

# DFT Matrix Properties (3A)

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# N=8 DFT Matrix

$$X[k] = \sum_{n=0}^7 W_8^{kn} x[n] \quad W_8^{kn} = e^{-j\left(\frac{2\pi}{8}\right)kn}$$

$$\begin{bmatrix} X[0] \\ X[1] \\ X[2] \\ X[3] \\ X[4] \\ X[5] \\ X[6] \\ X[7] \end{bmatrix} = \begin{bmatrix} e^{-j\frac{\pi}{4}\cdot 0} & e^{-j\frac{\pi}{4}\cdot 0} & e^{-j\frac{\pi}{4}\cdot 0} & e^{-j\frac{\pi}{4}\cdot 0} & e^{-j\frac{\pi}{4}\cdot 0} & e^{-j\frac{\pi}{4}\cdot 0} & e^{-j\frac{\pi}{4}\cdot 0} & e^{-j\frac{\pi}{4}\cdot 0} \\ e^{-j\frac{\pi}{4}\cdot 0} & e^{-j\frac{\pi}{4}\cdot 1} & e^{-j\frac{\pi}{4}\cdot 2} & e^{-j\frac{\pi}{4}\cdot 3} & e^{-j\frac{\pi}{4}\cdot 4} & e^{-j\frac{\pi}{4}\cdot 5} & e^{-j\frac{\pi}{4}\cdot 6} & e^{-j\frac{\pi}{4}\cdot 7} \\ e^{-j\frac{\pi}{4}\cdot 0} & e^{-j\frac{\pi}{4}\cdot 2} & e^{-j\frac{\pi}{4}\cdot 4} & e^{-j\frac{\pi}{4}\cdot 6} & e^{-j\frac{\pi}{4}\cdot 0} & e^{-j\frac{\pi}{4}\cdot 2} & e^{-j\frac{\pi}{4}\cdot 4} & e^{-j\frac{\pi}{4}\cdot 6} \\ e^{-j\frac{\pi}{4}\cdot 0} & e^{-j\frac{\pi}{4}\cdot 3} & e^{-j\frac{\pi}{4}\cdot 6} & e^{-j\frac{\pi}{4}\cdot 1} & e^{-j\frac{\pi}{4}\cdot 4} & e^{-j\frac{\pi}{4}\cdot 7} & e^{-j\frac{\pi}{4}\cdot 2} & e^{-j\frac{\pi}{4}\cdot 5} \\ e^{-j\frac{\pi}{4}\cdot 0} & e^{-j\frac{\pi}{4}\cdot 4} & e^{-j\frac{\pi}{4}\cdot 0} & e^{-j\frac{\pi}{4}\cdot 4} & e^{-j\frac{\pi}{4}\cdot 0} & e^{-j\frac{\pi}{4}\cdot 4} & e^{-j\frac{\pi}{4}\cdot 0} & e^{-j\frac{\pi}{4}\cdot 4} \\ e^{-j\frac{\pi}{4}\cdot 0} & e^{-j\frac{\pi}{4}\cdot 5} & e^{-j\frac{\pi}{4}\cdot 2} & e^{-j\frac{\pi}{4}\cdot 7} & e^{-j\frac{\pi}{4}\cdot 4} & e^{-j\frac{\pi}{4}\cdot 1} & e^{-j\frac{\pi}{4}\cdot 6} & e^{-j\frac{\pi}{4}\cdot 3} \\ e^{-j\frac{\pi}{4}\cdot 0} & e^{-j\frac{\pi}{4}\cdot 6} & e^{-j\frac{\pi}{4}\cdot 4} & e^{-j\frac{\pi}{4}\cdot 2} & e^{-j\frac{\pi}{4}\cdot 0} & e^{-j\frac{\pi}{4}\cdot 6} & e^{-j\frac{\pi}{4}\cdot 4} & e^{-j\frac{\pi}{4}\cdot 2} \\ e^{-j\frac{\pi}{4}\cdot 0} & e^{-j\frac{\pi}{4}\cdot 7} & e^{-j\frac{\pi}{4}\cdot 6} & e^{-j\frac{\pi}{4}\cdot 5} & e^{-j\frac{\pi}{4}\cdot 4} & e^{-j\frac{\pi}{4}\cdot 3} & e^{-j\frac{\pi}{4}\cdot 2} & e^{-j\frac{\pi}{4}\cdot 1} \end{bmatrix} \begin{bmatrix} x[0] \\ x[1] \\ x[2] \\ x[3] \\ x[4] \\ x[5] \\ x[6] \\ x[7] \end{bmatrix}$$

# N=8 IDFT Matrix

$$x[n] = \frac{1}{N} \sum_{k=0}^7 W_8^{-kn} X[k] \quad W_8^{-kn} = e^{+j\left(\frac{2\pi}{8}\right)kn}$$

