Mtg 19: Fri, 20 Feb 09

$$\alpha = \beta + \gamma$$

p. 16-1   p. 18-1   p. 18-3

div (grad u)

$$u_{xx} + u_{yy} = \underbrace{u_{rr} + \frac{1}{r^2} u_{\theta\theta}}_{\beta} + \underbrace{\underbrace{\begin{bmatrix} \frac{1}{r} & 0 \end{bmatrix} \begin{Bmatrix} \partial_r u \\ \partial_\theta u \end{Bmatrix}}_{\frac{1}{r} u_r}}_{\gamma}$$

Classification: 2nd order lin. PDEs

p. 8-1 Eq. (2)

p. 13-1 Eq (1)

Ref: LP, G. Scholarpedia

$ac - b^2 = \det \underline{A}$ Eq (2) p. 8-1

Gowtham.

$$\boxed{\begin{array}{lll} ac - b^2 & < 0 & \text{hyperbolic} \\ \text{———} & = 0 & \text{parabolic} \\ \text{———} & > 0 & \text{elliptic} \end{array}}$$

why? (these names)

$u_{xx} + u_{yy} \Rightarrow a = 1, \; b = 0, \; c = 1$

$\Rightarrow ac - b^2 = 1 > 0$ elliptic

Gabriel

Q: What would be the classification of diffusion op. div(grad(.)) in polar coord?

$$\underline{\bar{A}} := \underline{J}\,\underline{A}\,\underline{J}^T$$ p. 13-1 Eq (2)

p. 8-1

$$= \underbrace{\begin{bmatrix} A & B \\ B & C \end{bmatrix}}_{\text{LP}} = \underbrace{\begin{bmatrix} \bar{a} & \bar{b} \\ \bar{b} & \bar{c} \end{bmatrix}}_{\text{preferred (ours)}}$$

$$\det \underline{\bar{A}} = \bar{a}\bar{c} - \bar{b}^2$$

$$\bar{a} = 1\,,\quad \bar{b} = 0\,,\quad \bar{c} = \frac{1}{r^2}$$

$$\bar{a}\,\cancel{\bar{b}}\,\bar{c} - \bar{b}^2 = \frac{1}{r^2} > 0 \quad (r \neq 0)$$

$$\Rightarrow \text{elliptic (still)}$$

Obs: 1) diffusion op. remains elliptic in polar coord. How about a diff. transf. of coord? Would classification remain the same?

2) Does classification make sense if it changes under transf. of coord.?

L9-3

Why?

No

Gabriel

Heat cond. eq. with const conductivity : steady state

$$\operatorname{div}(\kappa \operatorname{grad} u) = 0$$

$$\boxed{\operatorname{div}(\operatorname{grad} u) = 0}$$

A: Physics must remain the same (dist. of temp as result of soln of heat eq.) ~~must be~~ regardless of how heat eq. was solved (under diff. coord. syst.) => classification better remains same under diff. coord. syst for it to make sense.

Mtg 20: Mon, 23 Feb 09

Q: (Andy) Canonical forms (theory later)

Laplace eq = heat eq for const. heat conductivity, w/o heat source:

$$\mathrm{div}\,(\mathrm{grad}\, u) = 0 = u_{rr} + \frac{1}{r^2} u_{\theta\theta} + \frac{1}{r} u_r$$

Ref: S. p. 195.

Laplace op. = diffusion op in polar coord.

Axisym. pbs $\Rightarrow$ soln indep. of θ

$$\Rightarrow u_\theta = u_{\theta\theta} = \cdots = 0$$

$$\mathrm{div}\,(\mathrm{grad}\, u) = \nabla^2 u = \Delta u$$

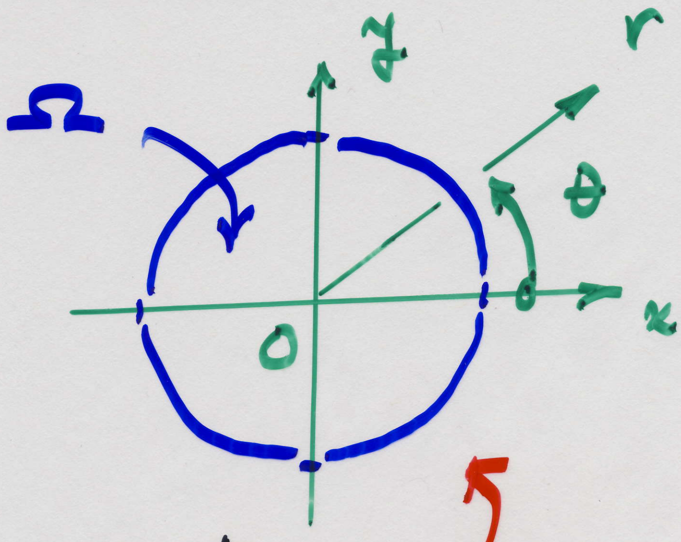
$$\nabla \cdot (\nabla u) = \underbrace{u_{rr} + \frac{1}{r} u_r}_{\frac{1}{r}\frac{d}{dr}\left(r \frac{du}{dr}\right)} = 0 \quad \text{ODE}$$

Int. $\boxed{u(r) = A_0 \ln r + B_0}$ HW

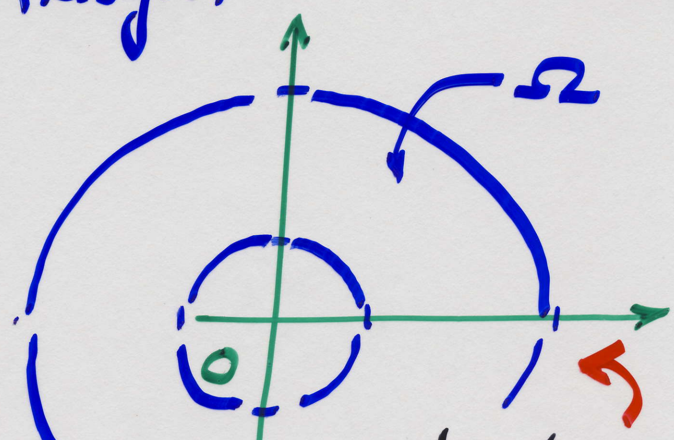
If domain Ω includes origin, $r = 0$, $A_0 = 0$ for $u(r) < +\infty$ (soln finite)

Domain Ω :

Axisym.

