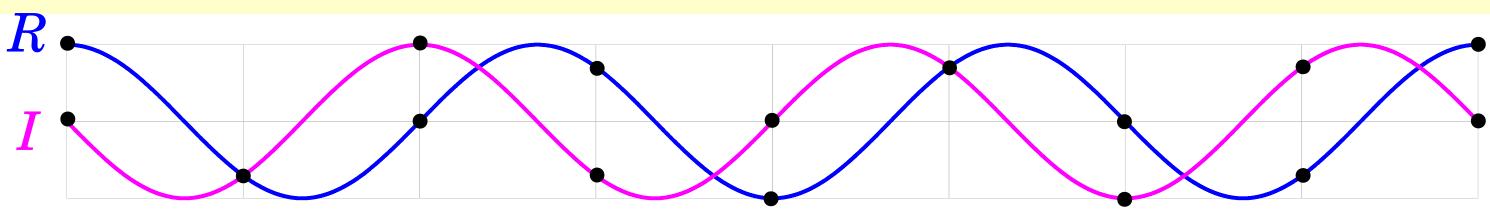
Sampling Frequency $f_s = \dfrac{1}{\tau}$

Sequence Time Length $T = N\tau$

2nd Harmonic Freq $f_2 = \dfrac{2}{T} = \dfrac{2}{N\tau} = \dfrac{2 f_s}{N}$

N=8 DFT : The 4th Row of the DFT Matrix

$$\begin{pmatrix} e^{-j\cdot\frac{\pi}{4}\cdot 0} & e^{-j\cdot\frac{\pi}{4}\cdot 3} & e^{-j\cdot\frac{\pi}{4}\cdot 6} & e^{-j\cdot\frac{\pi}{4}\cdot 1} & e^{-j\cdot\frac{\pi}{4}\cdot 4} & e^{-j\cdot\frac{\pi}{4}\cdot 7} & e^{-j\cdot\frac{\pi}{4}\cdot 2} & e^{-j\cdot\frac{\pi}{4}\cdot 5} \end{pmatrix}$$



3 cycles

$$W_8^{kn} = e^{-j\left(\frac{2\pi}{8}\right)kn} \qquad k = 3, \quad n = 0, 1, \ldots, 7$$

R ➡ *sampled values of* $\cos(-\omega t) = \cos(\omega t)$

I ➡ *sampled values of* $\sin(-\omega t) = -\sin(\omega t)$

$$\omega t = 2\pi f t$$
$$2\pi\cdot\left(\tfrac{3}{8}\right)\cdot f_s\cdot t$$

$\boldsymbol{X}[3]$ *measures how much of the above signal component is present in* $\boldsymbol{x}$.

τ

$T = N\tau$

Sampling Time τ

Sampling Frequency $f_s = \frac{1}{\tau}$

Sequence Time Length $T = N\tau$

3rd Harmonic Freq $f_3 = \frac{3}{T} = \frac{3}{N\tau} = \frac{3 f_s}{N}$

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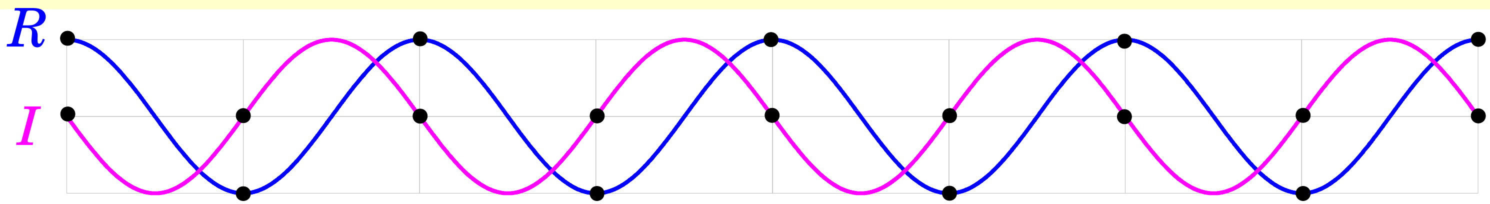
6/19/10

