DFT Frequency (4B)

- Negative Frequency
- Angular Frequency
- Fundamental Frequency
- Harmonic Frequency
- Sampling Frequency
- Normalized Frequency
- Examples of N=8 DFT Matrix

Please send corrections (or suggestions) to youngwlim@hotmail.com.

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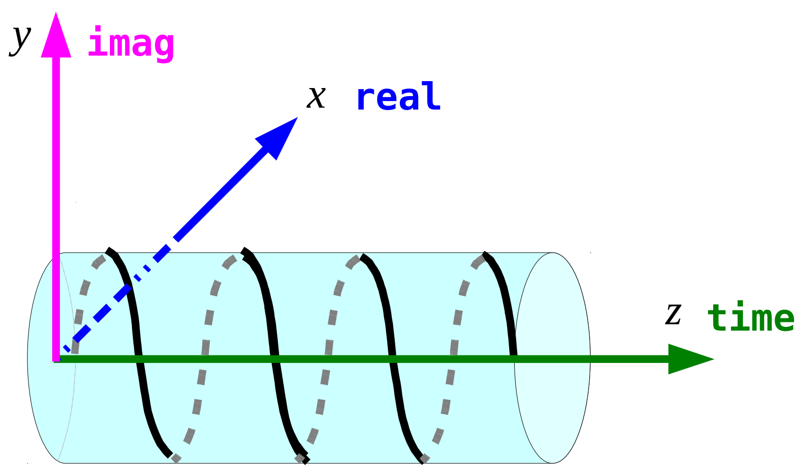
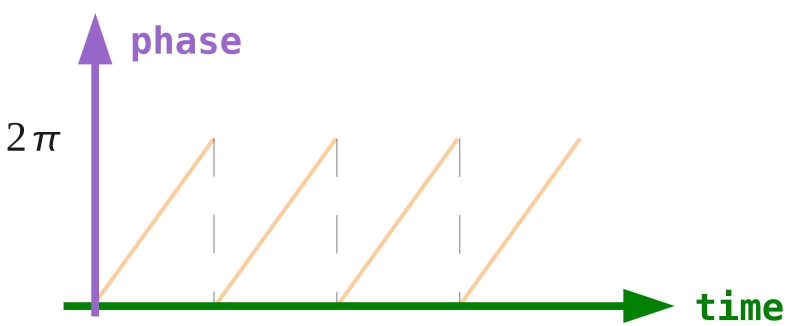
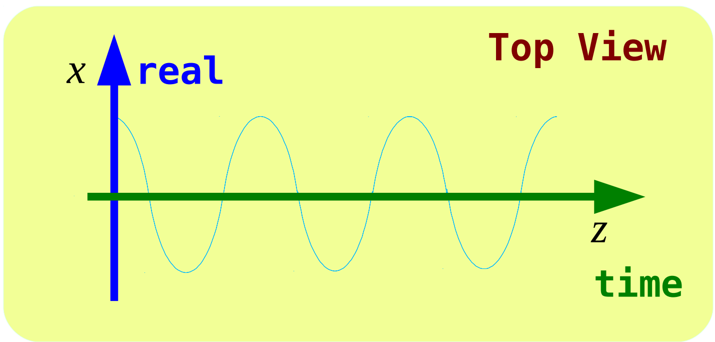
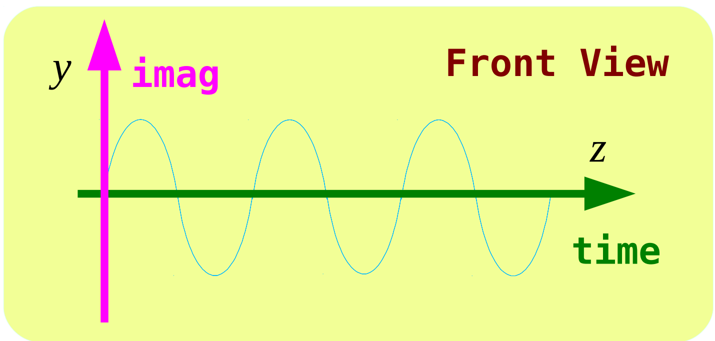
Euler Equation

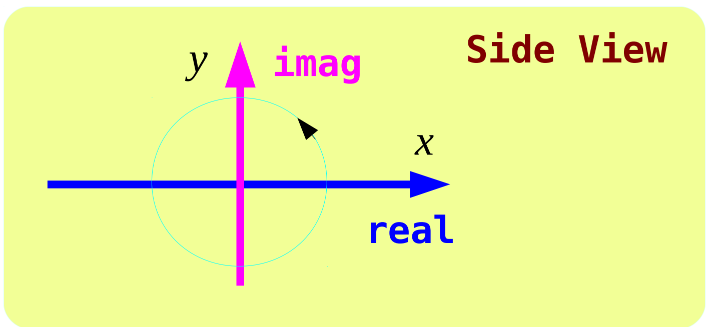
$$e^{+j\omega t} = \cos\omega t + j\sin\omega t$$







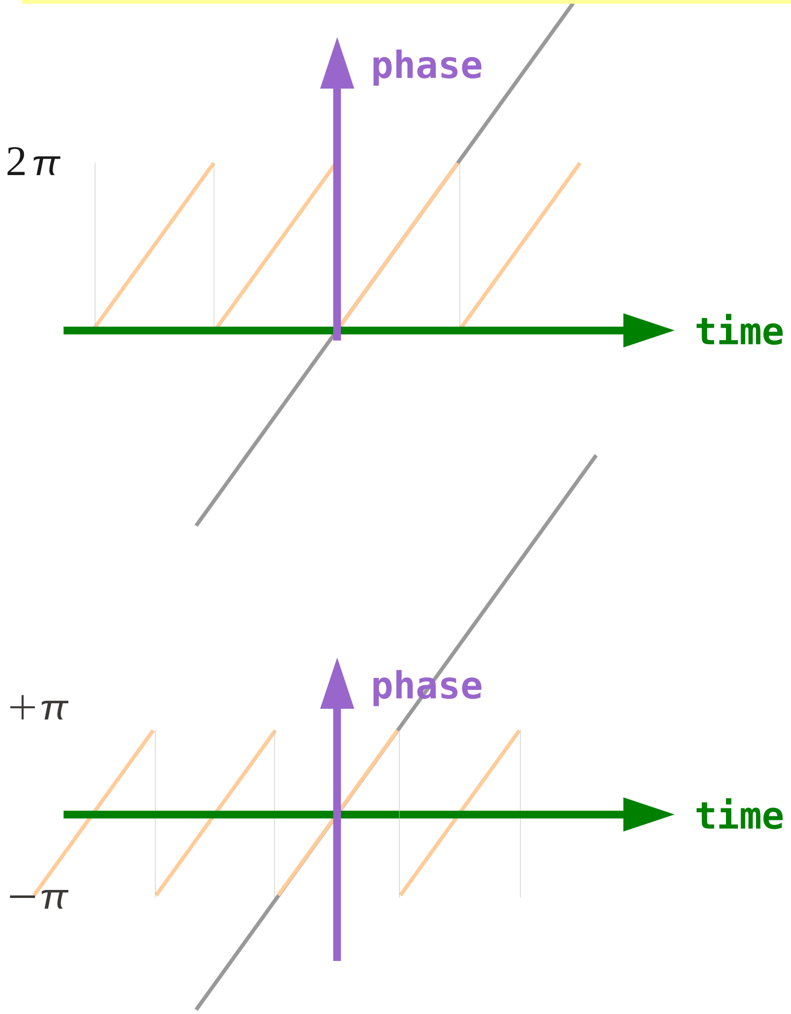


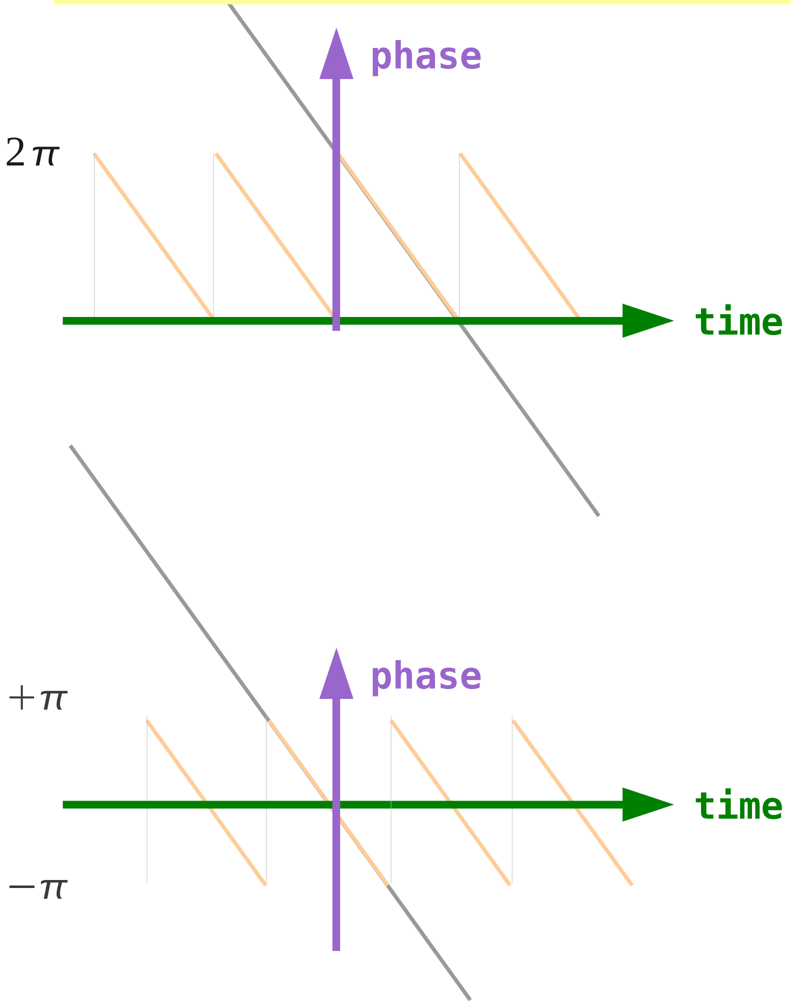


Linear Phase (1)

$$\Phi = \omega t \quad (\omega > 0)$$

$$\Phi = \omega' t \quad (\omega' < 0)$$

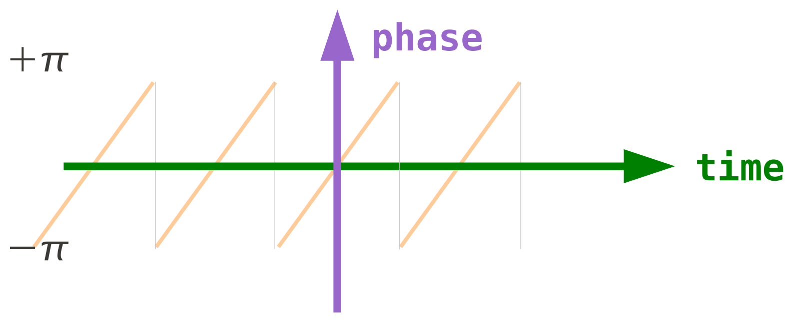


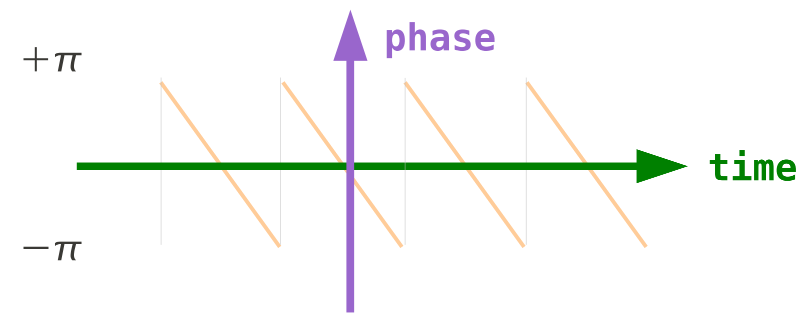


Linear Phase (2)

$$\Phi = \omega t \quad (\omega > 0)$$

$$\Phi = \omega' t \quad (\omega' < 0)$$





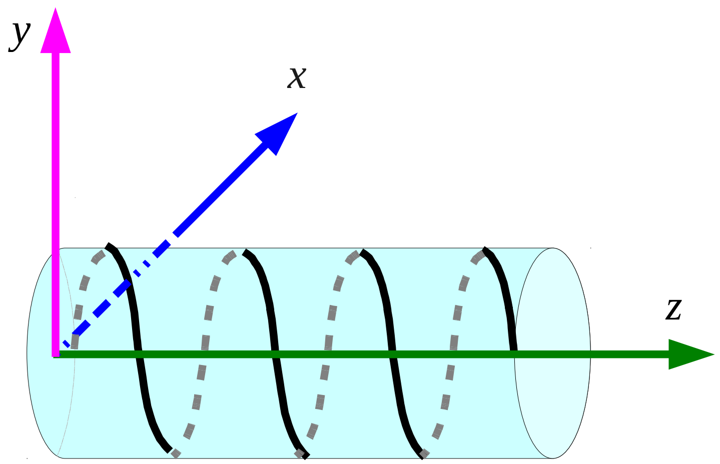
$$\omega' = -\omega$$

$$\cos(\omega' t) = \cos(-\omega t) \Rightarrow \cos(\omega' t) = \cos(\omega t)$$

$$\sin(\omega' t) = \sin(-\omega t) \Rightarrow \sin(\omega' t) = -\sin(\omega t)$$