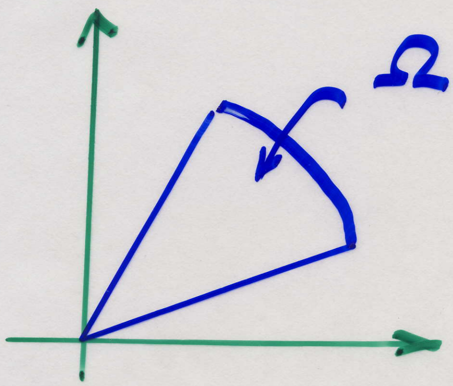


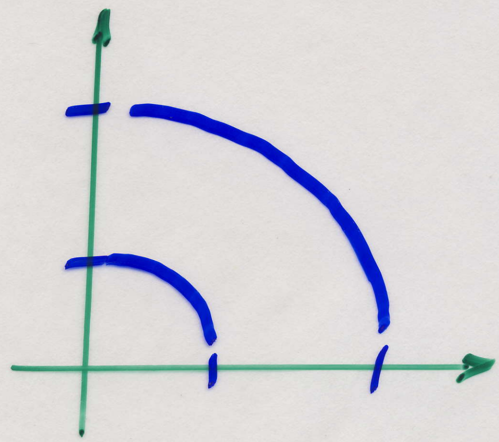
$A_0 = 0$



$A_0 \neq 0$

More general cases





Separation of variables:

$$\Delta u = \underbrace{u_{rr} + \frac{1}{r} u_r}_{} + \underbrace{\frac{1}{r^2} u_{\theta\theta}}_{} = 0$$

$$\Rightarrow \quad \underbrace{r^2 \left(u_{rr} + \frac{1}{r} u_r \right)}_{\mathcal{L}_1(u)} + \underbrace{u_{\theta\theta}}_{\mathcal{L}_2(u)} = 0$$

Try soln: $\boxed{u(r,\theta) = F(r)\, G(\theta)}$

$$\Rightarrow \quad \frac{G(\theta)}{r}\frac{d}{dr}\left(r\frac{dF(r)}{dr}\right) + \frac{F(r)}{r^2}\frac{d^2G}{d\theta^2} = 0$$

Divide by $F(r)\,G(\theta)$:

$$\Rightarrow \quad \underbrace{\frac{1}{F(r)}\left(r^2\frac{d^2F}{dr^2} + r\frac{dF}{dr}\right)}_{\text{only func. of } r} = \underbrace{-\frac{1}{G(\theta)}\frac{d^2G(\theta)}{d\theta^2}}_{\text{only func. of } \theta}$$

$$= \text{const} = n^2$$ (why pos. const.?) HW

$\Rightarrow$ 2 uncoupled ODEs:

$$\boxed{\begin{aligned} & r^2\frac{d^2F}{dr^2} + r\frac{dF}{dr} - n^2F = 0 \\ & \frac{d^2G}{d\theta^2} + n^2G = 0 \end{aligned}}$$

HW

$$\boxed{\begin{aligned} F(r) &= Ar^n + \frac{B}{r^n} \quad \text{for } n \neq 0 \\ G(\theta) &= C\cos n\theta + D\sin n\theta \quad \text{for } n \neq 0 \end{aligned}}$$

If $n=0$, $$\boxed{\begin{aligned} F(r) &= A_0 \ln r + B_0 \\ G(\theta) &= C_0\,\theta + D_0 \end{aligned}}$$

Rem: soln of Laplace eq. called harmonic soln. ← HW

Const "n" is deg. of the harmonic.

In polar coord, soln is called circular harmonics.

In general, want soln to be periodic

$$u(r, \theta + k\,2\pi) = u(r, \theta)$$

k ← const.

$$\boxed{u(r,\theta) = A_0 \ln r + \sum_{n=1}^{\infty} r^n (A_n \cos n\theta + B_n \sin n\theta) + \sum_{n=1}^{\infty} \frac{1}{r^n}(C_n \cos n\theta + D_n \sin n\theta) + C_0}$$

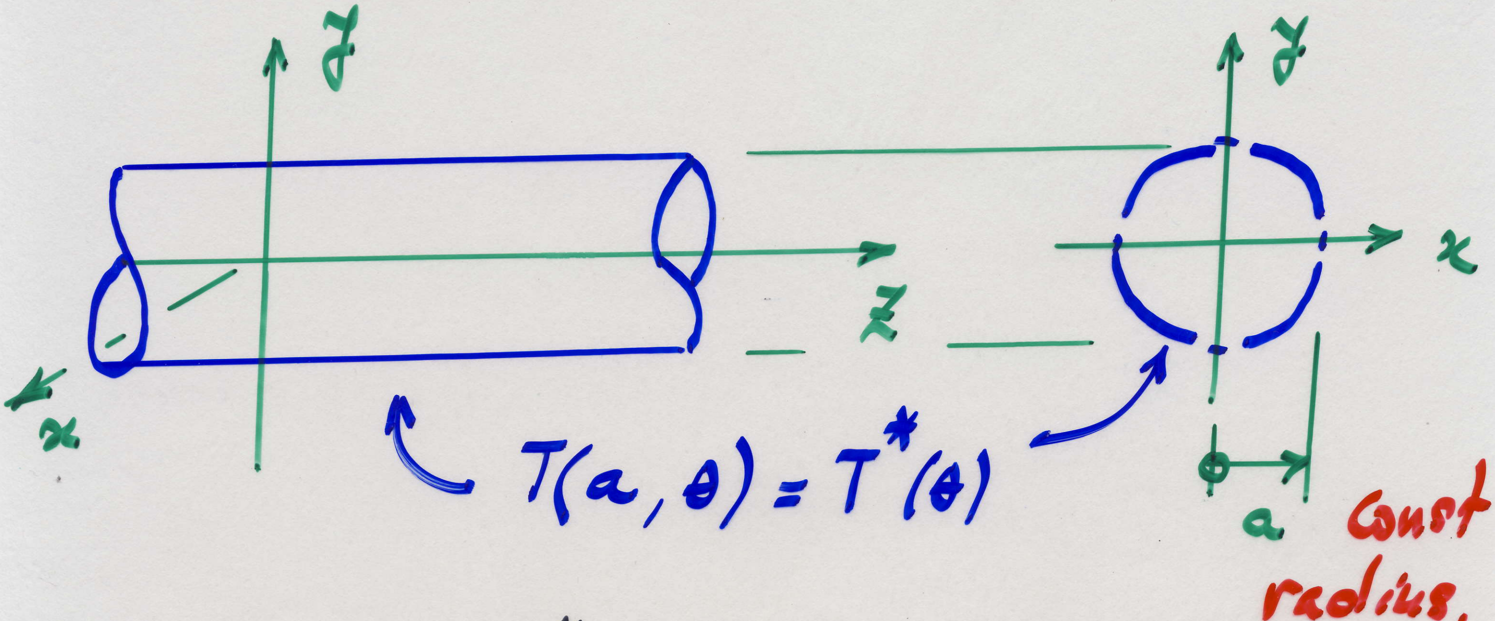
← Eq (2)

Mtg 21: Wed, 25 Feb 09

HW: p. 19-2 ; LP, p. 14, (1.2.13)

$$\bar{a}\bar{c} - \bar{b}^2 \overset{(1)}{=} (ac - b^2)\underbrace{(\phi_x \psi_y - \phi_y \psi_x)}_{p.\ 14-2}{}^2$$