N=8 DFT : The 5th Row of the DFT Matrix

$$\begin{pmatrix} e^{-j\cdot\frac{\pi}{4}\cdot 0} & e^{-j\cdot\frac{\pi}{4}\cdot 4} & e^{-j\cdot\frac{\pi}{4}\cdot 0} & e^{-j\cdot\frac{\pi}{4}\cdot 4} & e^{-j\cdot\frac{\pi}{4}\cdot 0} & e^{-j\cdot\frac{\pi}{4}\cdot 4} & e^{-j\cdot\frac{\pi}{4}\cdot 0} & e^{-j\cdot\frac{\pi}{4}\cdot 4} \end{pmatrix}$$



4 cycles

$$W_8^{kn} = e^{-j\left(\frac{2\pi}{8}\right)kn} \qquad k = 4, \quad n = 0, 1, \ldots, 7$$

R ➡ *sampled values of* $\cos(-\omega t) = \cos(\omega t)$

I ➡ *sampled values of* $\sin(-\omega t) = -\sin(\omega t)$

$$\omega t = 2\pi f t$$
$$2\pi\cdot\left(\tfrac{4}{8}\right)\cdot f_s\cdot t$$

$\boldsymbol{X}[4]$ *measures how much of the above signal component is present in* $\boldsymbol{x}$.

Sampling Time τ

Sampling Frequency $f_s = \dfrac{1}{\tau}$

Sequence Time Length $T = N\tau$

4th Harmonic Freq $f_4 = \dfrac{4}{T} = \dfrac{4}{N\tau} = \dfrac{4 f_s}{N}$

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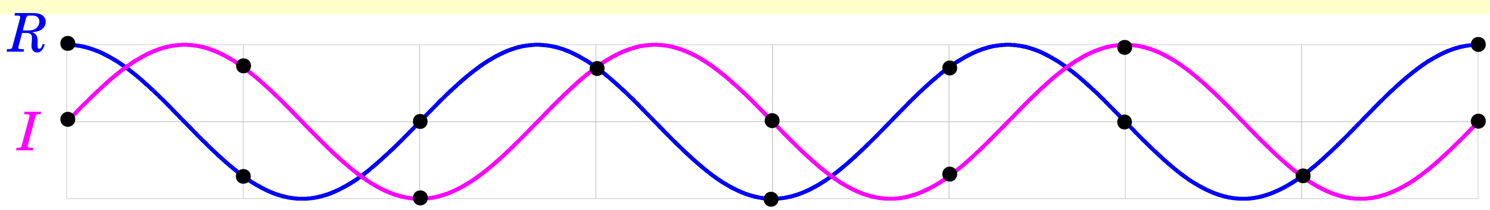
6/19/10

N=8 DFT : The 6th Row of the DFT Matrix

$$\begin{pmatrix} e^{-j\cdot\frac{\pi}{4}\cdot 0} & e^{-j\cdot\frac{\pi}{4}\cdot 5} & e^{-j\cdot\frac{\pi}{4}\cdot 2} & e^{-j\cdot\frac{\pi}{4}\cdot 7} & e^{-j\cdot\frac{\pi}{4}\cdot 4} & e^{-j\cdot\frac{\pi}{4}\cdot 1} & e^{-j\cdot\frac{\pi}{4}\cdot 6} & e^{-j\cdot\frac{\pi}{4}\cdot 3} \end{pmatrix}$$



3 cycles

$$W_8^{kn} = e^{-j\left(\frac{2\pi}{8}\right)kn} \qquad k = 5, \quad n = 0, 1, \ldots, 7$$

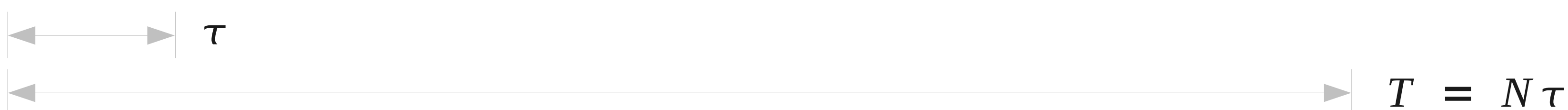
R ➡ *sampled values of* $\cos(\omega' t) = \cos(-(-\omega)t)$

I ➡ *sampled values of* $\sin(\omega' t) = \sin(-(-\omega)t)$

$$-\omega t = -2\pi f t$$

$$2\pi\cdot\left(\frac{-3}{8}\right)\cdot f_s\cdot t$$

$\boldsymbol{X}[5]$ *measures how much of the above signal component is present in* $\boldsymbol{x}$.