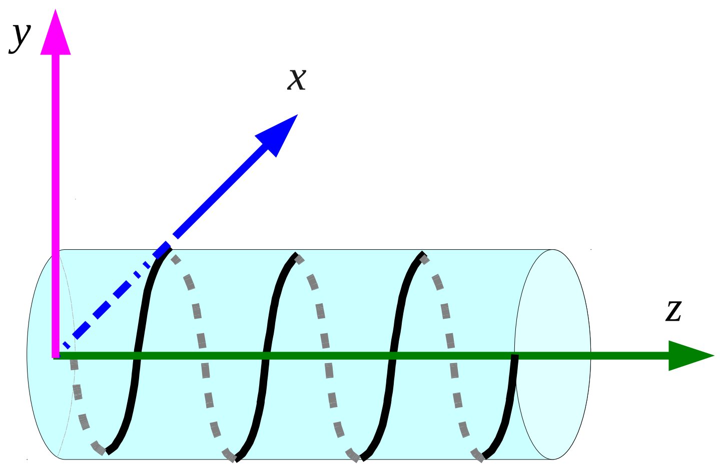
$$e^{j\omega' t} = e^{-j\omega t} \Rightarrow e^{j\omega' t} = \cos(\omega t) - j\sin(\omega t)$$

Linear Phase (3)



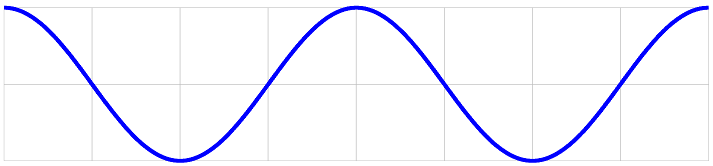
$$e^{j\omega t} = \cos(\omega t) + j\sin(\omega t)$$

$$\omega' = -\omega$$

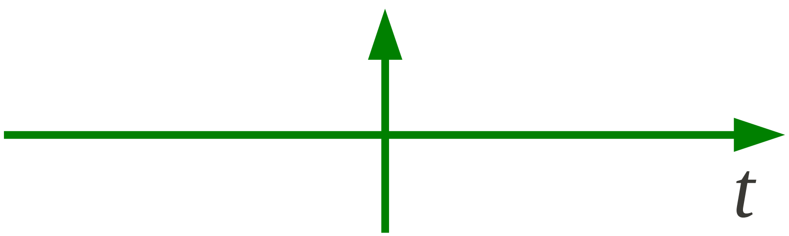


$$e^{j\omega' t} = \cos(\omega t) - j\sin(\omega t)$$

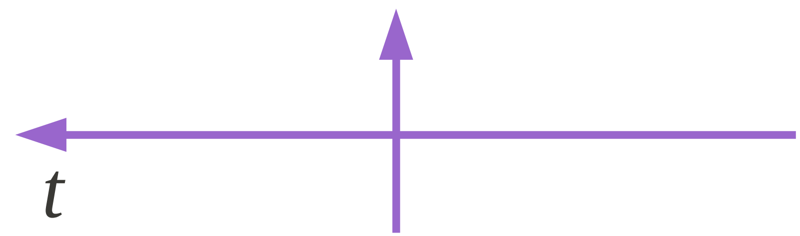
Negative Frequency (1)



***Coordinate* (A)**

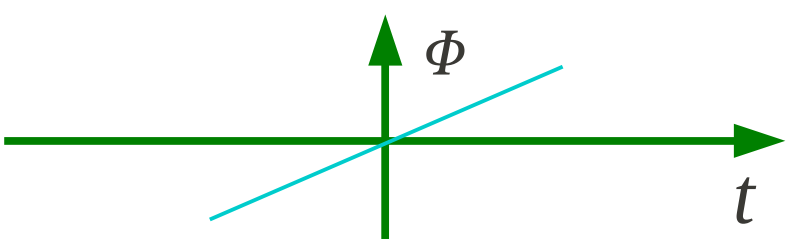


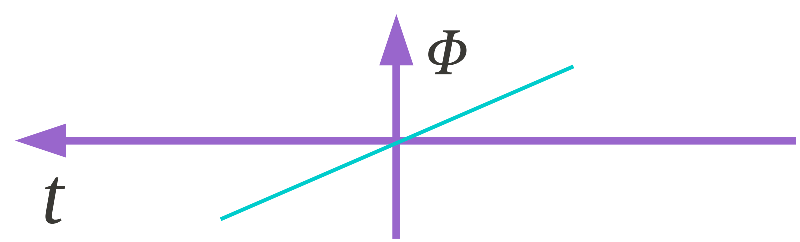
***Coordinate* (B)**



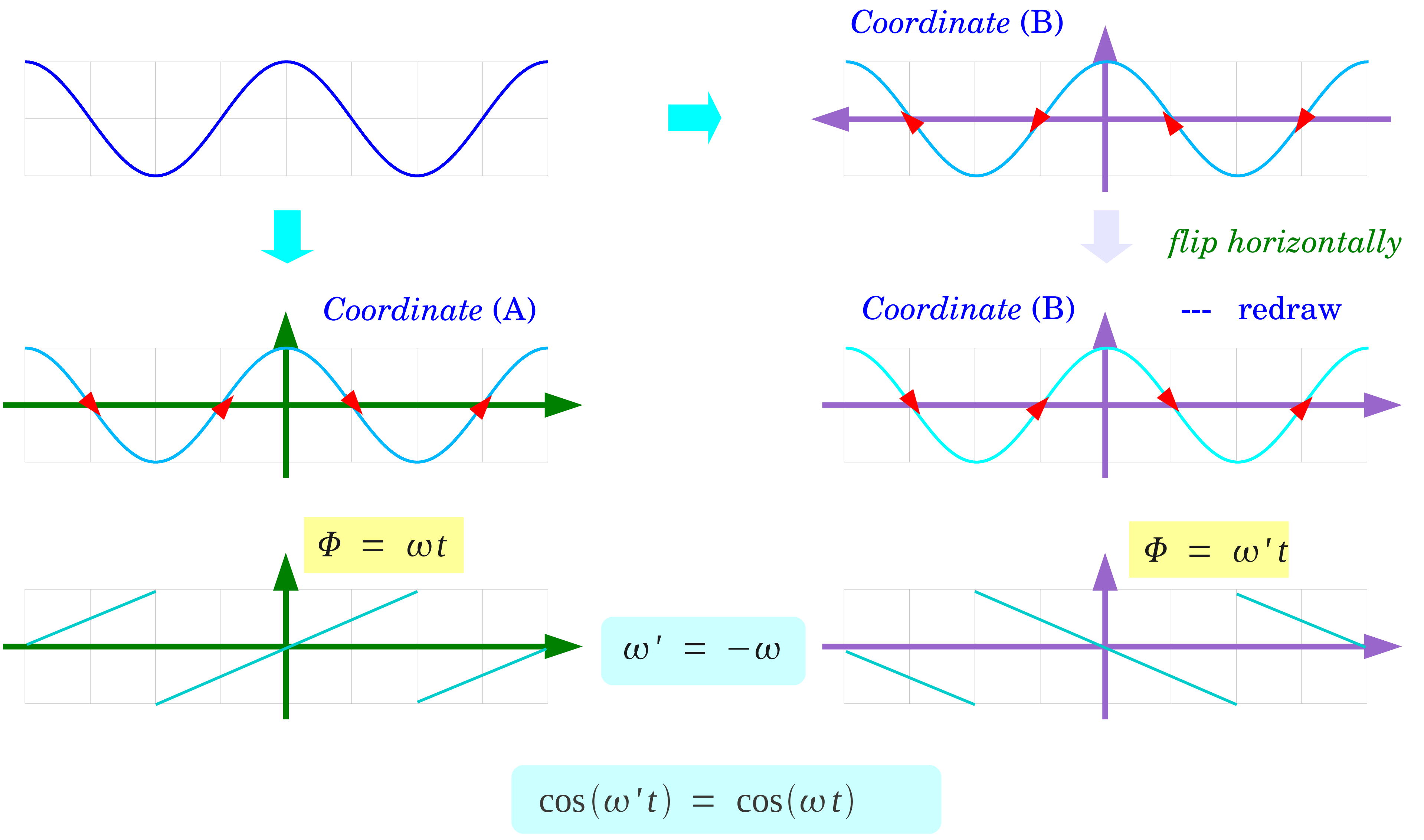
*As **t** increases,*
the phase increases.

*As **t** increases,*
the phase decreases.

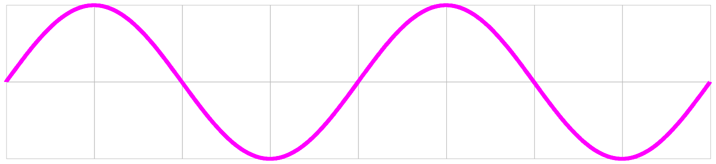




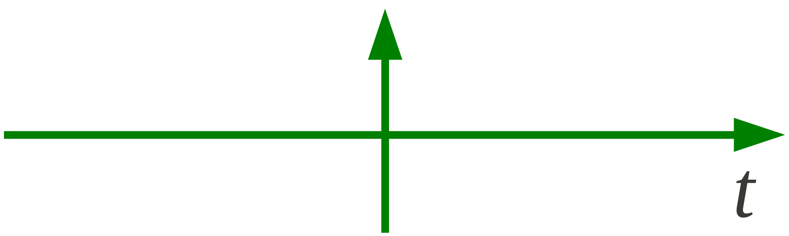
Negative Frequency (2)



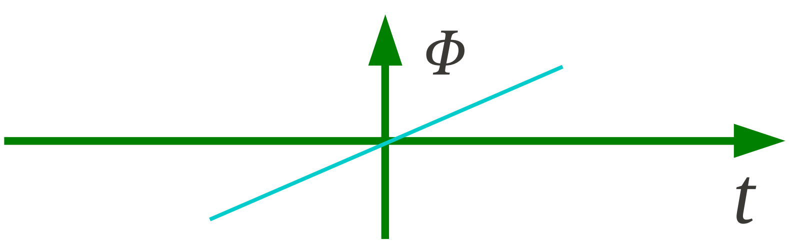
Negative Frequency (3)



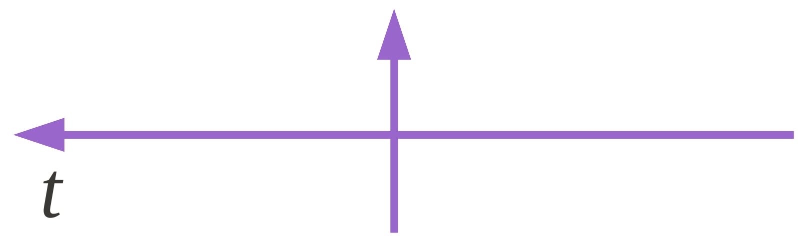
***Coordinate* (A)**



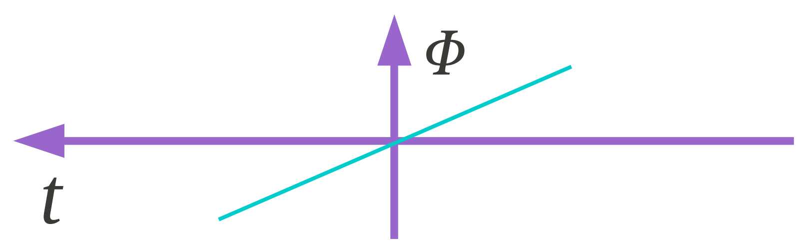
*As **t** increases,*
the phase increases.