Application:



$$T(a, \theta) = T^*(\theta)$$

Example: $T^*(\theta) = \underbrace{T_0}_{\text{const.}} (1 + \cos^2\theta)$

$$T^*(\theta = 0) = 2T_0 = T^*(\theta = 2\pi)$$

$$T^*\left(\theta = \frac{\pi}{2}\right) = T_0 = T^*\left(\theta = \frac{3\pi}{2}\right)$$

$$T^*(\theta) = \frac{3T_0}{2} + \frac{T_0}{2}\cos 2\theta \quad \text{HW}$$

Princ. of superposition:

soln $T(r,\theta)$ = soln for $T^*(\theta) = \frac{3T_0}{2} = T_1^*(\theta)$

⊕ soln for $T^*(\theta) = \frac{T_0}{2}\cos 2\theta = T_2^*(\theta)$

why? HW.

$$T(r,\theta) = T_1(r,\theta) + T_2(r,\theta)$$

$$T^*(\theta) = T_1^*(\theta) + T_2^*(\theta)$$

Prob. $\mathcal{P}$:

PDE: $\operatorname{div}(\operatorname{grad} T) = 0$

General soln: Eq. (2) p. 20-4.

st $T(r=a,\theta) = T^*(\theta)$

Superposition: $\mathcal{P} = \mathcal{P}_1 \oplus \mathcal{P}_2$

Prob. $\mathcal{P}_1$: $\operatorname{div}(\operatorname{grad} T_1) = 0$

st $T_1(r=a,\theta) = T_1^*(\theta)$

Pb $\mathcal{P}_2$: $\operatorname{div}(\operatorname{grad} T_2) = 0$

st $T_2(r=a,\theta) = T_2^*(\theta)$

HW: $T_1(r,\theta) = \frac{3T_0}{2}$

For Pb P_2: $T_2(a,\theta) = T_2^*(\theta) = \frac{T_0}{2}\cos 2\theta$

HW: $A_0 = 0$, $C_n = 0$, $D_n = 0$ why?

$$T_2(r,\theta) = \sum_{n=1}^{\infty} r^n \left\{ A_n \cos n\theta + B_n \sin n\theta \right\}$$

HW: Using b.c., show:

$A_1 = 0$, $A_2 \neq 0$, $A_3 = A_4 = \dots = 0$

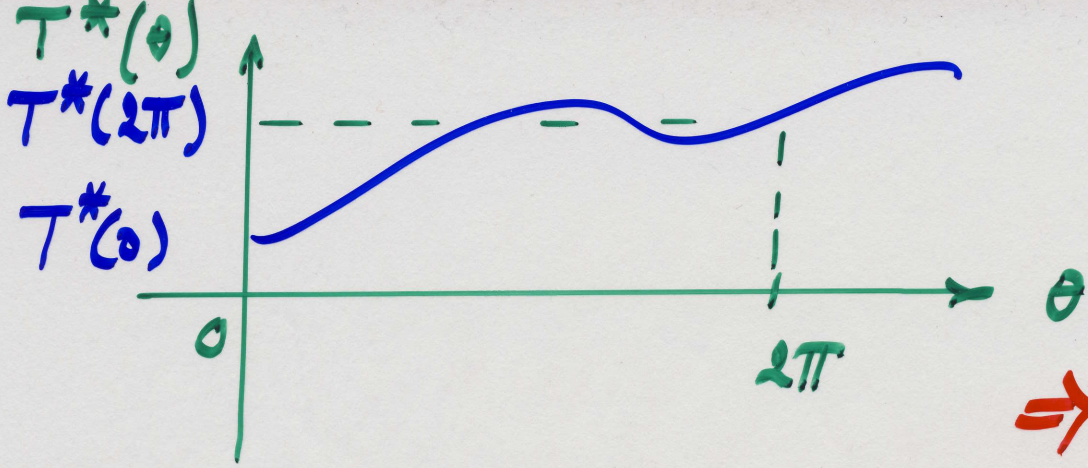
$B_n = 0$, $n = 1, \dots$

$$A_2 = \frac{T_0}{2a^2}$$

Final soln: $$\boxed{T(r,\theta) = T_0\left[\frac{3}{2} + \frac{1}{2}\left(\frac{r}{a}\right)^2 \cos 2\theta\right]}$$

In general, for arbitrary func. $T^*(\theta)$ but periodic, i.e.,

$$T^*(\theta + k2\pi) = T^*(\theta) \quad \forall\theta, \ \forall k \text{ const}$$



Not periodic $\Rightarrow$ not acceptable.

Gen. soln: HW

$$T(r,\theta) = C_0 + \sum_{n=1}^{\infty} r^n \{ A_n \cos n\theta + B_n \sin n\theta \}$$

$$T(a,\theta) = \underline{\qquad} \; a^n \; \underline{\qquad\qquad\qquad}$$

Find C_0, A_n, B_n Fourier coeff.

HW: Derive

$$C_0 = \frac{1}{2\pi} \int_{\theta=0}^{2\pi} T^*(\theta)\, d\theta$$

$$A_n = \frac{1}{2\pi} \int_{\theta=0}^{2\pi} T^*(\theta) \cos n\theta \, d\theta$$

$$B_n = \frac{1}{2\pi} \int_{\theta=0}^{2\pi} T^*(\theta) \sin n\theta \, d\theta$$

Due to orthog. of $\{1, \cos n\theta, \sin n\theta\}$

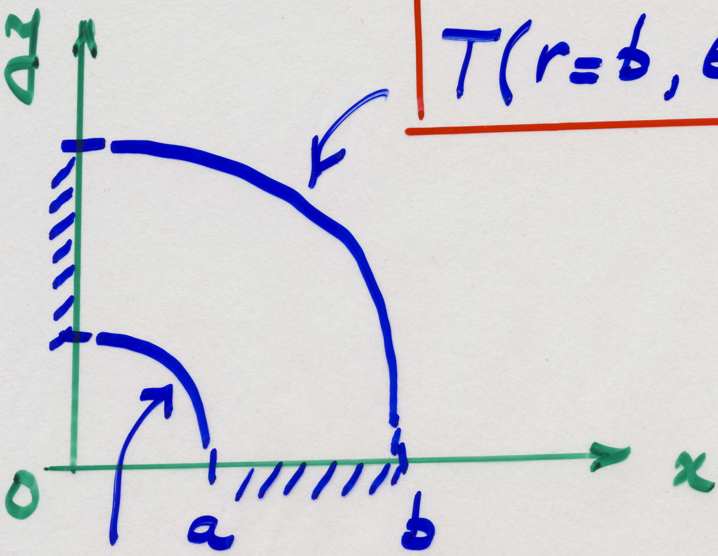
Nayfeh: Fourier basis func.

Mtg 22: Fri, 27 Feb 09

Another appl: Domain = quadrant of annulus

$$\boxed{T(r=b, \theta) = T_b \cos 4\theta} \quad (1)$$

T_b: given const



At $\theta = 0$, $\theta = \frac{\pi}{2}$

no heat flow

(insulated)

$$\boxed{T(a, \theta) = T_a \cos 4\theta} \quad (2)$$

T_a: given const.