Sampling Time τ

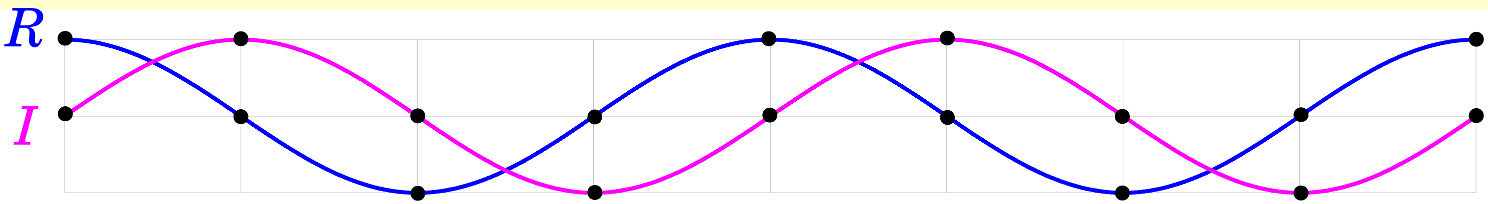
Sampling Frequency $f_s = \frac{1}{\tau}$

Sequence Time Length $T = N\tau$

3rd Harmonic Freq $f_3 = \frac{3}{T} = \frac{3}{N\tau} = \frac{3f_s}{N}$

N=8 DFT : The 7th Row of the DFT Matrix

$$\begin{pmatrix} e^{-j\cdot\frac{\pi}{4}\cdot 0} & e^{-j\cdot\frac{\pi}{4}\cdot 6} & e^{-j\cdot\frac{\pi}{4}\cdot 4} & e^{-j\cdot\frac{\pi}{4}\cdot 2} & e^{-j\cdot\frac{\pi}{4}\cdot 0} & e^{-j\cdot\frac{\pi}{4}\cdot 6} & e^{-j\cdot\frac{\pi}{4}\cdot 4} & e^{-j\cdot\frac{\pi}{4}\cdot 2} \end{pmatrix}$$



2 cycles

$$W_8^{kn} = e^{-j(\frac{2\pi}{8})kn} \qquad k = 2, \quad n = 0, 1, \dots, 7$$

R ➡ *sampled values of* $\cos(\omega' t) = \cos(-(-\omega)t)$

I ➡ *sampled values of* $\sin(\omega' t) = \sin(-(-\omega)t)$

$$-\omega t = -2\pi f t$$

$$2\pi\cdot(\tfrac{-2}{8})\cdot f_s \cdot t$$

$\mathbf{X}[6]$ *measures how much of the above signal component is present in* $\mathbf{x}$.

τ

$T = N\tau$

Sampling Time τ | Sampling Frequency $f_s = \frac{1}{\tau}$

Sequence Time Length $T = N\tau$ | 2nd Harmonic Freq $f_2 = \frac{2}{T} = \frac{2}{N\tau} = \frac{2 f_s}{N}$

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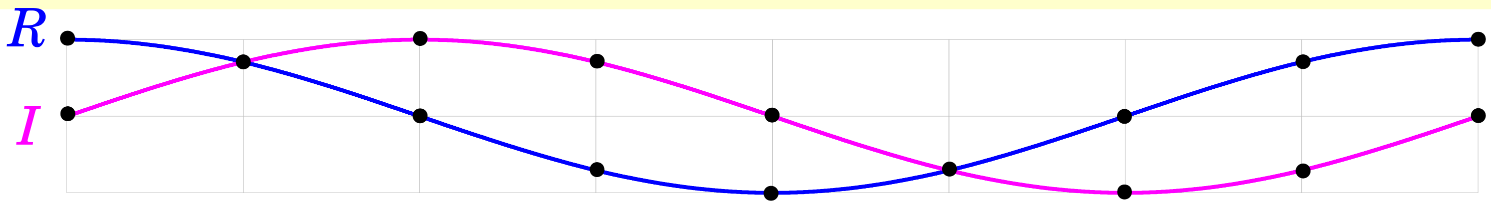
6/19/10

N=8 DFT : The 8th Row of the DFT Matrix

$$\begin{pmatrix} e^{-j\cdot\frac{\pi}{4}\cdot 0} & e^{-j\cdot\frac{\pi}{4}\cdot 7} & e^{-j\cdot\frac{\pi}{4}\cdot 6} & e^{-j\cdot\frac{\pi}{4}\cdot 5} & e^{-j\cdot\frac{\pi}{4}\cdot 4} & e^{-j\cdot\frac{\pi}{4}\cdot 3} & e^{-j\cdot\frac{\pi}{4}\cdot 2} & e^{-j\cdot\frac{\pi}{4}\cdot 1} \end{pmatrix}$$