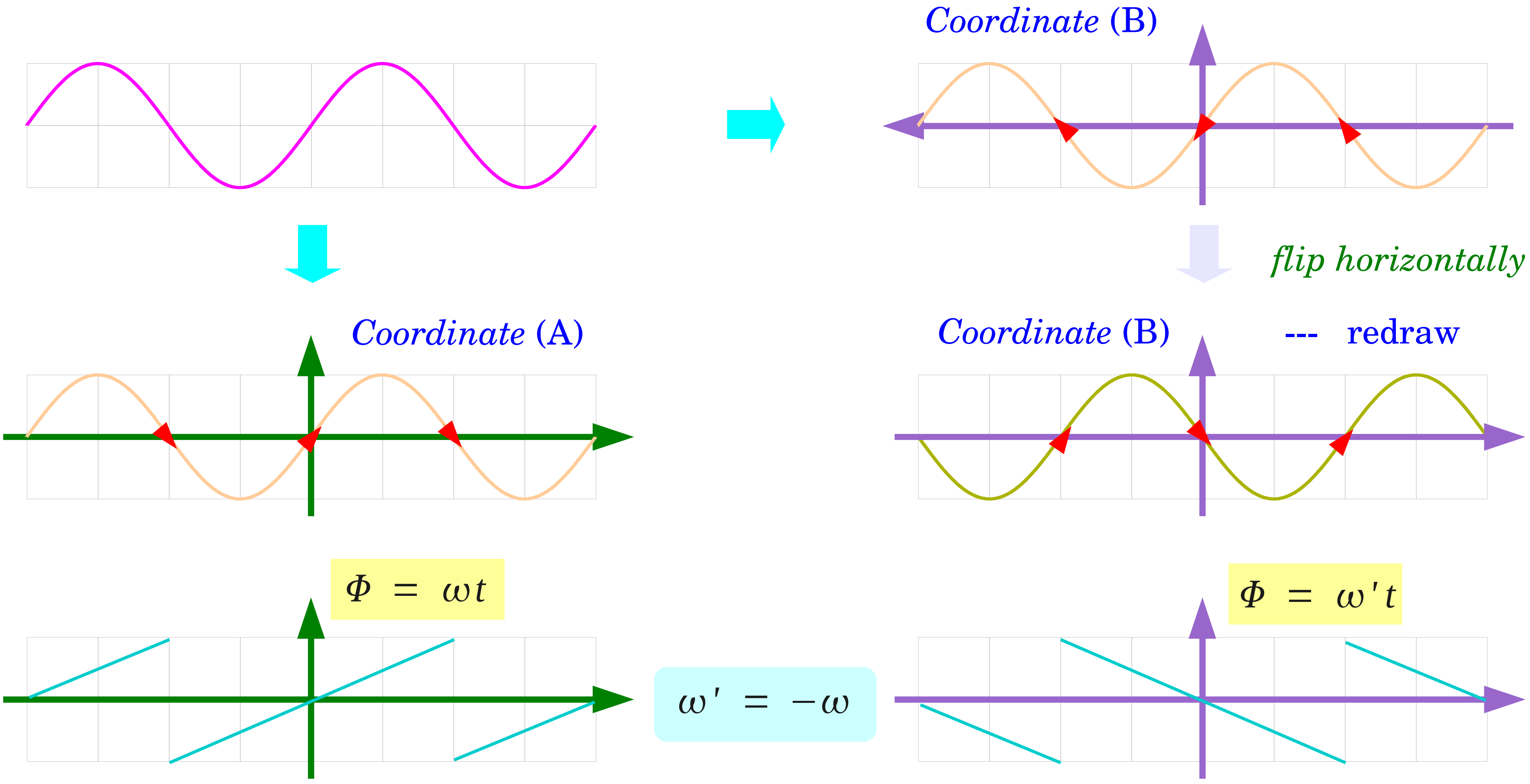
***Coordinate* (B)**



*As **t** increases,*
the phase <u>decreases</u>.



Negative Frequency (4)



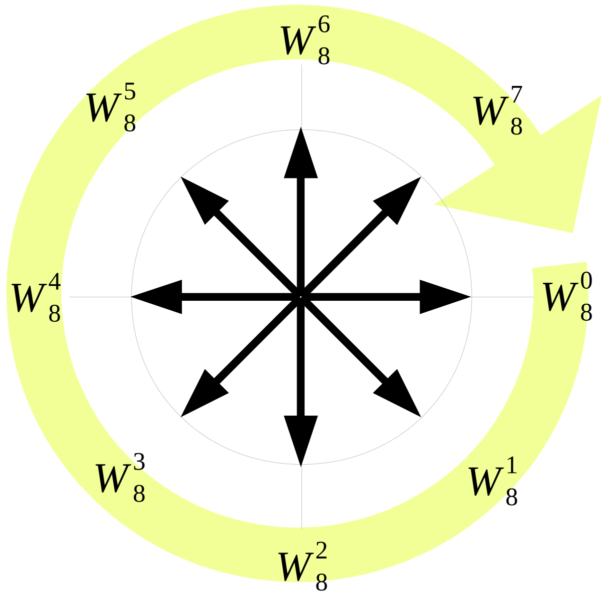
$\omega' = -\omega$

$\sin(\omega' t) = -\sin(\omega t)$

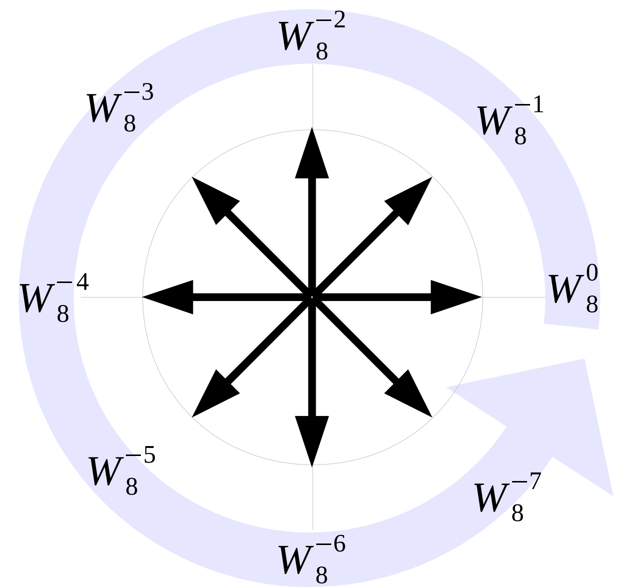
Complex Phase Factor

$$W_8^{k} = e^{-j\left(\frac{2\pi}{8}\right)k}$$

$$W_8^{-k} = e^{+j\left(\frac{2\pi}{8}\right)k}$$



$$W_8^1 = W_8^{-7} = W_8^1$$
$$W_8^2 = W_8^{-6} = W_8^2$$
$$W_8^3 = W_8^{-5} = W_8^3$$
$$W_8^4 = W_8^{-4} = W_8^4$$
$$W_8^5 = W_8^{-3} = W_8^{-3}$$
$$W_8^6 = W_8^{-2} = W_8^{-2}$$
$$W_8^7 = W_8^{-1} = W_8^{-1}$$



$$W_N^{k \pm N} = W_N^{k}$$

DFT Matrix (1)

$$
\begin{matrix}
W_8^0 & W_8^0 & W_8^0 & W_8^0 & W_8^0 & W_8^0 & W_8^0 & W_8^0 \\
W_8^0 & W_8^1 & W_8^2 & W_8^3 & W_8^4 & W_8^5 & W_8^6 & W_8^7 \\
W_8^0 & W_8^2 & W_8^4 & W_8^6 & W_8^0 & W_8^2 & W_8^4 & W_8^6 \\
W_8^0 & W_8^3 & W_8^6 & W_8^1 & W_8^4 & W_8^7 & W_8^2 & W_8^5 \\
W_8^0 & W_8^4 & W_8^0 & W_8^4 & W_8^0 & W_8^4 & W_8^0 & W_8^4 \\
W_8^0 & W_8^5 & W_8^2 & W_8^7 & W_8^4 & W_8^1 & W_8^6 & W_8^3 \\
W_8^0 & W_8^6 & W_8^4 & W_8^2 & W_8^0 & W_8^6 & W_8^4 & W_8^2 \\
W_8^0 & W_8^7 & W_8^6 & W_8^5 & W_8^4 & W_8^3 & W_8^2 & W_8^1
\end{matrix}
$$

$$W_8^k = e^{-j(\frac{2\pi}{8})k}$$

	n=0		n=1		n=2		n=3		n=4		n=5		n=6		n=7
k=0	0		0		0		0		0		0		0		0
		-0		-0		-0		-0		-0		-0		-0	
k=1	0		-1		-2		-3		-4		-5		-6		-7
		-1		-1		-1		-1		-1		-1		-1	
k=2	0		-2		-4		-6		0		-2		-4		-6
		-2		-2		-2		-2		-2		-2		-2	
k=3	0		-3		-6		-1		-4		-7		-2		-5
		-3		-3		-3		-3		-3		-3		-3	
k=4	0		-4		0		-4		0		-4		0		-4
		-4		-4		-4		-4		-4		-4		-4	
k=5	0		-5		-2		-7		-4		-1		-6		-3
		-5		-5		-5		-5		-5		-5		-5	
k=6	0		-6		-4		-2		0		-6		-4		-2
		-6		-6		-6		-6		-6		-6		-6	
k=7	0		-7		-6		-5		-4		-3		-2		-1
		-7		-7		-7		-7		-7		-7		-7	

DFT Matrix (2)

$$
\begin{array}{cccccccc}
W_8^0 & W_8^0 & W_8^0 & W_8^0 & W_8^0 & W_8^0 & W_8^0 & W_8^0 \\
W_8^0 & W_8^1 & W_8^2 & W_8^3 & W_8^4 & W_8^5 & W_8^6 & W_8^7 \\
W_8^0 & W_8^2 & W_8^4 & W_8^6 & W_8^0 & W_8^2 & W_8^4 & W_8^6 \\
W_8^0 & W_8^3 & W_8^6 & W_8^1 & W_8^4 & W_8^7 & W_8^2 & W_8^5 \\
W_8^0 & W_8^4 & W_8^0 & W_8^4 & W_8^0 & W_8^4 & W_8^0 & W_8^4 \\
W_8^0 & W_8^{-3} & W_8^{-6} & W_8^{-1} & W_8^{-4} & W_8^{-7} & W_8^{-2} & W_8^{-5} \\
W_8^0 & W_8^{-2} & W_8^{-4} & W_8^{-6} & W_8^0 & W_8^{-2} & W_8^{-4} & W_8^{-6} \\
W_8^0 & W_8^{-1} & W_8^{-2} & W_8^{-3} & W_8^{-4} & W_8^{-5} & W_8^{-6} & W_8^{-7}
\end{array}
$$

still symmetric matrix

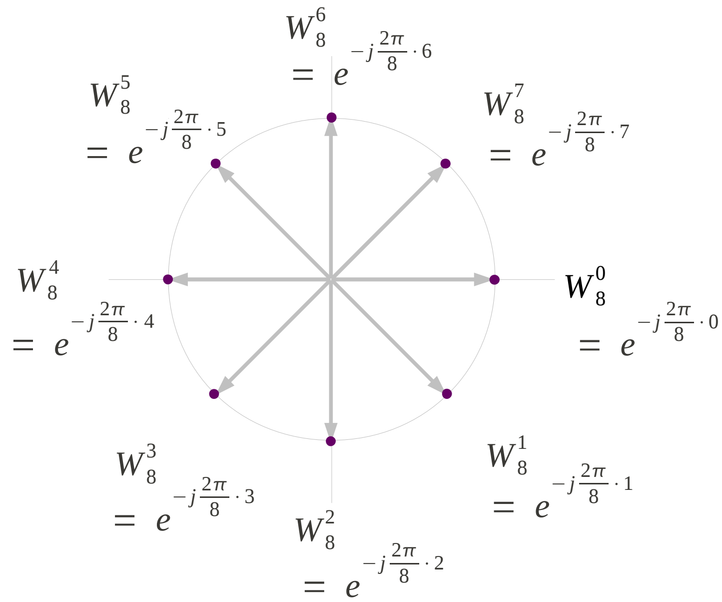
$$W_N^{k \pm N} = W_N^k$$

$$W_8^k = e^{-j\left(\frac{2\pi}{8}\right)k}$$

	n=0		n=1		n=2		n=3		n=4		n=5		n=6		n=7
k=0	0	-0	0	-0	0	-0	0	-0	0	-0	0	-0	0	-0	0
k=1	0	-1	-1	-1	-2	-1	-3	-1	-4	-1	-5	-1	-6	-1	-7
k=2	0	-2	-2	-2	-4	-2	-6	-2	0	-2	-2	-2	-4	-2	-6
k=3	0	-3	-3	-3	-6	-3	-1	-3	-4	-3	-7	-3	-2	-3	-5
k=4	0	-4	-4	-4	0	-4	-4	-4	0	-4	-4	-4	0	-4	-4
k=5	0	+3	+3	+3	+6	+3	+1	+3	+4	+3	+7	+3	+2	+3	+5
k=6	0	+2	+2	+2	+4	+2	+6	+2	0	+2	+2	+2	+4	+2	+6
k=7	0	+1	+1	+1	+2	+1	+3	+1	+4	+1	+5	+1	+6	+1	+7

DFT Matrix (3)

	n=0		n=1		n=2		n=3		n=4		n=5		n=6		n=7
k=0	0	-0	0	-0	0	-0	0	-0	0	-0	0	-0	0	-0	0
k=1	0	-1	-1	-1	-2	-1	-3	-1	-4	-1	-5	-1	-6	-1	-7
k=2	0	-2	-2	-2	-4	-2	-6	-2	0	-2	-2	-2	-4	-2	-6
k=3	0	-3	-3	-3	-6	-3	-1	-3	-4	-3	-7	-3	-2	-3	-5
k=4	0	-4	-4	-4	0	-4	-4	-4	0	-4	-4	-4	0	-4	-4
k=5	0	-5	-5	-5	-2	-5	-7	-5	-4	-5	-1	-5	-6	-5	-3
k=6	0	-6	-6	-6	-4	-6	-2	-6	0	-6	-6	-6	-4	-6	-2
k=7	0	-7	-7	-7	-6	-7	-5	-7	-4	-7	-3	-7	-2	-7	-1