Fourier's law:

$$\underline{q} = -\underline{K} \cdot \text{grad}\, T$$

$$\underline{q} = \underline{0} \Rightarrow \text{grad}\, T = 0 \Leftrightarrow \boxed{\frac{\partial T}{\partial \theta} = 0} \quad \text{why?}$$

at $\theta = 0$, $\theta = \frac{\pi}{2}$

HW.

at $\theta = 0$, $\theta = \frac{\pi}{2}$ (3)

Gen. soln: p. 20-4 Eq (1)

Eliminate terms that do not satisfy b.c.'s. **Obs**: Soln $T(r, \theta)$

1) $A_0 \ln r + C_0$: indep. of θ eliminate

2) $B_n \sin n\theta$, $D_n \sin n\theta$ cannot satis-

$$\text{f}y \quad \frac{\partial T}{\partial \theta}(r, \theta = 0) = 0 \qquad \text{HW.}$$

Soln now has the form:

$$T(r,\theta) = \sum_{n=1}^{\infty}\left(A_n r^n + \frac{C_n}{r^n}\right)\cos n\theta$$

Find A_n, C_n, $n = 1, \ldots, \infty$.

B.c.'s: $T(r=a, \theta) = T_a \cos 4\theta$

$T(r=b, \theta) = T_b \cos 4\theta$

HW: A_n, C_n equal zero except $n=4$.

$A_4 \neq 0$, $C_4 \neq 0$.

$$\begin{bmatrix} a^8 & 1 \\ b^8 & 1 \end{bmatrix}\begin{Bmatrix} A_4 \\ C_4 \end{Bmatrix} = \begin{Bmatrix} a^4 T_a \\ b^4 T_b \end{Bmatrix}$$

$$\boxed{T(r,\theta) = \left\{\frac{a^4 b^4 T_b}{(b^8 - a^8)}\left[\frac{r^4}{a^4} - \frac{a^4}{r^4}\right] - \frac{a^4 b^4 T_a}{(b^8 - a^8)}\left[\frac{r^4}{b^4} - \frac{b^4}{r^4}\right]\right\}\cos 4\theta}$$

Data: $a = 1$ $\quad b = 2$

$T_a = 5$ $\quad T_b = 20$

Plot $T(r,\theta)$.

Example of nonlinear heat eq. (S. Phillpot slide)

$$\operatorname{div}\left(K(u)\operatorname{grad} u\right) = 0$$

$$+ f(x,y)$$ heat source.

HW: Show nonlinearity.

Appl: Deflection of circ. membrane under transv. dist. load. Ref: S., p. 226

Pb. 5.11: Deriv. gov. eq. (Laplace eq.)

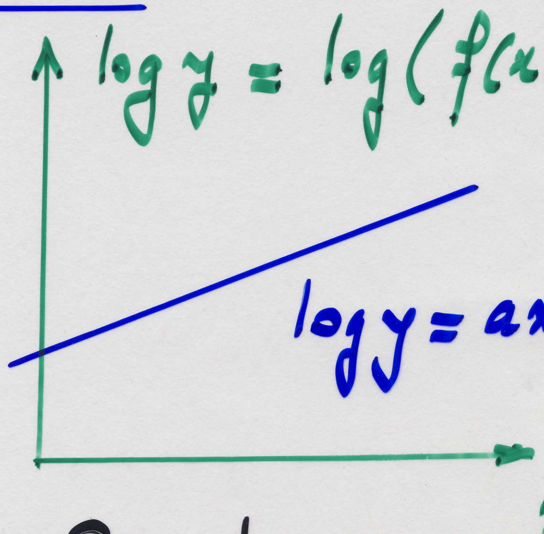
Pb. 5.12: HW.

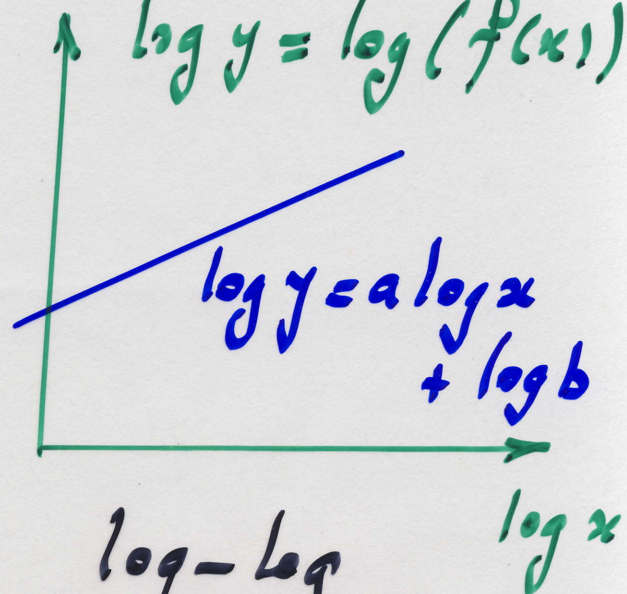
w/o dist load: $\operatorname{div}(\operatorname{grad} w) = 0$

w = trans. deflection of membrane.

Mtg 23: Mon, 2 Mar 09

Rem: $\kappa(u) \equiv \kappa(T)$ thermal cond.





Semi log

$$y = \exp[ax + \log b] = b \exp(ax)$$

log-log

$$y = \exp[a \log x + \log b]$$

$$\boxed{y = b x^a}$$

Power law

In nature:

Electrostatics, ..., Gravitational force: $a = -2$ (inverse square law)

Stefan-Boltzmann: $a = 4$

many other areas in nature (biology)

$\kappa(T)$ for diamond: $\kappa(T = 1\,^\circ K) \cong 0.4\ W/mK$

$\kappa(T \cong 10.7\,^\circ K) = 1000\ W/mK$

Fourier's law: $\boxed{q = -K \underbrace{\text{grad}\, T}_{\underbrace{\frac{dT}{dx}}}}$

heat flux: power / unit area

$[q] = \dfrac{W}{m^2}$

$\left[\dfrac{dT}{dx}\right] = \dfrac{°K}{m}$

$$\Rightarrow [K] = \frac{W/m^2}{K/m} = \frac{W}{mK}.$$

$$a = \frac{\log K_2 - \log K_1}{\log T_2 - \log T_1} = \frac{\log 1000 - \log 0.4}{\log(10.7) - \log(1)}$$