1 cycle

$$W_8^{kn} = e^{-j\left(\frac{2\pi}{8}\right)kn} \qquad k = 7, \quad n = 0, 1, \ldots, 7$$

R ➡ *sampled values of* $\cos(\omega' t) = \cos(-(-\omega)t)$

I ➡ *sampled values of* $\sin(\omega' t) = \sin(-(-\omega)t)$

$$-\omega t = -2\pi f t$$
$$2\pi\cdot\left(\frac{-1}{8}\right)\cdot f_s\cdot t$$

$X[7]$ *measures how much of the above signal component is present in* x.

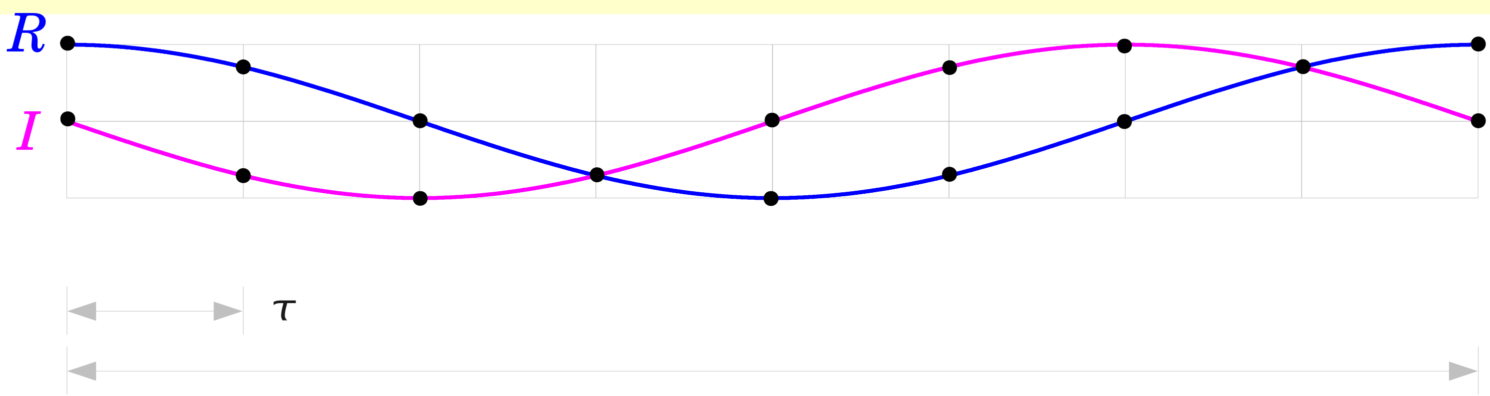
τ

$T = N\tau$

Sampling Time	τ	Sampling Frequency	$f_s = \frac{1}{\tau}$
Sequence Time Length	$T = N\tau$	1st Harmonic Freq	$f_1 = \frac{1}{T} = \frac{1}{N\tau} = \frac{f_s}{N}$

Fundamental Frequency

$$\left(e^{-j\cdot\frac{\pi}{4}\cdot 0} \quad e^{-j\cdot\frac{\pi}{4}\cdot 1} \quad e^{-j\cdot\frac{\pi}{4}\cdot 2} \quad e^{-j\cdot\frac{\pi}{4}\cdot 3} \quad e^{-j\cdot\frac{\pi}{4}\cdot 4} \quad e^{-j\cdot\frac{\pi}{4}\cdot 5} \quad e^{-j\cdot\frac{\pi}{4}\cdot 6} \quad e^{-j\cdot\frac{\pi}{4}\cdot 7} \right)$$