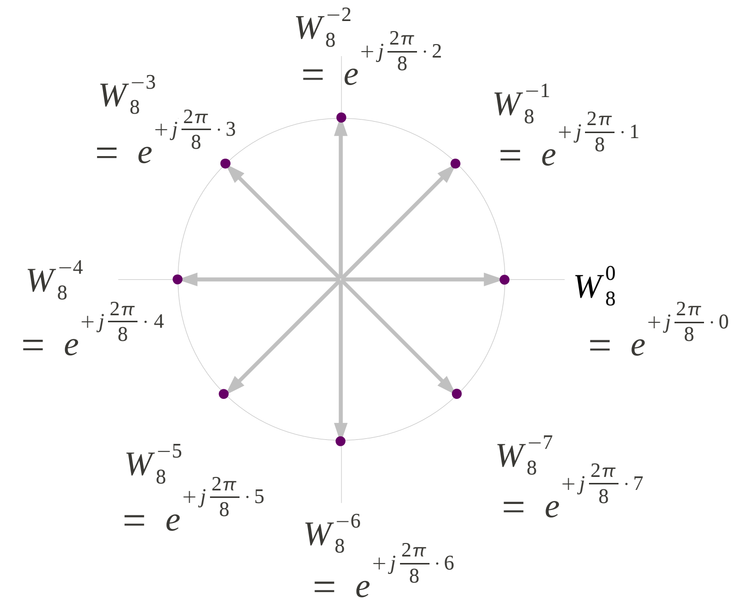


k=0	stride = 0	ccw angular speed = 0
k=1	stride = -1	ccw angular speed = 1ω
k=2	stride = -2	ccw angular speed = 2ω
k=3	stride = -3	ccw angular speed = 3ω
k=4	stride = -4	ccw angular speed = 4ω
k=5	stride = -5	ccw angular speed = 5ω
k=6	stride = -6	ccw angular speed = 6ω
k=7	stride = -7	ccw angular speed = 7ω

-7

DFT Matrix (4)

	n=0		n=1		n=2		n=3		n=4		n=5		n=6		n=7
k=0	0	-0	0	-0	0	-0	0	-0	0	-0	0	-0	0	-0	0
k=1	0	-1	-1	-1	-2	-1	-3	-1	-4	-1	-5	-1	-6	-1	-7
k=2	0	-2	-2	-2	-4	-2	-6	-2	0	-2	-2	-2	-4	-2	-6
k=3	0	-3	-3	-3	-6	-3	-1	-3	-4	-3	-7	-3	-2	-3	-5
k=4	0	-4	-4	-4	0	-4	-4	-4	0	-4	-4	-4	0	-4	-4
k=5	0	+3	+3	+3	+6	+3	+1	+3	+4	+3	+7	+3	+2	+3	+5
k=6	0	+2	+2	+2	+4	+2	+6	+2	0	+2	+2	+2	+4	+2	+6
k=7	0	+1	+1	+1	+2	+1	+3	+1	+4	+1	+5	+1	+6	+1	+7



k=0	stride = 0	ccw angular speed = 0			
k=1	stride = -1	ccw angular speed = 1ω			
k=2	stride = -2	ccw angular speed = 2ω			
k=3	stride = -3	ccw angular speed = 3ω			
k=4	stride = -4	ccw angular speed = 4ω			
k=5	stride = -5	ccw angular speed = 5ω	⟺	stride = +3	cw angular speed = 3ω
k=6	stride = -6	ccw angular speed = 6ω	⟺	stride = +2	cw angular speed = 2ω
k=7	stride = -7	ccw angular speed = 7ω	⟺	stride = +1	cw angular speed = 1ω

Angular Frequency

Frequency $f = \frac{1}{T}$ (Hz: cycles per second)

1Hz → event repeats once per second

Angular Frequency $\omega = \frac{2\pi}{T}$ (radians per second)

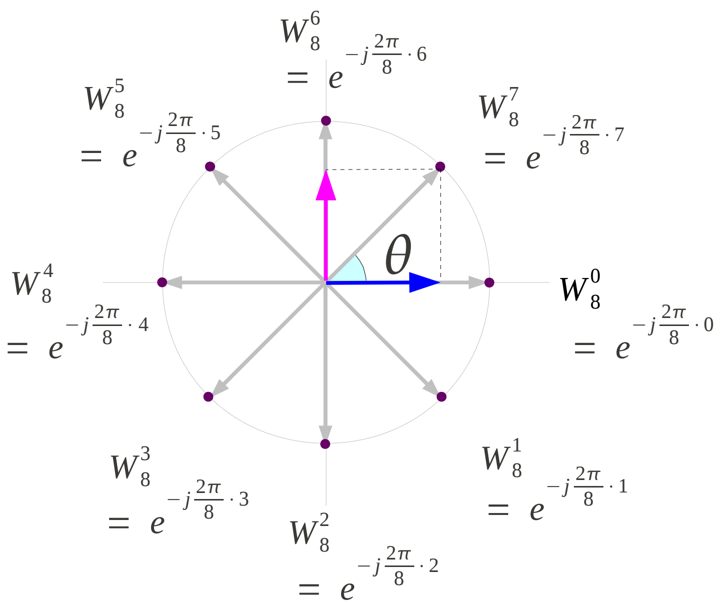
One revolution = 2π radian

$$\omega = 2\pi f = 2\pi \frac{1}{T}$$ → **Angular Speed**

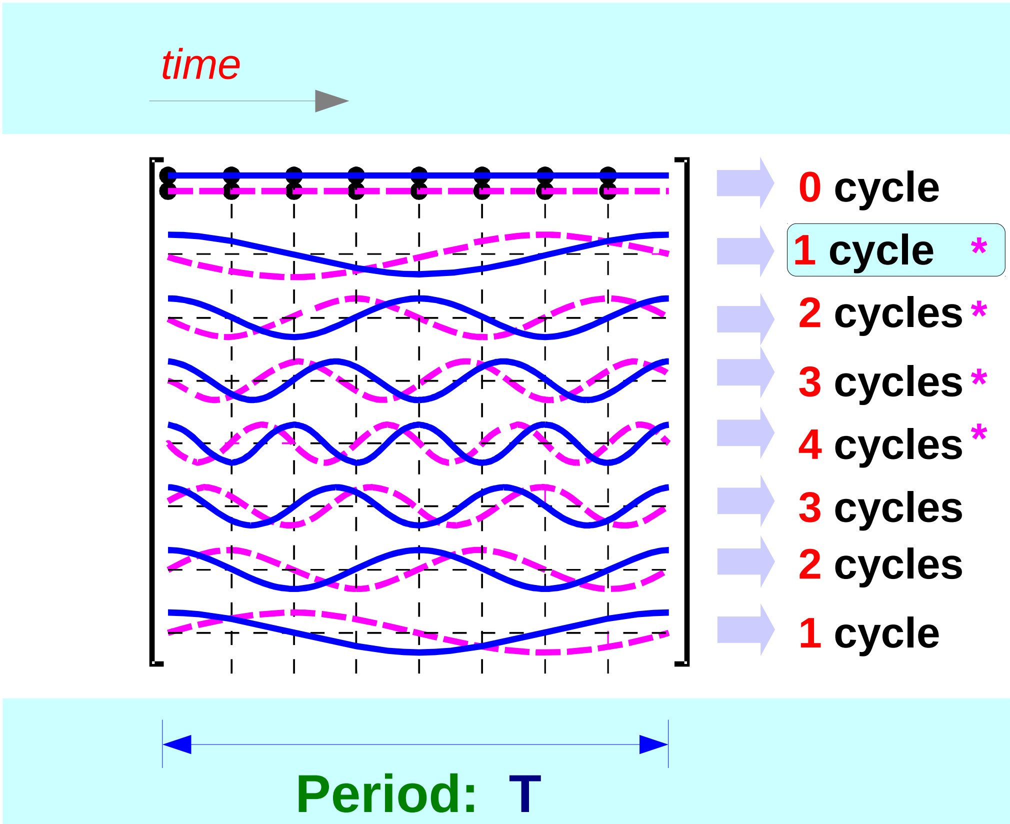
$$\theta = \omega t = 2\pi f t$$ → **Phase**

Fundamental Frequency

N=8 DFT → 8 complex phases → View as 8 samples in time domain



$$W_N^{nk} \triangleq e^{-j(2\pi/N)nk}$$

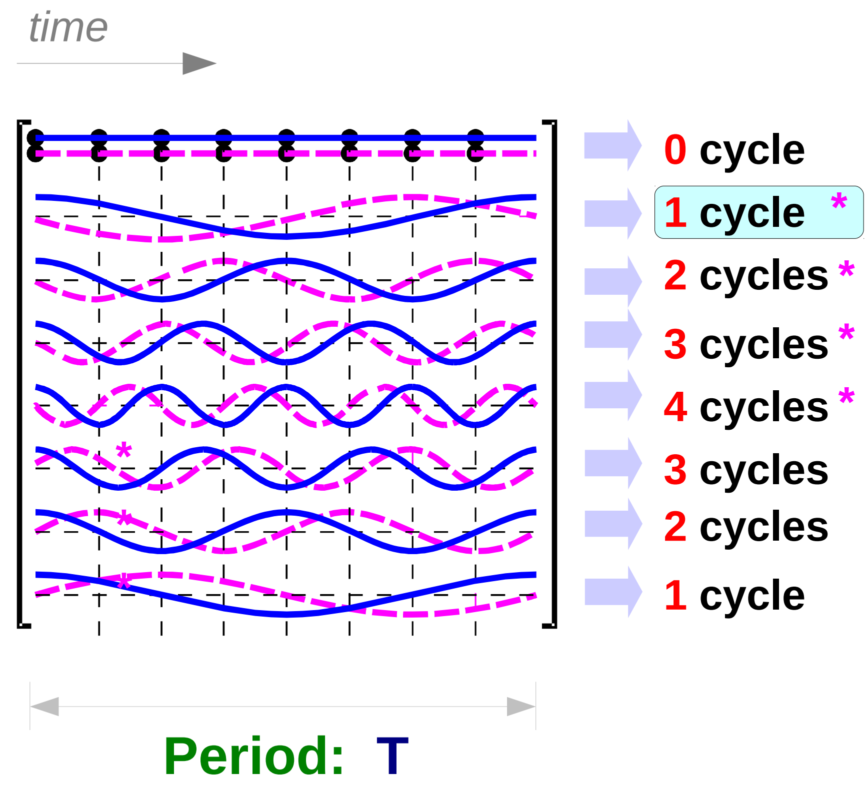


Fundamental Frequency : f = 1 / T

Harmonic Frequency

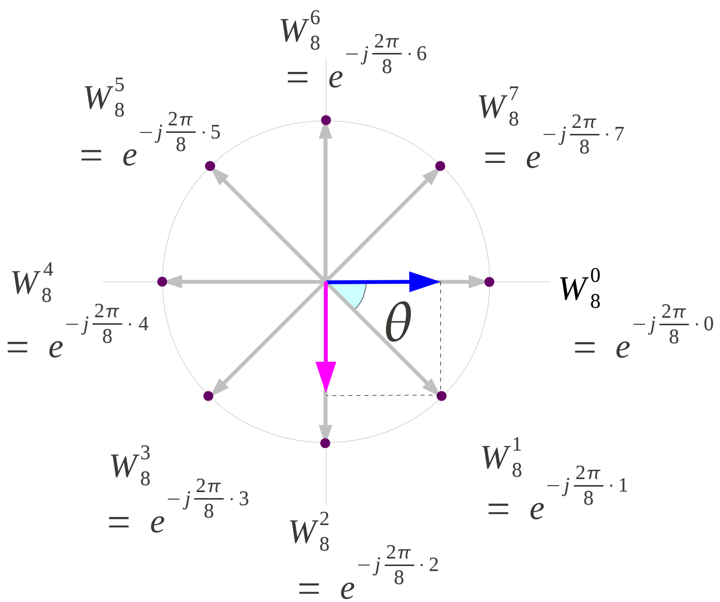
N=8 DFT ➡ 8 complex phases ➡ View as 8 samples in time domain

	frequency
	0
1^{st} harmonic *	-f
2^{nd} harmonic *	-2f
3^{rd} harmonic *	-3f
4^{th} harmonic *	-4f
3^{rd} harmonic	3f
2^{nd} harmonic	2f
1^{st} harmonic	f



Fundamental Frequency : f = 1 /T

Sampling Time



N=8 8 complex phases

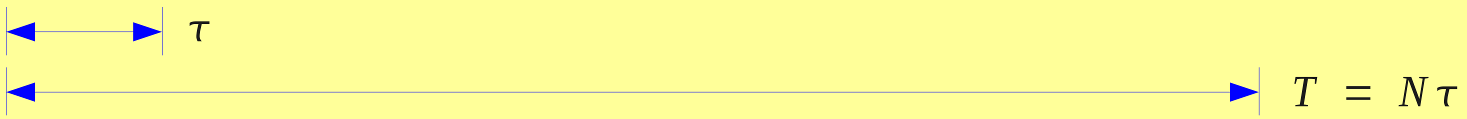
DFT 8 samples in time domain

Sampling Time : τ

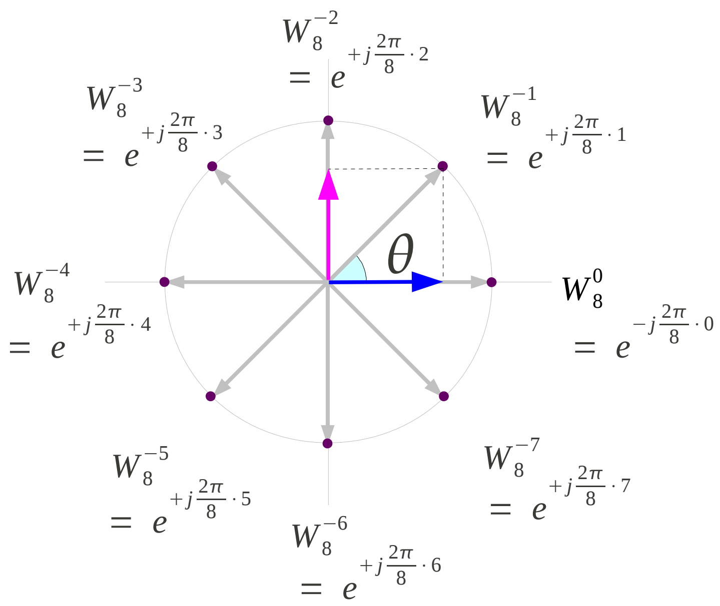
Period: T $T = N\tau$

n=0	n=1	n=2	n=3	n=4	n=5	n=6	n=7

k=1 $\left(e^{-j\cdot\frac{\pi}{4}\cdot 0} \quad e^{-j\cdot\frac{\pi}{4}\cdot 1} \quad e^{-j\cdot\frac{\pi}{4}\cdot 2} \quad e^{-j\cdot\frac{\pi}{4}\cdot 3} \quad e^{-j\cdot\frac{\pi}{4}\cdot 4} \quad e^{-j\cdot\frac{\pi}{4}\cdot 5} \quad e^{-j\cdot\frac{\pi}{4}\cdot 6} \quad e^{-j\cdot\frac{\pi}{4}\cdot 7} \right)$ → **1 cycle**