$\cong 3.39$

$b \cong 0.4\ W/mK$

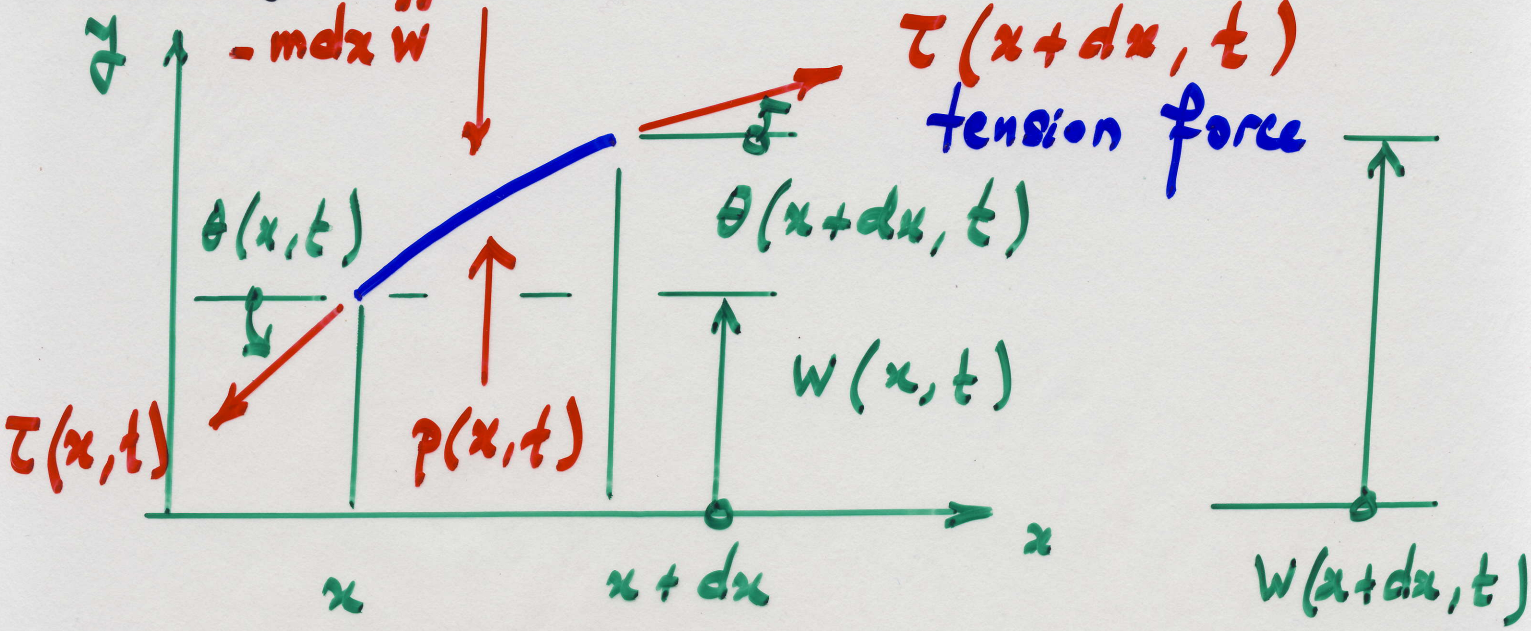
For diamond:

$K(T) = 0.4\ T^{3.39}\ W/mK$

$T \in [1°K, 10.7°K]$

HW: Find power law for diamond for $T \in [100°K, 1000°K]$ using graphite ($\parallel$ to layers) to estimate slope "a".

String vib: (1-D spatial case)



m = mass / unit length

$$\Sigma F_x = -\tau(x,t)\cos\theta(x,t) + \tau(x+dx,t)\cos\theta(x+dx,t)$$

θ small $\Rightarrow$ $\cos\theta \cong 1 - \dfrac{\theta^2}{2}$

Retain 1st terms : $\tau(x+dx,t) = \tau(x,t) = \tau$ const

$$\Sigma F_y = 0 = -\tau\sin\theta(x,t) + \tau\sin\theta(x+dx,t) + p(x,t)\,dx - m\,dx\,\underbrace{\ddot{w}}_{w_{tt}}$$

θ small: $\theta(x+dx,t) - \theta(x,t)$

$$\simeq \frac{d\theta}{dx}\,dx$$

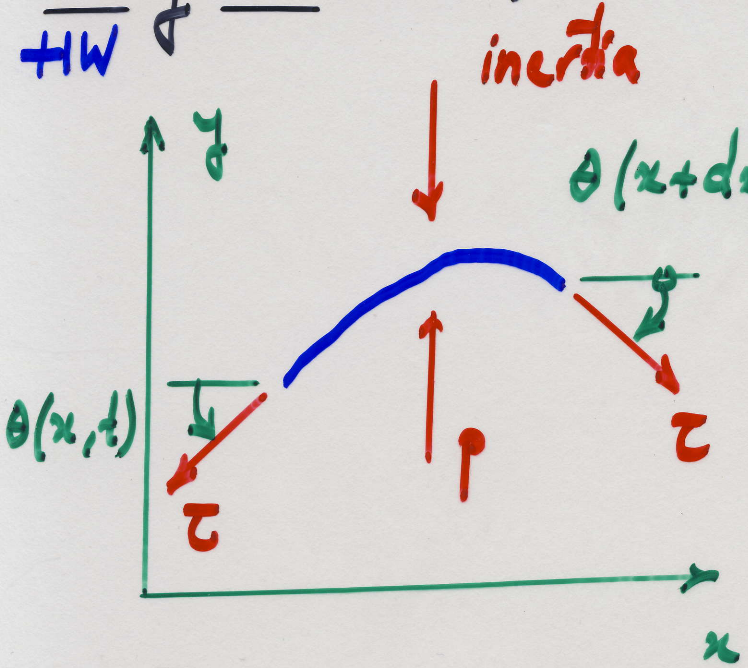
$$= (w_x)_x\,dx$$

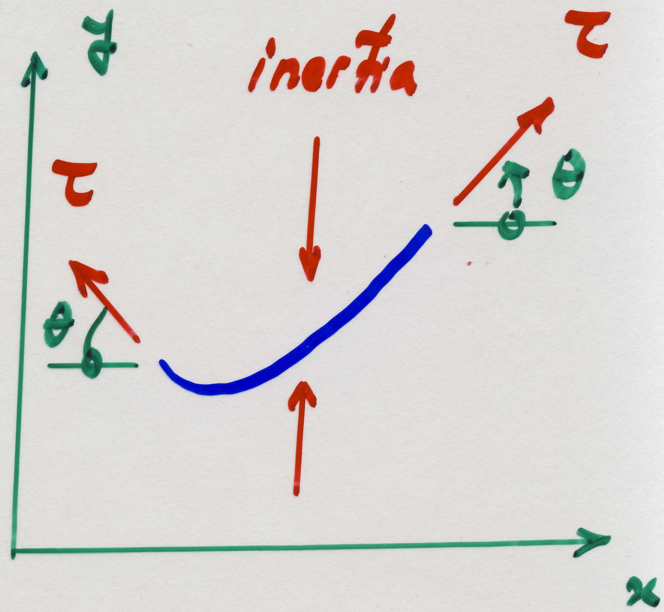
$$= w_{xx}\,dx$$

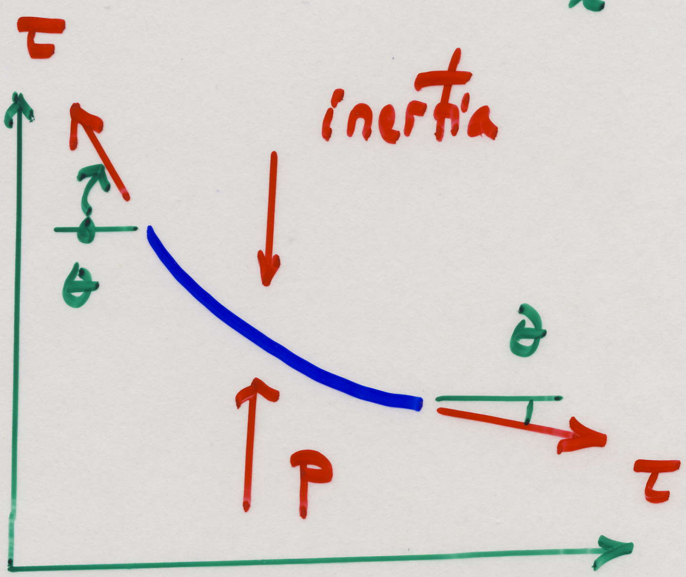
$$\boxed{\tau w_{xx} + p = m\,w_{tt}} \qquad (1)$$

m: mass / unit length

HW







Derive eom.