1 cycle

Sampling Time τ

Sequence Time Length $T = N\tau$

Sampling Frequency $f_s = \frac{1}{\tau}$

1st Harmonic Freq $f_1 = \frac{1}{T} = \frac{1}{N\tau} = \frac{f_s}{N}$

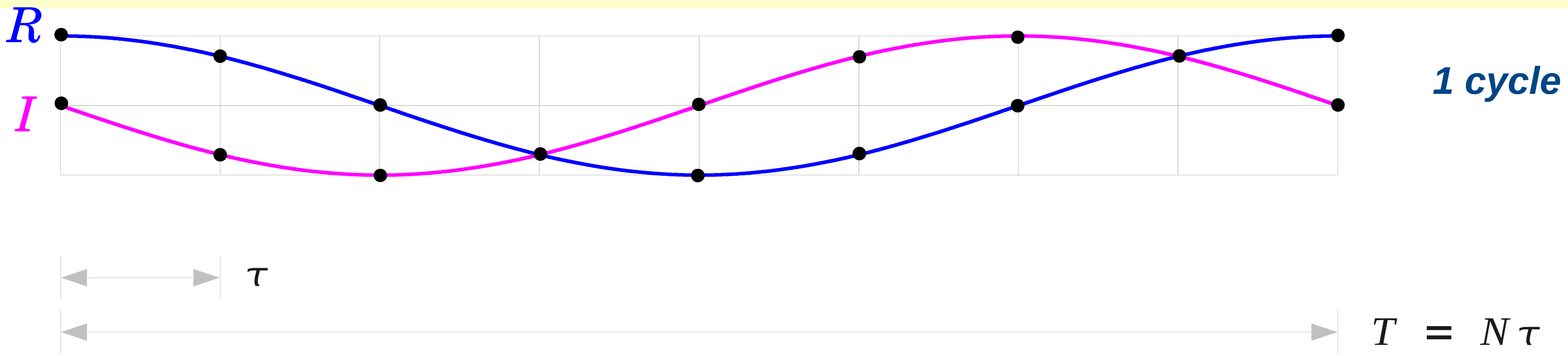
Fundamental Frequency f_o

The Lowest Frequency
in a harmonic series.

$$f_0 = f_1 = \frac{f_s}{N}$$

Normalized Frequency

$$\left(e^{-j\cdot\frac{\pi}{4}\cdot 0} \quad e^{-j\cdot\frac{\pi}{4}\cdot 1} \quad e^{-j\cdot\frac{\pi}{4}\cdot 2} \quad e^{-j\cdot\frac{\pi}{4}\cdot 3} \quad e^{-j\cdot\frac{\pi}{4}\cdot 4} \quad e^{-j\cdot\frac{\pi}{4}\cdot 5} \quad e^{-j\cdot\frac{\pi}{4}\cdot 6} \quad e^{-j\cdot\frac{\pi}{4}\cdot 7} \right)$$



Sampling Time τ

Sequence Time Length $T = N\tau$

Sampling Frequency $f_s = \frac{1}{\tau}$ (samples per second)

1st Harmonic Freq $f_1 = \frac{1}{T} = \frac{1}{N\tau} = \frac{f_s}{N}$

Normalized Frequency (cycles per sample)

$n = 0,\ 1,\ 2,\ \ldots,\ N-1$

$f_n = \frac{n \cdot f_s}{N}$ ⇒ $\frac{f_n}{f_s} = \frac{n}{N}$

N=8 DFT : DFT Matrix in + or – Frequencies

$\omega_0 = 2\pi \cdot \frac{f_s}{N}$

0th row:	*samples of*	$\cos 0\omega_0 t + j \cdot \sin 0\omega_0 t$	(0 cycle)
1th row:	*samples of*	$\cos 1\omega_0 t + j \cdot \sin 1\omega_0 t$	(1 cycle)
2th row:	*samples of*	$\cos 2\omega_0 t + j \cdot \sin 2\omega_0 t$	(2 cycles)
3th row:	*samples of*	$\cos 3\omega_0 t + j \cdot \sin 3\omega_0 t$	(3 cycles)
4th row:	*samples of*	$\cos 4\omega_0 t + j \cdot \sin 4\omega_0 t$	(4 cycles)
5th row:	*samples of*	$\cos 5\omega_0 t + j \cdot \sin 5\omega_0 t$	(5 cycles)
6th row:	*samples of*	$\cos 6\omega_0 t + j \cdot \sin 6\omega_0 t$	(6 cycles)
7th row:	*samples of*	$\cos 7\omega_0 t + j \cdot \sin 7\omega_0 t$	(7 cycles)