Sampling Frequency



Sampling Time τ

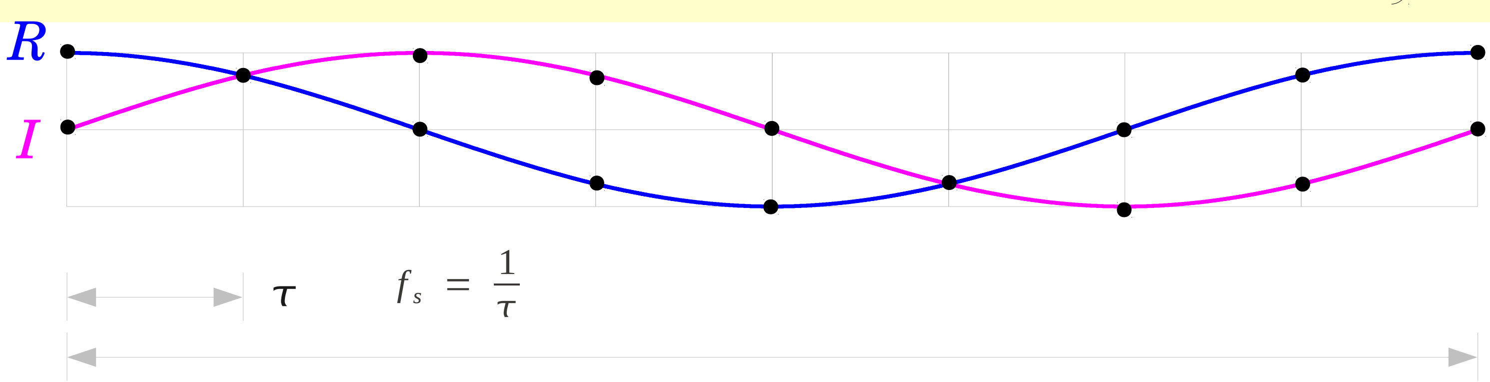
Sampling Frequency $f_s = \frac{1}{\tau}$ *(samples per second)*

Period $T = N\tau$

Fundamental Freq $f = \frac{1}{T}$ *(cycles per second)*

$$f = \frac{f_s}{N} \quad \left(= \frac{1}{N\tau} \right)$$

$$\left(e^{+j\cdot\frac{\pi}{4}\cdot 0} \quad e^{+j\cdot\frac{\pi}{4}\cdot 1} \quad e^{+j\cdot\frac{\pi}{4}\cdot 2} \quad e^{+j\cdot\frac{\pi}{4}\cdot 3} \quad e^{+j\cdot\frac{\pi}{4}\cdot 4} \quad e^{+j\cdot\frac{\pi}{4}\cdot 5} \quad e^{+j\cdot\frac{\pi}{4}\cdot 6} \quad e^{+j\cdot\frac{\pi}{4}\cdot 7} \right)$$

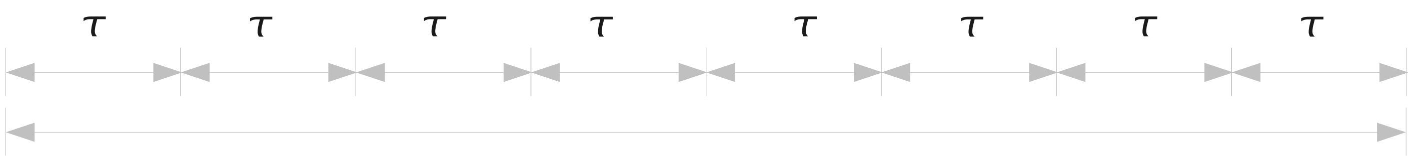


1 cycle

$f = \frac{f_s}{N}$

$T = N\tau$

Cycles / Sample



$$T = N\tau$$

τ *second / sample*

$1/\tau$ *sample / second*

$\frac{0}{N\tau}$	(*cycles / second*)	➡	**0 cycle**	**over N sample periods**	**= 0 / N** (*cycles / sample*)
$\frac{1}{N\tau}$	(*cycles / second*)	➡	**1 cycle**	" "	**= 1 / N** (*cycles / sample*)
$\frac{2}{N\tau}$	(*cycles / second*)	➡	**2 cycles**	" "	**= 2 / N** (*cycles / sample*)
$\frac{3}{N\tau}$	(*cycles / second*)	➡	**3 cycles**	" "	**= 3 / N** (*cycles / sample*)
$\frac{4}{N\tau}$	(*cycles / second*)	➡	**4 cycles**	" "	**= 4 / N** (*cycles / sample*)

Normalized Frequency

$$\frac{f_n}{f_s} = \frac{n}{N}$$

(cycles per sample)

$$\frac{\text{(cycles per second)}}{\text{(samples per second)}}$$

Normalized Frequency (1)

τ

$T = N\tau$

Sampling Time	τ	
Sequence Time Length	$T = N\tau$	
Sampling Frequency	$f_s = \frac{1}{\tau}$	(samples per second)

Normalized Frequency

$$\frac{f_n}{f_s} = \frac{n}{N}$$

(cycles per sample)

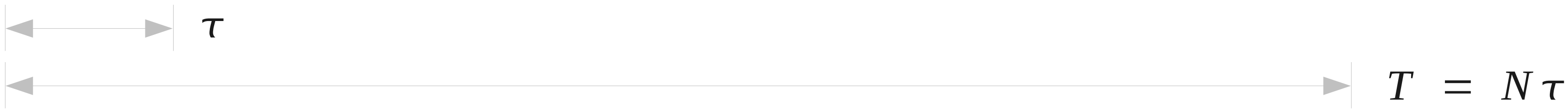
Fundamental Frequency → f_1

Harmonic Frequencies:

$$f_1 = 1 \cdot f_1 \qquad 1/N$$
$$f_2 = 2 \cdot f_1 \qquad 2/N$$
$$f_3 = 3 \cdot f_1 \qquad 3/N$$
$$\cdots \qquad \cdots$$
$$f_{N-1} = \frac{N}{2} \cdot f_1 \qquad 1/2$$

Normalized Frequencies

Normalized Frequency (2)



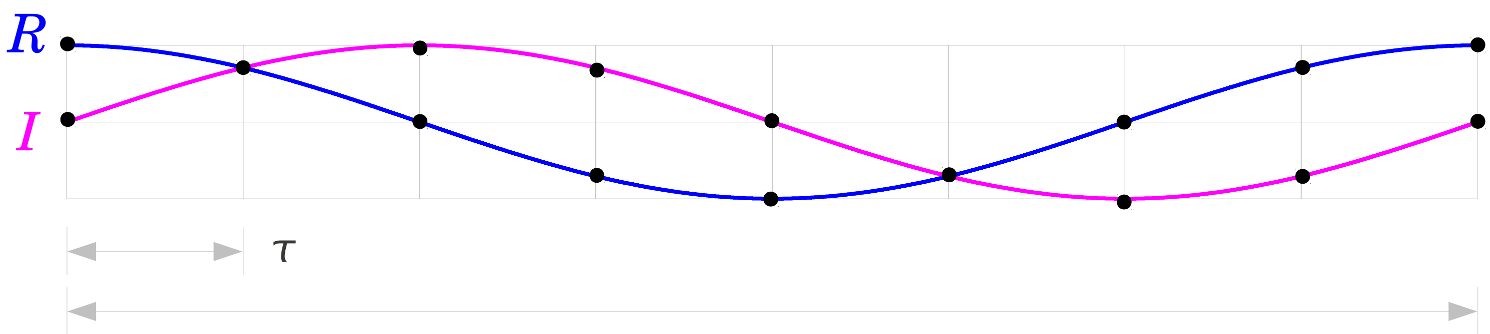
Sampling Time	τ	*(seconds per sample)*
Sequence Time Length	$T = N\tau$	
Sampling Frequency	$f_s = \frac{1}{\tau}$	*(samples per second)*

1st Harmonic Freq $\quad f_1 = \frac{1}{T} = \frac{1}{N\tau} = \frac{1}{N} f_s$

nth Harmonic Freq $\quad f_n = \frac{n}{T} = \frac{n}{N\tau} = \frac{n}{N} f_s \qquad n = 0,\ 1,\ 2,\ \ldots,\ \frac{N}{2}$

Normalized Frequency $\quad \frac{f_n}{f_s} = \frac{n}{N}$ *(cycles per sample)* $\quad \frac{\text{(cycles per second)}}{\text{(samples per second)}}$

Normalized Frequency (Ex 1)

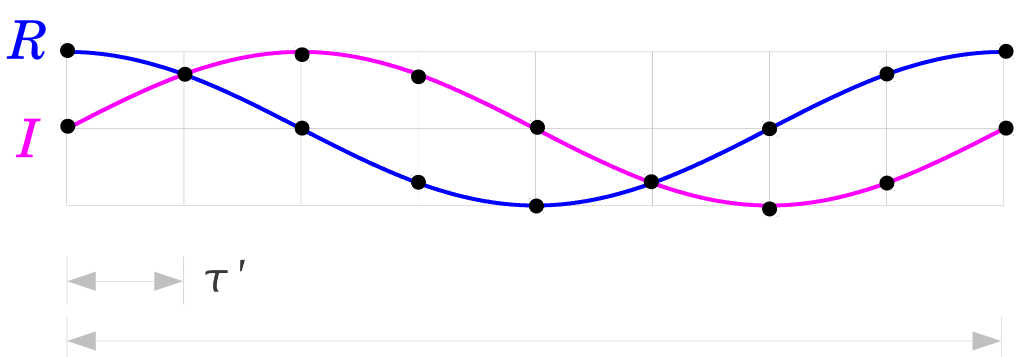


1 cycle

$$T = N\tau$$

1st Harmonic Freq $f_1 = \dfrac{1}{T} = \dfrac{1}{N\tau} = \dfrac{f_s}{N}$

Normalized Freq $\dfrac{f_1}{f_s} = \dfrac{1}{N}$



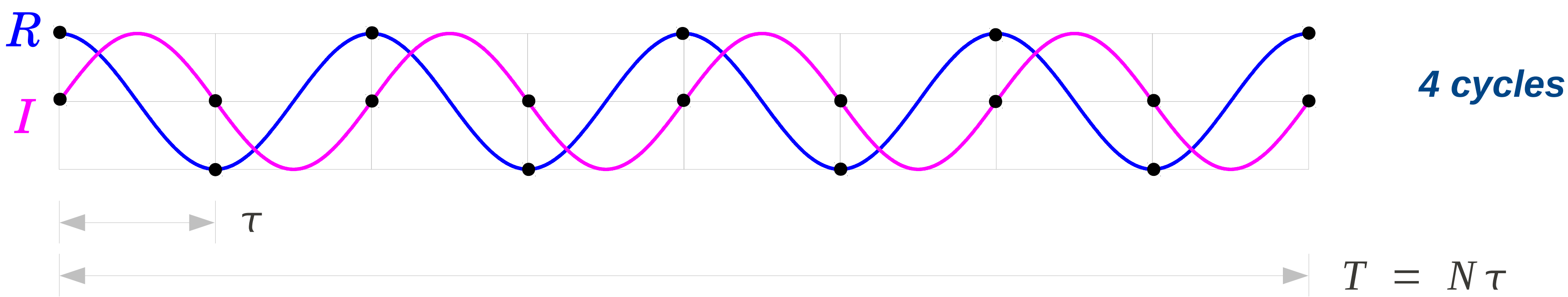
1 cycle

$$T' = N\tau'$$

1st Harmonic Freq $f_1' = \dfrac{1}{T'} = \dfrac{1}{N\tau'} = \dfrac{f_s'}{N}$

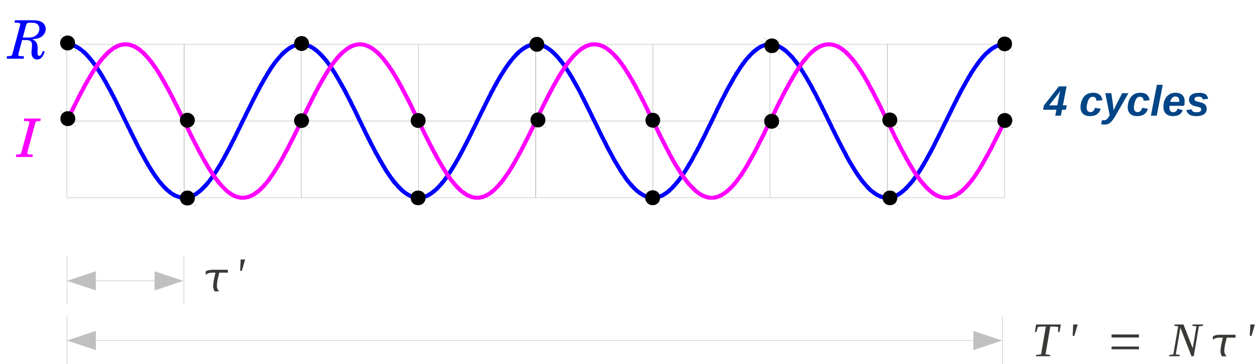
Normalized Freq $\dfrac{f_1'}{f_s'} = \dfrac{1}{N}$