HW: Derive eom for stretched membrane in cartesian coord. (beyond 5.11 in S. p.226)

$$\boxed{\tau\,(\underbrace{\text{div}\,(\text{grad}\,w)}_{w_{xx}+w_{yy}}) + p(x,y) = m\,w_{tt}}$$

m: mass/unit area

Wave eq in 1-D space (actually a 2-D (x, t) pb) p. 23-4 Eq (1).

$$\tau W_{xx} - m \underbrace{W_{yy}}_{W_{tt}} + p = 0$$

(Andy)

$$\underline{A} = \begin{bmatrix} a & b \\ b & c \end{bmatrix}, \quad \det \underline{A} = ac - b^2$$

$a = \tau > 0$, $b = 0$, $c = -m < 0$

Andy Gabriel

$\det \underline{A} < 0 \Rightarrow$ hyperbolic

Unsteady heat eq:

1-D space: $\frac{d}{dx}\left(\kappa \frac{du}{dx}\right) + f = c \frac{du}{dt}$

temp.

heat cond. heat source heat capac.

κ const assumed:

$\kappa u_{xx} - c u_y + f = 0$

$a = \kappa > 0$, $b = 0$, $c = 0$, $\det \underline{A} = 0$
parabolic

1D

2D space: div (K grad u) + f = $c\dfrac{du}{dt}$

3D

2D space: (K const assumed)

$$K(u_{xx} + u_{yy}) - c\underbrace{u_t}_{u_z} + f = 0$$

p. 8-1: Gen. to 3 ind. var. (x, y, z)

$$\lfloor \partial_x \;\; \partial_y \;\; \partial_z \rfloor \underbrace{\begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}}_{[A_{ij}]_{3\times3}} \begin{Bmatrix} \partial_x \\ \partial_y \\ \partial_z \end{Bmatrix}$$

row ↗ i, col. ↖ j

$A_{ij} = 0 \quad \forall i,j$ except $A_{11} = A_{22} = K > 0$

$\det \underline{A} = 0$ parabolic.

Soln of unsteady heat eq. w/o heat source ($f = 0$) in polar coord. Ref. Z, p. 441.

Separation of var.

$$\underbrace{\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}}_{\text{div(grad } u)} = \frac{\partial u}{\partial t} \quad (1)$$

$$\boxed{u(r,\theta,t) = R(r)\,\Theta(\theta)\,T(t)} \quad (2)$$

Put (2) into (1):

$$\underbrace{\frac{1}{rR}\frac{d}{dr}\left(r\frac{dR}{dr}\right)}_{\text{only func of } r} + \underbrace{\frac{1}{r^2\Theta}\frac{d^2\Theta}{d\theta^2}}_{\text{only func. of } \theta, r} - \underbrace{\frac{1}{T}\frac{dT}{dt}}_{\text{only func of } t} = 0$$

depend only on (r,θ)

(3)

Comments on R3:

underbar ≡ boldface letters in wiki
in notes

$$\overrightarrow{OP} = \overrightarrow{O\bar{O}} + \overrightarrow{\bar{O}P}$$

$$\overrightarrow{OP} = x\,\underline{i} + y\,\underline{j}$$

$$\overrightarrow{O\bar{O}} = x_0\,\underline{i} + y_0\,\underline{j}$$

$$\overrightarrow{\bar{O}P} = \bar{x}\,\underline{\bar{i}} + \bar{y}\,\underline{\bar{j}} = \bar{x}\,(m\,\underline{i} + n\,\underline{j}) + \bar{y}\,(p\,\underline{i} + q\,\underline{j})$$

$$= (m\bar{x} + p\bar{y})\,\underline{i} + (n\bar{x} + q\bar{y})\,\underline{j}$$

$$\begin{cases} x = m\bar{x} + p\bar{y} + x_0 \\ y = n\bar{x} + q\bar{y} + y_0 \end{cases}$$

$$\Rightarrow \boxed{\begin{Bmatrix} x \\ y \end{Bmatrix} = \begin{bmatrix} m & p \\ n & q \end{bmatrix} \begin{Bmatrix} \bar{x} \\ \bar{y} \end{Bmatrix} + \begin{Bmatrix} x_0 \\ y_0 \end{Bmatrix}}$$

p. 17-2: $\underline{J}^{-1} = \left[\dfrac{\partial x_i}{\partial \bar{x}_j} \right] = \begin{bmatrix} m & p \\ n & q \end{bmatrix}$

p. 17-3: $\underline{J} = \dfrac{1}{\det \underline{J}^{-1}} \begin{bmatrix} q & -p \\ -n & m \end{bmatrix}$

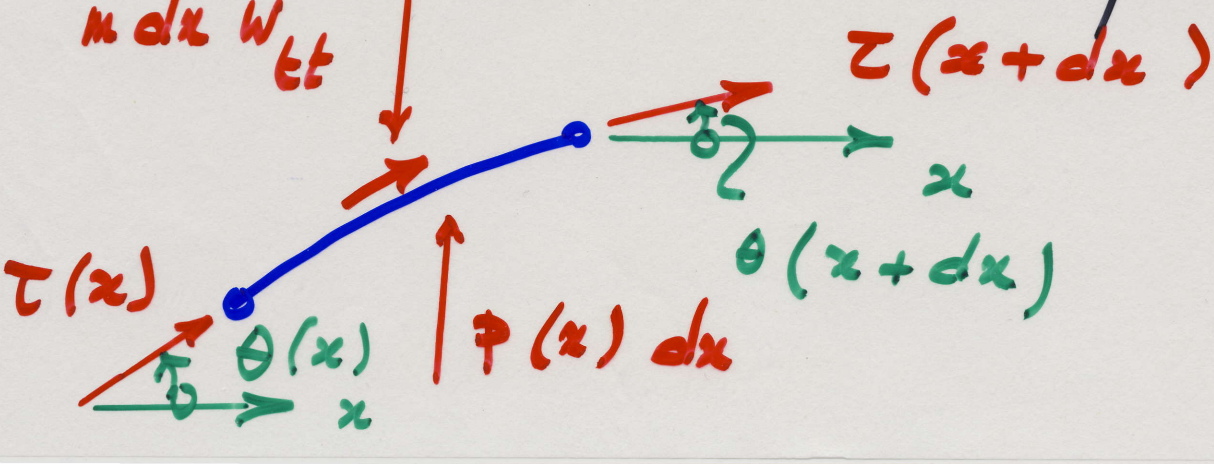
$$\det \underline{J}^{-1} = mq - np$$

$$\boxed{\begin{Bmatrix} \bar{x} \\ \bar{y} \end{Bmatrix} = \underline{J} \left[\begin{Bmatrix} x \\ y \end{Bmatrix} - \begin{Bmatrix} x_0 \\ y_0 \end{Bmatrix} \right]}$$

p. 18-1 Find $\underline{E} = \underline{J}$, $\underline{F} = -\underline{J} \begin{Bmatrix} x_0 \\ y_0 \end{Bmatrix}$