=

0th row:	*samples of*	$\cos(\ 0\omega_0)t + j \cdot \sin(\ 0\omega_0)t$	(0 cycle)
1th row:	*samples of*	$\cos(-7\omega_0)t + j \cdot \sin(-7\omega_0)t$	(7 cycles)
2th row:	*samples of*	$\cos(-6\omega_0)t + j \cdot \sin(-6\omega_0)t$	(6 cycles)
3th row:	*samples of*	$\cos(-5\omega_0)t + j \cdot \sin(-5\omega_0)t$	(5 cycles)
4th row:	*samples of*	$\cos(-4\omega_0)t + j \cdot \sin(-4\omega_0)t$	(4 cycles)
5th row:	*samples of*	$\cos(-3\omega_0)t + j \cdot \sin(-3\omega_0)t$	(3 cycles)
6th row:	*samples of*	$\cos(-2\omega_0)t + j \cdot \sin(-2\omega_0)t$	(2 cycles)
7th row:	*samples of*	$\cos(-1\omega_0)t + j \cdot \sin(-1\omega_0)t$	(1 cycles)

N=8 DFT : DFT Matrix in Both Frequencies

$\omega_0 = 2\pi \cdot \frac{f_s}{N}$

0th row:	*samples of*	$\cos 0\,\omega_0 t + j \cdot \sin 0\,\omega_0 t$	(0 cycle)
1th row:	*samples of*	$\cos 1\,\omega_0 t + j \cdot \sin 1\,\omega_0 t$	(1 cycle)
2th row:	*samples of*	$\cos 2\,\omega_0 t + j \cdot \sin 2\,\omega_0 t$	(2 cycles)
3th row:	*samples of*	$\cos 3\,\omega_0 t + j \cdot \sin 3\,\omega_0 t$	(3 cycles)
4th row:	*samples of*	$\cos 4\,\omega_0 t + j \cdot \sin 4\,\omega_0 t$	(4 cycles)
5th row:	*samples of*	$\cos 5\,\omega_0 t + j \cdot \sin 5\,\omega_0 t$	(5 cycles)
6th row:	*samples of*	$\cos 6\,\omega_0 t + j \cdot \sin 6\,\omega_0 t$	(6 cycles)
7th row:	*samples of*	$\cos 7\,\omega_0 t + j \cdot \sin 7\,\omega_0 t$	(7 cycles)

$=$

0th row:	*samples of*	$\cos 0\,\omega_0 t + j \cdot \sin 0\,\omega_0 t$	(0 cycle)
1th row:	*samples of*	$\cos 1\,\omega_0 t + j \cdot \sin 1\,\omega_0 t$	(1 cycle)
2th row:	*samples of*	$\cos 2\,\omega_0 t + j \cdot \sin 2\,\omega_0 t$	(2 cycles)
3th row:	*samples of*	$\cos 3\,\omega_0 t + j \cdot \sin 3\,\omega_0 t$	(3 cycles)
4th row:	*samples of*	$\cos 4\,\omega_0 t + j \cdot \sin 4\,\omega_0 t$	(4 cycles)
5th row:	*samples of*	$\cos 3\,\omega_0 t - j \cdot \sin 3\,\omega_0 t$	(3 cycles)
6th row:	*samples of*	$\cos 2\,\omega_0 t - j \cdot \sin 2\,\omega_0 t$	(2 cycles)
7th row:	*samples of*	$\cos 1\,\omega_0 t - j \cdot \sin 1\,\omega_0 t$	(1 cycles)

Frequency View of a DFT Matrix

			Normalized Frequency
	$f = 0$	$(0\,cycle)$	0
$\frac{N}{2}-1$	$f = 1 * f_o$	$(1\,cycle)$	$\frac{1}{N}$
	$f = 2 * f_o$	$(2\,cycles)$	$\frac{2}{N}$
	$f = 3 * f_o$	$(3\,cycles)$	$\frac{3}{N}$
	$f = (\frac{N}{2} - 1) * f_o$	$(\frac{N}{2} - 1\,cycles)$	$\frac{1}{2} - \frac{1}{N}$
	$f = (\frac{N}{2}) * f_o$	$(\frac{N}{2}\quad cycles)$	$\frac{1}{2}$
$\frac{N}{2}-1$	$f = -(\frac{N}{2} - 1) + * f_o$	$(\frac{N}{2} - 1\,cycles)$	$\frac{1}{2} + \frac{1}{N}$
	$f = -2 * f_o$	$(2\,cycles)$	$1 - \frac{2}{N}$
	$f = -1 * f_o$	$(1\,cycle)$	$1 - \frac{1}{N}$

References

[1] http://en.wikipedia.org/
[2] J.H. McClellan, et al., Signal Processing First, Pearson Prentice Hall, 2003
[3] A “graphical interpretation” of the DFT and FFT, by Steve Mann