Normalized Frequency (Ex 2)



4th Harmonic Freq $f_1 = \frac{4}{T} = \frac{4}{N\tau} = \frac{4 f_s}{N}$

Normalized Freq $\frac{f_1}{f_s} = \frac{4}{N}$



4th Harmonic Freq $f_1' = \frac{4}{T'} = \frac{4}{N\tau'} = \frac{4 f_s'}{N}$

Normalized Freq $\frac{f_1'}{f_s'} = \frac{4}{N}$

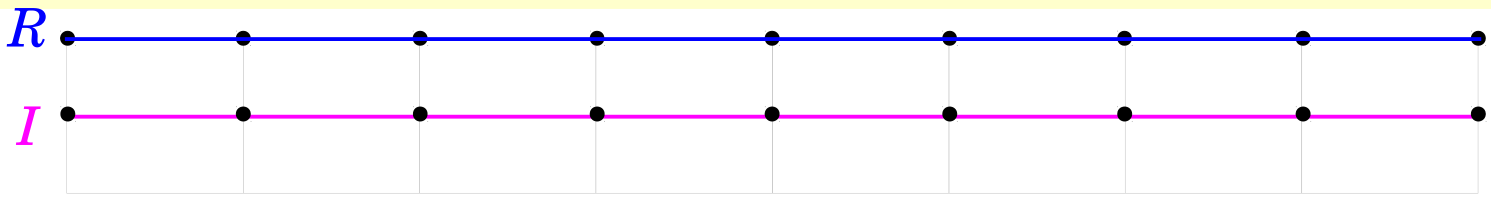
N=8 DFT

$$X[k] = \sum_{n=0}^{7} W_8^{kn}\, x[n] \qquad W_8^{kn} = e^{-j(\frac{2\pi}{8})kn}$$

$$\begin{bmatrix} X[0] \\ X[1] \\ X[2] \\ X[3] \\ X[4] \\ X[5] \\ X[6] \\ X[7] \end{bmatrix} = \begin{bmatrix} W_8^0 & W_8^0 & W_8^0 & W_8^0 & W_8^0 & W_8^0 & W_8^0 & W_8^0 \\ W_8^0 & W_8^1 & W_8^2 & W_8^3 & W_8^4 & W_8^5 & W_8^6 & W_8^7 \\ W_8^0 & W_8^2 & W_8^4 & W_8^6 & W_8^0 & W_8^2 & W_8^4 & W_8^6 \\ W_8^0 & W_8^3 & W_8^6 & W_8^1 & W_8^4 & W_8^7 & W_8^2 & W_8^5 \\ W_8^0 & W_8^4 & W_8^0 & W_8^4 & W_8^0 & W_8^4 & W_8^0 & W_8^4 \\ W_8^0 & W_8^5 & W_8^2 & W_8^7 & W_8^4 & W_8^1 & W_8^6 & W_8^3 \\ W_8^0 & W_8^6 & W_8^4 & W_8^2 & W_8^0 & W_8^6 & W_8^4 & W_8^2 \\ W_8^0 & W_8^7 & W_8^6 & W_8^5 & W_8^4 & W_8^3 & W_8^2 & W_8^1 \end{bmatrix} \begin{bmatrix} x[0] \\ x[1] \\ x[2] \\ x[3] \\ x[4] \\ x[5] \\ x[6] \\ x[7] \end{bmatrix}$$

N=8 DFT : The 1st Row of the DFT Matrix

$$\begin{pmatrix} e^{-j\cdot\frac{\pi}{4}\cdot 0} & e^{-j\cdot\frac{\pi}{4}\cdot 0} & e^{-j\cdot\frac{\pi}{4}\cdot 0} & e^{-j\cdot\frac{\pi}{4}\cdot 0} & e^{-j\cdot\frac{\pi}{4}\cdot 0} & e^{-j\cdot\frac{\pi}{4}\cdot 0} & e^{-j\cdot\frac{\pi}{4}\cdot 0} & e^{-j\cdot\frac{\pi}{4}\cdot 0} \end{pmatrix}$$



0 cycle

$$W_8^{kn} = e^{-j\left(\frac{2\pi}{8}\right)kn} \qquad k = 0, \quad n = 0, 1, \ldots, 7$$

R ➡ *samples of* $\cos(-\omega t) = \cos(\omega t)$

I ➡ *samples of* $\sin(-\omega t) = -\sin(\omega t)$

measure →

$$\omega t = 2\pi f t$$
$$2\pi\cdot\left(\tfrac{0}{8}\right)\cdot f_s\cdot t$$

$X[0]$ *measures how much of the* $+0\cdot\omega$ *component is present in* x.

τ

$T = N\tau$

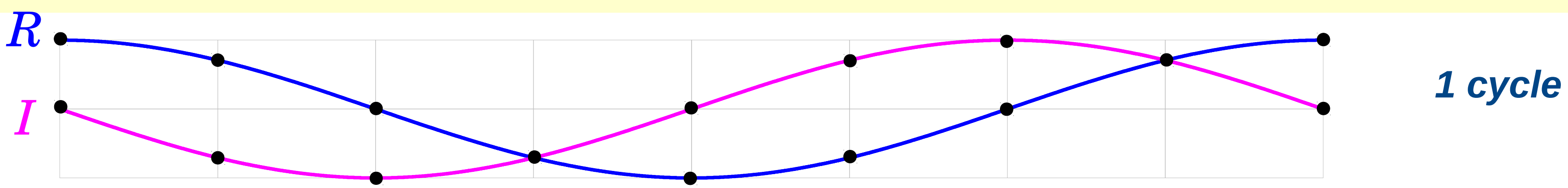
Sampling Time τ

Sampling Frequency $f_s = \frac{1}{\tau}$

Sequence Time Length $T = N\tau$

zero Frequency

$$\begin{pmatrix} e^{-j\cdot\frac{\pi}{4}\cdot 0} & e^{-j\cdot\frac{\pi}{4}\cdot 1} & e^{-j\cdot\frac{\pi}{4}\cdot 2} & e^{-j\cdot\frac{\pi}{4}\cdot 3} & e^{-j\cdot\frac{\pi}{4}\cdot 4} & e^{-j\cdot\frac{\pi}{4}\cdot 5} & e^{-j\cdot\frac{\pi}{4}\cdot 6} & e^{-j\cdot\frac{\pi}{4}\cdot 7} \end{pmatrix}$$



$$W_8^{kn} = e^{-j\left(\frac{2\pi}{8}\right)kn} \qquad k = 1, \quad n = 0, 1, \ldots, 7$$

R ➡ *samples of* $\cos(-\omega t) = \cos(\omega t)$

I ➡ *samples of* $\sin(-\omega t) = -\sin(\omega t)$

measure →

$$\omega t = 2\pi f t$$

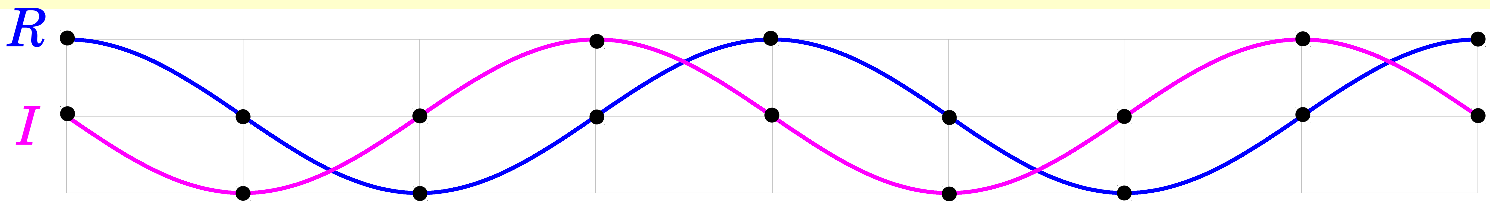
$$2\pi\cdot\left(\tfrac{1}{8}\right)\cdot f_s\cdot t$$

$X[1]$ *measures how much of the* $+1\cdot\omega$ *component is present in* x.

Sampling Time	τ	*Sampling Frequency*	$f_s = \frac{1}{\tau}$
Sequence Time Length	$T = N\tau$	*1st Harmonic Freq*	$f_1 = \frac{1}{T} = \frac{1}{N\tau} = \frac{f_s}{N}$

N=8 DFT : The 3rd Row of the DFT Matrix

$$\left(e^{-j\cdot\frac{\pi}{4}\cdot 0} \quad e^{-j\cdot\frac{\pi}{4}\cdot 2} \quad e^{-j\cdot\frac{\pi}{4}\cdot 4} \quad e^{-j\cdot\frac{\pi}{4}\cdot 6} \quad e^{-j\cdot\frac{\pi}{4}\cdot 0} \quad e^{-j\cdot\frac{\pi}{4}\cdot 2} \quad e^{-j\cdot\frac{\pi}{4}\cdot 4} \quad e^{-j\cdot\frac{\pi}{4}\cdot 6} \right)$$



2 cycles

$$W_8^{kn} = e^{-j\left(\frac{2\pi}{8}\right)kn} \qquad k = 2, \quad n = 0, 1, \ldots, 7$$

R ➡ *samples of* $\cos(-2\omega t) = \cos(2\omega t)$

I ➡ *samples of* $\sin(-2\omega t) = -\sin(2\omega t)$

measure →

$\omega t = 2\pi f t$

$2\pi\cdot\left(\frac{2}{8}\right)\cdot f_s\cdot t$

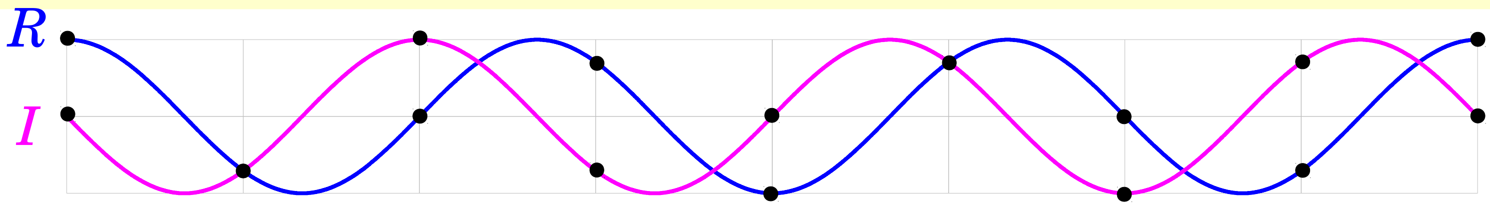
$\boldsymbol{X}[2]$ *measures how much of the* $\boldsymbol{+2\cdot\omega}$ *component is present in* $\boldsymbol{x}$.

Sampling Time τ | Sampling Frequency $f_s = \frac{1}{\tau}$

Sequence Time Length $T = N\tau$ | 2nd Harmonic Freq $f_2 = \frac{2}{T} = \frac{2}{N\tau} = \frac{2 f_s}{N}$

N=8 DFT : The 4th Row of the DFT Matrix

$$\left(e^{-j\cdot\frac{\pi}{4}\cdot 0} \quad e^{-j\cdot\frac{\pi}{4}\cdot 3} \quad e^{-j\cdot\frac{\pi}{4}\cdot 6} \quad e^{-j\cdot\frac{\pi}{4}\cdot 1} \quad e^{-j\cdot\frac{\pi}{4}\cdot 4} \quad e^{-j\cdot\frac{\pi}{4}\cdot 7} \quad e^{-j\cdot\frac{\pi}{4}\cdot 2} \quad e^{-j\cdot\frac{\pi}{4}\cdot 5} \right)$$



3 cycles

$$W_8^{kn} = e^{-j\left(\frac{2\pi}{8}\right)kn} \qquad k = 3, \quad n = 0, 1, \ldots, 7$$

R ➡ *samples of* $\cos(-3\omega t) = \cos(3\omega t)$

I ➡ *samples of* $\sin(-3\omega t) = -\sin(3\omega t)$

measure →

$\omega t = 2\pi f t$

$2\pi\cdot\left(\frac{3}{8}\right)\cdot f_s\cdot t$

$X[3]$ *measures how much of the* $+3\cdot\omega$ *component is present in* x.