Comments on R4:

Treat tension $\tau(x,t)$ } algebraic
slope angle $\theta(x,t)$ } quantities.



(1) $\sum F_x = + \tau(x)\cos\theta(x) + \tau(x+dx)\cos\theta(x+dx) = 0$

(2) $\sum F_y = + \tau(x)\sin\theta(x) + \tau(x+dx)\sin\theta(x+dx)$

$+ p(x)dx - m\,dx\, W_{tt} = 0$

Eq (1) and θ small $\Rightarrow \cos\theta \cong 1$

$$\Rightarrow \underbrace{\tau(x+dx) = -\tau(x)}_{\forall x} = \tau \text{ Const}$$

$$\Rightarrow \begin{cases} \tau(x+dx) = \tau \\ \tau(x) = -\tau \end{cases}$$

Eq (2) : $\sum F_y = -\tau\sin\theta(x) + \tau\sin\theta(x+dx)$

$+ p(x)dx - m\,dx\, W_{tt} = 0$

θ small $\Rightarrow \sum F_y = 0 = \tau \dfrac{\partial\theta}{\partial x} dx + \text{hot}$

$+ p(x)dx - m\,dx\, W_{tt}$

$$\Rightarrow \boxed{\tau W_{xx} + p = m W_{tt}} \quad (3)$$

p. 24-4 : cont'd

$$f(r,\theta) + g(t) = 0 \qquad (1)$$

$$\Rightarrow \quad f(r,\theta) = -g(t) = \underline{\lambda = \text{const}} \qquad (2)$$

since (r, θ, t) are <u>indep. var.</u>

Eq (3) p. 24-4 : $\underbrace{\frac{1}{T}\frac{dT}{dt}}_{g(t)} = -\lambda$

why?

sign of λ i.e., $\lambda > 0$ or $\lambda < 0$?

$$\frac{dT}{T} = -\lambda \, dt$$

$$\Rightarrow \quad \log T = -\lambda t + k_1$$

$$T(t) = \exp(-\lambda t + k_1)$$

$$= \exp(-\lambda t) \underbrace{\exp k_1}_{k_2}$$

$$\boxed{T(t) = k_2 \exp(-\lambda t)}$$

Gabriel: $\lambda > 0$ and $\exp(-\lambda t) \to 0$ as $t \to +\infty$

Selection of const in Eq. (2) so to obtain (or elimination) physically meaningful soln.

$$f(r,\theta) = \frac{1}{rR}\frac{d}{dr}\left(r\frac{dR}{dr}\right) + \frac{1}{r^2\Theta}\frac{d^2\Theta}{d\theta^2}$$

$$= -\lambda \quad \text{const}$$

$$\Rightarrow \boxed{\underbrace{\frac{r}{R}\frac{d}{dr}\left(r\frac{dR}{dr}\right)}_{\text{func. of } r \text{ only} \;\; \ell(r)} + \overbrace{\frac{1}{\Theta}\frac{d^2\Theta}{d\theta^2}}^{m(\theta)} + \lambda r^2 = 0} \qquad (1)$$

$$\Rightarrow \ell(r) + m(\theta) = 0$$

$$\Rightarrow \ell(r) = -m(\theta) = +\rho \quad \text{const} \qquad (2)$$

why? $(\rho > 0)$?

From (1) and (2):

$$m(\theta) = \frac{1}{\Theta}\frac{d^2\Theta}{d\theta^2} = -\rho \qquad (3)$$

$$\boxed{\Theta'' + \rho\,\Theta = 0} \qquad \text{Navya} \quad (4)$$

Assume:

$$(5) \quad \boxed{\Theta(\theta) = \exp(\alpha\theta)}, \quad \alpha \text{ unknown const.}$$

$$\Theta' = \frac{d\Theta}{d\theta} = \alpha\exp(\alpha\theta) = \alpha\,\Theta$$

$\Theta'' = \alpha^2 \Theta$

Eq (4), p. 26-2: $\alpha^2 \Theta + \rho \Theta = 0$

$\alpha^2 + \rho = 0 \Rightarrow \underbrace{\alpha = \pm i\sqrt{\rho}}_{\text{2 poss. soln.}}$

if $\rho > 0$

(4)

$$\Theta(\theta) = \underbrace{k_3}_{\text{const}} \exp(+i\sqrt{\rho}\,\theta) + \underbrace{k_4}_{\text{const}} \exp(-i\sqrt{\rho}\,\theta)$$

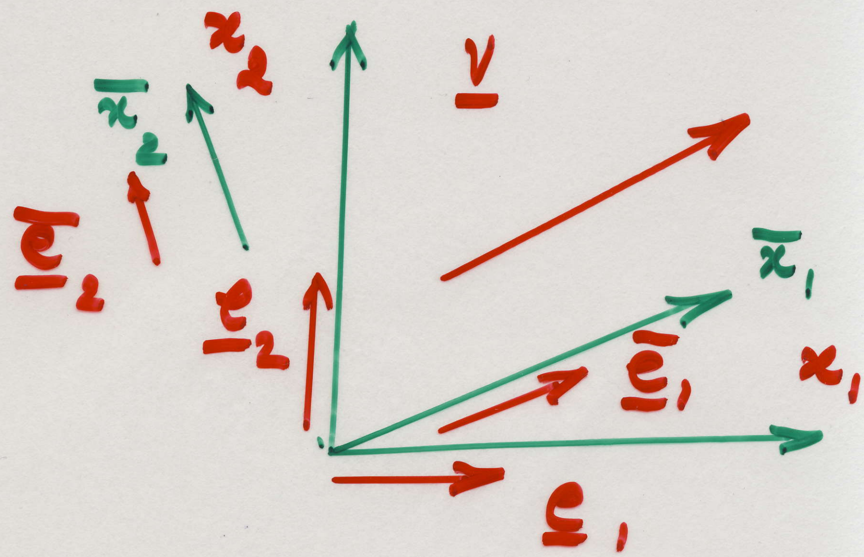
Sydni: superposition.

Gowtham: Eq (4) p. 26-2 (ODE) linear.