τ

$T = N\tau$

Sampling Time τ

Sampling Frequency $f_s = \frac{1}{\tau}$

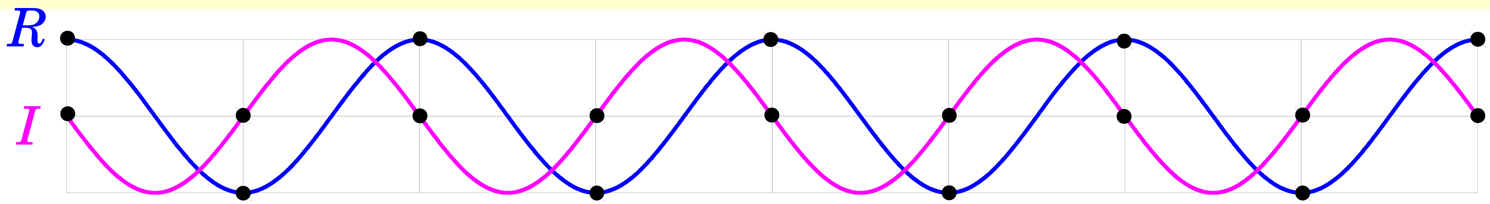
Sequence Time Length $T = N\tau$

3rd Harmonic Freq $f_3 = \frac{3}{T} = \frac{3}{N\tau} = \frac{3 f_s}{N}$

Young Won Lim
1/27/11

N=8 DFT : The 5th Row of the DFT Matrix

$$\left(e^{-j\cdot\frac{\pi}{4}\cdot 0} \quad e^{-j\cdot\frac{\pi}{4}\cdot 4} \quad e^{-j\cdot\frac{\pi}{4}\cdot 0} \quad e^{-j\cdot\frac{\pi}{4}\cdot 4} \quad e^{-j\cdot\frac{\pi}{4}\cdot 0} \quad e^{-j\cdot\frac{\pi}{4}\cdot 4} \quad e^{-j\cdot\frac{\pi}{4}\cdot 0} \quad e^{-j\cdot\frac{\pi}{4}\cdot 4} \right)$$



4 cycles

$$W_8^{kn} = e^{-j\left(\frac{2\pi}{8}\right)kn} \qquad k = 4, \quad n = 0, 1, \ldots, 7$$

R ➡ *samples of* $\cos(-4\omega t) = \cos(4\omega t)$

I ➡ *samples of* $\sin(-4\omega t) = -\sin(4\omega t)$

measure →

$$\omega t = 2\pi f t$$

$$2\pi\cdot\left(\tfrac{4}{8}\right)\cdot f_s\cdot t$$

X[4] *measures how much of the* ***+4·ω*** *component is present in* ***x***.

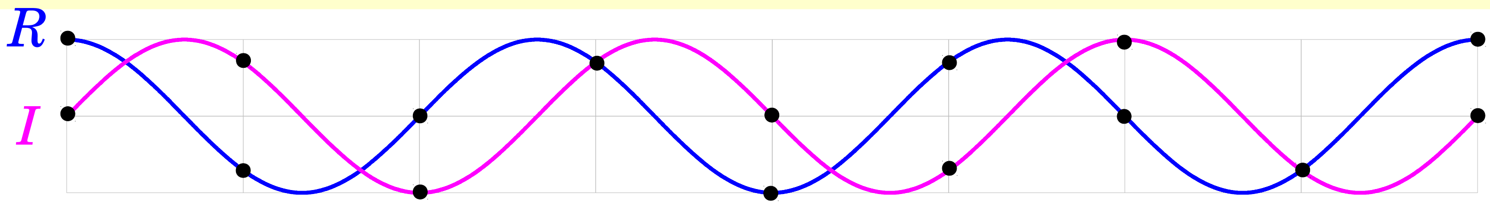
τ

$T = N\tau$

Sampling Time	τ	*Sampling Frequency*	$f_s = \dfrac{1}{\tau}$
Sequence Time Length	$T = N\tau$	*4th Harmonic Freq*	$f_4 = \dfrac{4}{T} = \dfrac{4}{N\tau} = \dfrac{4 f_s}{N}$

N=8 DFT : The 6th Row of the DFT Matrix

$$\left(e^{-j\cdot\frac{\pi}{4}\cdot 0} \quad e^{-j\cdot\frac{\pi}{4}\cdot 5} \quad e^{-j\cdot\frac{\pi}{4}\cdot 2} \quad e^{-j\cdot\frac{\pi}{4}\cdot 7} \quad e^{-j\cdot\frac{\pi}{4}\cdot 4} \quad e^{-j\cdot\frac{\pi}{4}\cdot 1} \quad e^{-j\cdot\frac{\pi}{4}\cdot 6} \quad e^{-j\cdot\frac{\pi}{4}\cdot 3} \right)$$



3 cycles

$$W_8^{kn} = e^{-j(\frac{2\pi}{8})kn} \qquad k = 5, \quad n = 0, 1, \ldots, 7$$

R ➡ *samples of* $\cos(-(-3\omega)t) = \cos(3\omega t)$

I ➡ *samples of* $\sin(-(-3\omega)t) = \sin(3\omega t)$

measure →

$$-\omega t = -2\pi f t$$

$$2\pi\cdot(\tfrac{-3}{8})\cdot f_s\cdot t$$

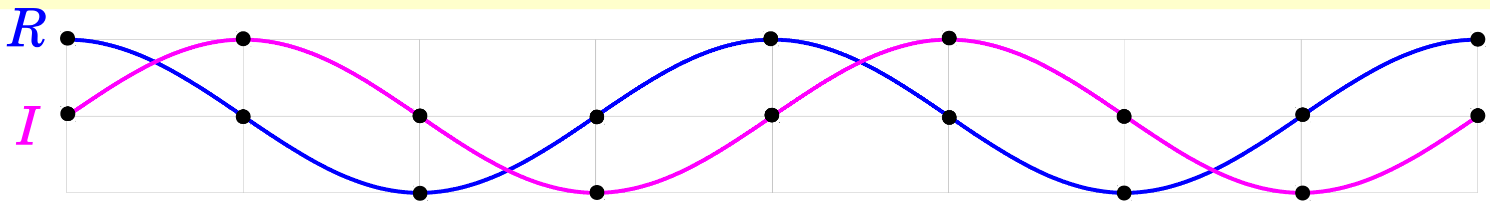
X[5] *measures how much of the* ***–3·ω*** *component is present in* ***x***.

Sampling Time τ *Sampling Frequency* $f_s = \frac{1}{\tau}$

Sequence Time Length $T = N\tau$ *-3rd Harmonic Freq* $f_{-3} = \frac{-3}{T} = \frac{-3}{N\tau} = \frac{-3 f_s}{N}$

N=8 DFT : The 7th Row of the DFT Matrix

$$\begin{pmatrix} e^{-j\cdot\frac{\pi}{4}\cdot 0} & e^{-j\cdot\frac{\pi}{4}\cdot 6} & e^{-j\cdot\frac{\pi}{4}\cdot 4} & e^{-j\cdot\frac{\pi}{4}\cdot 2} & e^{-j\cdot\frac{\pi}{4}\cdot 0} & e^{-j\cdot\frac{\pi}{4}\cdot 6} & e^{-j\cdot\frac{\pi}{4}\cdot 4} & e^{-j\cdot\frac{\pi}{4}\cdot 2} \end{pmatrix}$$



2 cycles

$$W_8^{kn} = e^{-j\left(\frac{2\pi}{8}\right)kn} \qquad k = 2, \quad n = 0, 1, \ldots, 7$$

R ➡ *samples of* $\cos(-(-2\omega)t) = \cos(2\omega t)$

I ➡ *samples of* $\sin(-(-2\omega)t) = \sin(2\omega t)$

measure →

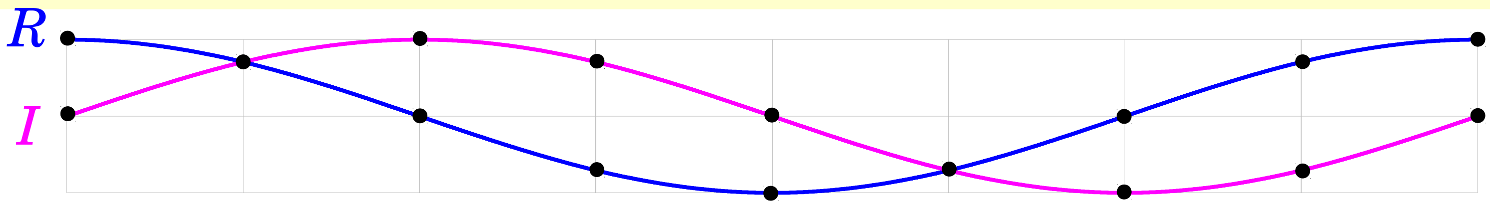
$$-\omega t = -2\pi f t$$
$$2\pi\cdot\left(\tfrac{-2}{8}\right)\cdot f_s\cdot t$$

$\boldsymbol{X}[6]$ *measures how much of the* $-2\cdot\omega$ *component is present in* $\boldsymbol{x}$.

Sampling Time	τ	*Sampling Frequency*	$f_s = \dfrac{1}{\tau}$
Sequence Time Length	$T = N\tau$	*-2nd Harmonic Freq*	$f_{-2} = \dfrac{-2}{T} = \dfrac{-2}{N\tau} = \dfrac{-2 f_s}{N}$

N=8 DFT : The 8th Row of the DFT Matrix

$$\begin{pmatrix} e^{-j\cdot\frac{\pi}{4}\cdot 0} & e^{-j\cdot\frac{\pi}{4}\cdot 7} & e^{-j\cdot\frac{\pi}{4}\cdot 6} & e^{-j\cdot\frac{\pi}{4}\cdot 5} & e^{-j\cdot\frac{\pi}{4}\cdot 4} & e^{-j\cdot\frac{\pi}{4}\cdot 3} & e^{-j\cdot\frac{\pi}{4}\cdot 2} & e^{-j\cdot\frac{\pi}{4}\cdot 1} \end{pmatrix}$$



1 cycle

$$W_8^{kn} = e^{-j(\frac{2\pi}{8})kn} \qquad k = 7, \quad n = 0, 1, \ldots, 7$$

R → *samples of* $\cos(-(-\omega)t) = \cos(\omega t)$

I → *samples of* $\sin(-(-\omega)t) = \sin(\omega t)$

measure →

$$-\omega t = -2\pi f t$$
$$2\pi\cdot(\tfrac{-1}{8})\cdot f_s\cdot t$$

X*[7]** *measures how much of the* $-1\cdot\omega$ *component is present in* ***x.

τ

$T = N\tau$

Sampling Time	τ	Sampling Frequency	$f_s = \frac{1}{\tau}$
Sequence Time Length	$T = N\tau$	-1st Harmonic Freq	$f_{-1} = \frac{-1}{T} = \frac{-1}{N\tau} = \frac{f_s}{N}$

N=8 DFT : DFT Matrix in + or – Frequencies

$$\omega_0 = 2\pi \cdot \frac{f_s}{N}$$

0th row:	*samples of*	$\cos(\ 0\,\omega_0)t + j\cdot\sin(\ 0\,\omega_0)t$	(0 cycle)
1th row:	*samples of*	$\cos(-1\,\omega_0)t + j\cdot\sin(-1\,\omega_0)t$	(1 cycle)
2th row:	*samples of*	$\cos(-2\,\omega_0)t + j\cdot\sin(-2\,\omega_0)t$	(2 cycles)
3th row:	*samples of*	$\cos(-3\,\omega_0)t + j\cdot\sin(-3\,\omega_0)t$	(3 cycles)
4th row:	*samples of*	$\cos(-4\,\omega_0)t + j\cdot\sin(-4\,\omega_0)t$	(4 cycles)
5th row:	*samples of*	$\cos(-5\,\omega_0)t + j\cdot\sin(-5\,\omega_0)t$	(5 cycles)
6th row:	*samples of*	$\cos(-6\,\omega_0)t + j\cdot\sin(-6\,\omega_0)t$	(6 cycles)
7th row:	*samples of*	$\cos(-7\,\omega_0)t + j\cdot\sin(-7\,\omega_0)t$	(7 cycles)

=

0th row:	*samples of*	$\cos(\ 0\,\omega_0)t + j\cdot\sin(\ 0\,\omega_0)t$	(0 cycle)
1th row:	*samples of*	$\cos(+7\,\omega_0)t + j\cdot\sin(+7\,\omega_0)t$	(7 cycles)
2th row:	*samples of*	$\cos(+6\,\omega_0)t + j\cdot\sin(+6\,\omega_0)t$	(6 cycles)
3th row:	*samples of*	$\cos(+5\,\omega_0)t + j\cdot\sin(+5\,\omega_0)t$	(5 cycles)
4th row:	*samples of*	$\cos(+4\,\omega_0)t + j\cdot\sin(+4\,\omega_0)t$	(4 cycles)
5th row:	*samples of*	$\cos(+3\,\omega_0)t + j\cdot\sin(+3\,\omega_0)t$	(3 cycles)
6th row:	*samples of*	$\cos(+2\,\omega_0)t + j\cdot\sin(+2\,\omega_0)t$	(2 cycles)
7th row:	*samples of*	$\cos(+1\,\omega_0)t + j\cdot\sin(+1\,\omega_0)t$	(1 cycles)

N=8 DFT : DFT Matrix in Both Frequencies

$$\omega_0 = 2\pi \cdot \frac{f_s}{N}$$

0th row:	*samples of*	$\cos(\ 0\omega_0)t + j\cdot\sin(\ 0\omega_0)t$	(0 cycle)
1th row:	*samples of*	$\cos(-1\omega_0)t + j\cdot\sin(-1\omega_0)t$	(1 cycle)
2th row:	*samples of*	$\cos(-2\omega_0)t + j\cdot\sin(-2\omega_0)t$	(2 cycles)
3th row:	*samples of*	$\cos(-3\omega_0)t + j\cdot\sin(-3\omega_0)t$	(3 cycles)
4th row:	*samples of*	$\cos(-4\omega_0)t + j\cdot\sin(-4\omega_0)t$	(4 cycles)
5th row:	*samples of*	$\cos(-5\omega_0)t + j\cdot\sin(-5\omega_0)t$	(5 cycles)
6th row:	*samples of*	$\cos(-6\omega_0)t + j\cdot\sin(-6\omega_0)t$	(6 cycles)
7th row:	*samples of*	$\cos(-7\omega_0)t + j\cdot\sin(-7\omega_0)t$	(7 cycles)

$=$

0th row:	*samples of*	$\cos(\ 0\omega_0)t + j\cdot\sin(\ 0\omega_0)t$	(0 cycle)
1th row:	*samples of*	$\cos(-1\omega_0)t + j\cdot\sin(-1\omega_0)t$	(1 cycle)
2th row:	*samples of*	$\cos(-2\omega_0)t + j\cdot\sin(-2\omega_0)t$	(2 cycles)
3th row:	*samples of*	$\cos(-3\omega_0)t + j\cdot\sin(-3\omega_0)t$	(3 cycles)
4th row:	*samples of*	$\cos(-4\omega_0)t + j\cdot\sin(-4\omega_0)t$	(4 cycles)
5th row:	*samples of*	$\cos(+3\omega_0)t + j\cdot\sin(+3\omega_0)t$	(3 cycles)
6th row:	*samples of*	$\cos(+2\omega_0)t + j\cdot\sin(+2\omega_0)t$	(2 cycles)
7th row:	*samples of*	$\cos(+1\omega_0)t + j\cdot\sin(+1\omega_0)t$	(1 cycles)

Frequency View of a DFT Matrix

$row\ \ 0$	$f\ =\ 0$	$(0\,cycle)$	0
$row\ \ 1$	$f\ =\ -1 * f_o$	$(1\,cycle)$	$\frac{-1}{N}$
$row\ \ 2$	$f\ =\ -2 * f_o$	$(2\,cycles)$	$\frac{-2}{N}$
$row\ \ (\frac{N}{2}-1)$	$f\ =\ -(\frac{N}{2}-1) * f_o$	$(\frac{N}{2}-1\,cycles)$	$\frac{-1}{2}+\frac{1}{N}$
$row\ \ (\frac{N}{2})$	$f\ =\ -(\frac{N}{2}) * f_o$	$(\frac{N}{2}\ \ cycles)$	$\frac{-1}{2}$
$row\ \ (\frac{N}{2}+1)$	$f\ =\ +(\frac{N}{2}-1) * f_o$	$(\frac{N}{2}-1\,cycles)$	$\frac{1}{2}-\frac{1}{N}$
$row\ \ N-2$	$f\ =\ +2 * f_o$	$(2\,cycles)$	$\frac{2}{N}$
$row\ \ N-1$	$f\ =\ +1 * f_o$	$(1\,cycle)$	$\frac{1}{N}$

$\frac{N}{2}-1$ (rows 1 to $\frac{N}{2}-1$)

$\frac{N}{2}-1$ (rows $\frac{N}{2}+1$ to $N-1$)

Normalized Frequency

$$f_o\ =\ \frac{f_s}{N}$$

Frequency View of a X[i] Vector

				Normalized Frequency	
	$X[0]$	$f = 0$	0	0	$(0\, cycle)$
	$X[1]$	$f = +1 * f_o$	$\frac{1}{N} f_s$	$\frac{1}{N}$	$(1\, cycle)$
	$X[2]$	$f = +2 * f_o$	$\frac{2}{N} f_s$	$\frac{2}{N}$	$(2\, cycles)$
$\frac{N}{2} - 1$					
	$X[\frac{N}{2} - 1]$	$f = +(\frac{N}{2} - 1) * f_o$	$\frac{(\frac{N}{2} - 1)}{N} f_s$	$\frac{1}{2} - \frac{1}{N}$	$(\frac{N}{2} - 1\, cycles)$
	$X[\frac{N}{2}]$	$f = +(\frac{N}{2}) * f_o$	$\frac{1}{2} f_s$	$\frac{1}{2}$	$(\frac{N}{2}\, cycles)$
	$X[\frac{N}{2} + 1]$	$f = -(\frac{N}{2} - 1) * f_o$	$-\frac{(\frac{N}{2} - 1)}{N} f_s$	$\frac{-1}{2} + \frac{1}{N}$	$(\frac{N}{2} - 1\, cycles)$
$\frac{N}{2} - 1$					
	$X[N-2]$	$f = -2 * f_o$	$-\frac{2}{N} f_s$	$\frac{-2}{N}$	$(2\, cycles)$
	$X[N-1]$	$f = -1 * f_o$	$-\frac{1}{N} f_s$	$\frac{-1}{N}$	$(1\, cycle)$

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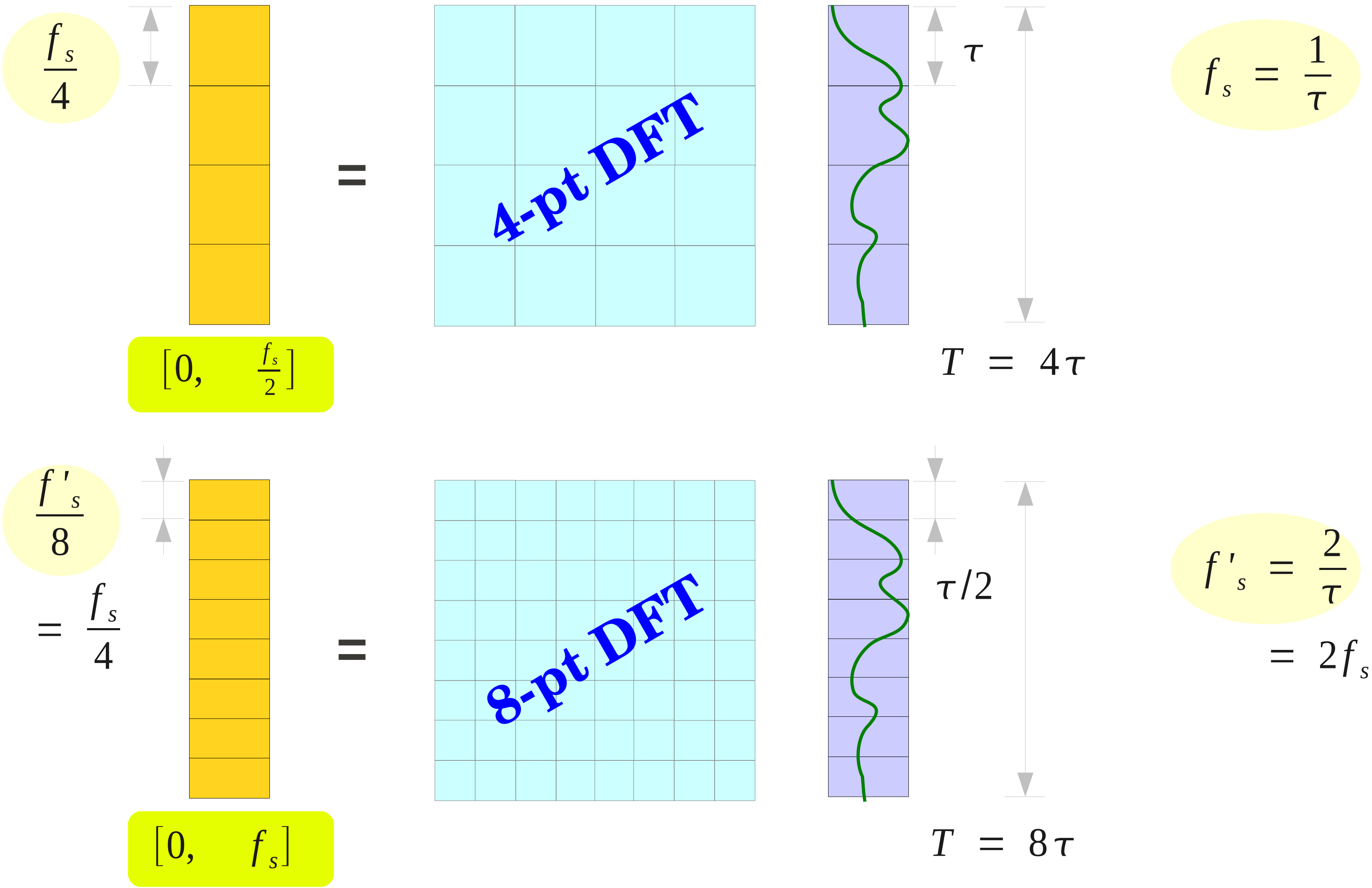
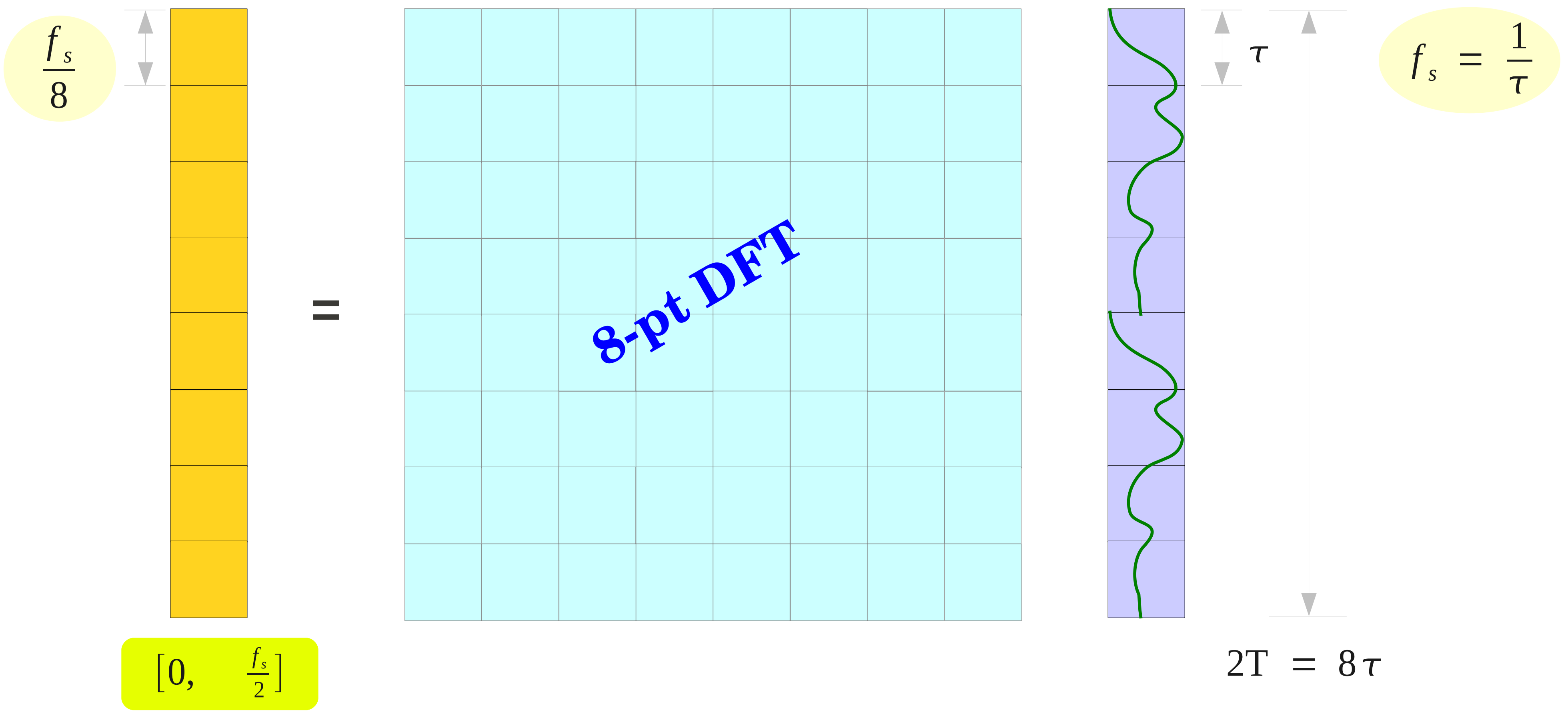
1/27/11



$\frac{f_s}{4}$
=
4-pt DFT
τ
$f_s = \frac{1}{\tau}$
$[0, \frac{f_s}{2}]$
$T = 4\tau$
$\frac{f'_s}{8}$
$= \frac{f_s}{4}$
=
8-pt DFT
$\tau/2$
$f'_s = \frac{2}{\tau}$
$= 2f_s$
$[0, f_s]$
$T = 8\tau$


$\frac{f_s}{8}$
=
8-pt DFT
τ
$f_s = \frac{1}{\tau}$
$[0, \frac{f_s}{2}]$
$2T = 8\tau$

References

[1] http://en.wikipedia.org/
[2] J.H. McClellan, et al., Signal Processing First, Pearson Prentice Hall, 2003
[3] A “graphical interpretation” of the DFT and FFT, by Steve Mann