Diff. ~~basis~~ funcs:

$\underline{V} = v_i \, \underline{e}_i$

$= \bar{v}_i \, \underline{\bar{e}}_i$



$\{ \underline{e}_i \}$ lin. indep.

$\{ \underline{\bar{e}}_i \}$ lin. indep.

Similarly for funcs: lin. indep. funcs.

$\{ \exp(+i\sqrt{\rho}), \exp(-i\sqrt{\rho}) \}$ lin. indep.

How? Deeper subject. (e.g. Fourier series)

Method 1 Navya

$$\boxed{\begin{cases} \exp(i\theta) = \cos\theta + i\sin\theta \quad (\text{de Moivre}) \\ \exp(-i\theta) = \cos(-\theta) + i\sin(-\theta) \\ \qquad\qquad = \cos\theta - i\sin\theta \end{cases}}$$

Solve for $C = \cos\theta$ and $S = \sin\theta$ in terms of $\exp(\pm i\theta)$:

$$\cos\theta = \frac{1}{2}\left[e^{i\theta} + e^{-i\theta}\right]$$

$$\sin\theta = \frac{1}{2i}\left[e^{i\theta} - e^{-i\theta}\right]$$

$$\boxed{\begin{aligned} \Theta(\theta) = {} & k_3\left[\cos(\sqrt{p}\,\theta) + i\sin(\sqrt{p}\,\theta)\right] \\ & + k_4\left[\cos(\sqrt{p}\,\theta) - \sin(\sqrt{p}\,\theta)\right] \end{aligned}}$$

(3)

Mtg 27: Fri, 20 Mar 09

Recall (use notation in Z.)

(1) $T(t) = \underbrace{A(\lambda)}_{\text{const. but func. of const } \lambda} \exp(-\lambda t)$ p. 26-1

(2) $\Theta(\theta) = \underbrace{B(\rho)}\cos(\sqrt{\rho}\,\theta) + \underbrace{C(\rho)}\sin(\sqrt{\rho}\,\theta)$

Const, but func. of const ρ

p. 26-3

$$u(r,\theta,t) = R(r)\,\Theta(\theta)\,T(t)$$ p. 24-4

Init. cond. $\boxed{u(r,\theta,t=t_0) = \bar{u}(r,\theta)}$ (3)

Assume for simplicity $\bar{u}(r,\theta) = T_0$ const.

$$u(r,\theta,t_0) = R(r)\,\Theta(\theta)\,\underbrace{T(t_0)} = \underbrace{T_0}\ \text{const}$$

$$\Rightarrow T(t_0) = T_0 = A\exp(-\lambda t_0)$$

$$\boxed{A = T_0 \exp(\lambda t_0) = A(\lambda)} \quad (4)$$

Q: (Andy) How about the more general init. cond. Eq. (3) ? $A = A(\lambda)$?

$$u(r,\theta,t_0) = R(r)\,\Theta(\theta)\,\underbrace{T(t_0)}_{\text{const}} = \bar{u}(r,\theta)$$

$$= \Big[\underbrace{\frac{1}{k} R(r)}_{\bar{R}(r)}\,\Theta(\theta)\Big]\,k = \bar{u}(r,\theta)$$

Can select $k = T(t_0) = 1 = \exp(-\lambda t_0)$

$k \equiv T_0$

$$\boxed{A(\lambda) = \exp(\lambda t_0)}$$

$$\bar{u}(r,\theta) = R(r)\,\Theta(\theta)$$

$$= \underbrace{\frac{1}{k} R(r)}_{\bar{R}(r)}\,\Theta(\theta)\,\underbrace{k}_{T(t_0)}$$

Since k is arb., select $k = 1$. $\equiv$

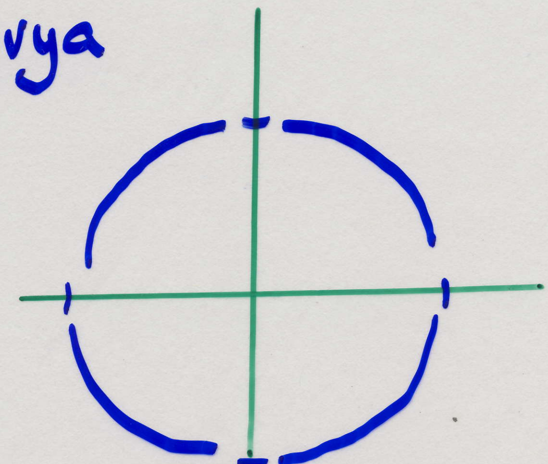
Now $\Theta(\theta)$ in Eq (2), p. 27-1

why $B = B(\rho)$, $C = C(\rho)$?

Gen. init. $u(r,\theta,t_0) = \bar{u}(r,\theta)$

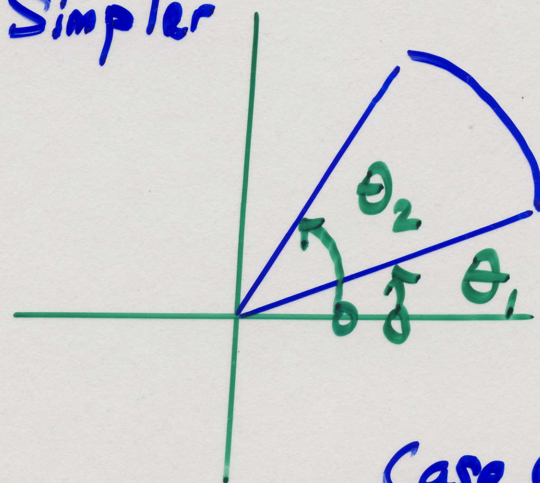
Simplify $u(r,\theta,t_0) = \bar{u}(\theta)$

Navya



Case 1: $\theta \in [0, 2\pi]$

Simpler



Case 2

$\theta \in [\theta_1, \theta_2]$

Case 2: $u(r,\theta_1,t_0) = \bar{u}(\theta_1)$

$$u(r,\theta_2,t_0) = \bar{u}(\theta_2)$$

$$\underbrace{\bar{u}(\theta) = \cancel{k}\,\Theta(\theta)}_{\text{Method 1}} = \underbrace{\frac{1}{k}\Theta(\theta)}_{\bar{\Theta}(\theta)}\ \underbrace{k}_{T(t_0)}$$

$$\bar{u}(\theta_1) = B\sin(\sqrt{p}\,\theta_1) + C\cos(\sqrt{p}\,\theta_1)$$

$$\underbrace{\bar{u}(\theta_2)}_{\text{known}} = B\underbrace{\sin(\sqrt{p}\,\theta_2)}_{\text{known}} + C\underbrace{\cos(\sqrt{p}\,\theta_2)}_{\text{known}}$$

Solve for $\mathbb{B}$ and $\mathbb{C}$; clearly

27-4

$$\mathbb{B} = \mathbb{B}(\rho, \theta_1, \theta_2)$$

$$\mathbb{C} = \mathbb{C}(\rho, \theta_1, \theta_2)$$