$$\begin{bmatrix} x[0] \\ x[1] \\ x[2] \\ x[3] \\ x[4] \\ x[5] \\ x[6] \\ x[7] \end{bmatrix} = \begin{bmatrix} e^{+j\frac{\pi}{4}\cdot 0} & e^{+j\frac{\pi}{4}\cdot 0} & e^{+j\frac{\pi}{4}\cdot 0} & e^{+j\frac{\pi}{4}\cdot 0} & e^{+j\frac{\pi}{4}\cdot 0} & e^{+j\frac{\pi}{4}\cdot 0} & e^{+j\frac{\pi}{4}\cdot 0} & e^{+j\frac{\pi}{4}\cdot 0} \\ e^{+j\frac{\pi}{4}\cdot 0} & e^{+j\frac{\pi}{4}\cdot 1} & e^{+j\frac{\pi}{4}\cdot 2} & e^{+j\frac{\pi}{4}\cdot 3} & e^{+j\frac{\pi}{4}\cdot 4} & e^{+j\frac{\pi}{4}\cdot 5} & e^{+j\frac{\pi}{4}\cdot 6} & e^{+j\frac{\pi}{4}\cdot 7} \\ e^{+j\frac{\pi}{4}\cdot 0} & e^{+j\frac{\pi}{4}\cdot 2} & e^{+j\frac{\pi}{4}\cdot 4} & e^{+j\frac{\pi}{4}\cdot 6} & e^{+j\frac{\pi}{4}\cdot 0} & e^{+j\frac{\pi}{4}\cdot 2} & e^{+j\frac{\pi}{4}\cdot 4} & e^{+j\frac{\pi}{4}\cdot 6} \\ e^{+j\frac{\pi}{4}\cdot 0} & e^{+j\frac{\pi}{4}\cdot 3} & e^{+j\frac{\pi}{4}\cdot 6} & e^{+j\frac{\pi}{4}\cdot 1} & e^{+j\frac{\pi}{4}\cdot 4} & e^{+j\frac{\pi}{4}\cdot 7} & e^{+j\frac{\pi}{4}\cdot 2} & e^{+j\frac{\pi}{4}\cdot 5} \\ e^{+j\frac{\pi}{4}\cdot 0} & e^{+j\frac{\pi}{4}\cdot 4} & e^{+j\frac{\pi}{4}\cdot 0} & e^{+j\frac{\pi}{4}\cdot 4} & e^{+j\frac{\pi}{4}\cdot 0} & e^{+j\frac{\pi}{4}\cdot 4} & e^{+j\frac{\pi}{4}\cdot 0} & e^{+j\frac{\pi}{4}\cdot 4} \\ e^{+j\frac{\pi}{4}\cdot 0} & e^{+j\frac{\pi}{4}\cdot 5} & e^{+j\frac{\pi}{4}\cdot 2} & e^{+j\frac{\pi}{4}\cdot 7} & e^{+j\frac{\pi}{4}\cdot 4} & e^{+j\frac{\pi}{4}\cdot 1} & e^{+j\frac{\pi}{4}\cdot 6} & e^{+j\frac{\pi}{4}\cdot 3} \\ e^{+j\frac{\pi}{4}\cdot 0} & e^{+j\frac{\pi}{4}\cdot 6} & e^{+j\frac{\pi}{4}\cdot 4} & e^{+j\frac{\pi}{4}\cdot 2} & e^{+j\frac{\pi}{4}\cdot 0} & e^{+j\frac{\pi}{4}\cdot 6} & e^{+j\frac{\pi}{4}\cdot 4} & e^{+j\frac{\pi}{4}\cdot 2} \\ e^{+j\frac{\pi}{4}\cdot 0} & e^{+j\frac{\pi}{4}\cdot 7} & e^{+j\frac{\pi}{4}\cdot 6} & e^{+j\frac{\pi}{4}\cdot 5} & e^{+j\frac{\pi}{4}\cdot 4} & e^{+j\frac{\pi}{4}\cdot 3} & e^{+j\frac{\pi}{4}\cdot 2} & e^{+j\frac{\pi}{4}\cdot 1} \end{bmatrix} \begin{bmatrix} \frac{X[0]}{N} \\ \frac{X[1]}{N} \\ \frac{X[2]}{N} \\ \frac{X[3]}{N} \\ \frac{X[4]}{N} \\ \frac{X[5]}{N} \\ \frac{X[6]}{N} \\ \frac{X[7]}{N} \end{bmatrix}$$

# Symmetric Matrices

$$A = A^T$$

DFT

$$A = \left[ \begin{array}{c} \text{Cyan square with a white diagonal line} \end{array} \right]$$

$$B = B^T$$

IDF

$$B = \left[ \begin{array}{c} \text{Green square with a white diagonal line} \end{array} \right]$$

# Conjugate Transpose Matrices

$$A = B^H \quad \rightarrow \quad A = B^*$$

DFT

$$A = \begin{bmatrix} \text{cyan triangle} \\ \text{green diagonal} \end{bmatrix} = \begin{bmatrix} \text{green triangle} \\ \text{cyan diagonal} \end{bmatrix}^H$$

$$B = A^H \quad \rightarrow \quad B = A^*$$

IDFT

$$B = \begin{bmatrix} \text{green triangle} \\ \text{cyan diagonal} \end{bmatrix} = \begin{bmatrix} \text{cyan triangle} \\ \text{green diagonal} \end{bmatrix}^H$$

# Product AB

$$A \cdot B \Rightarrow A \cdot A^H \Rightarrow A \cdot A^* \Rightarrow NI$$

DFT

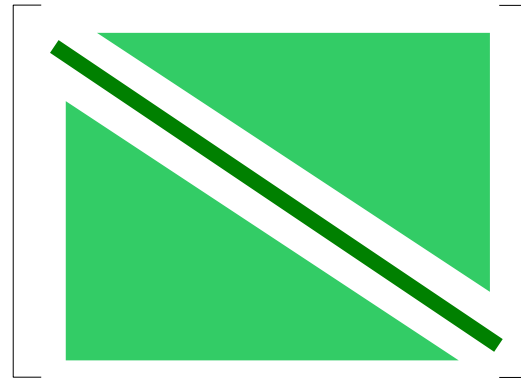
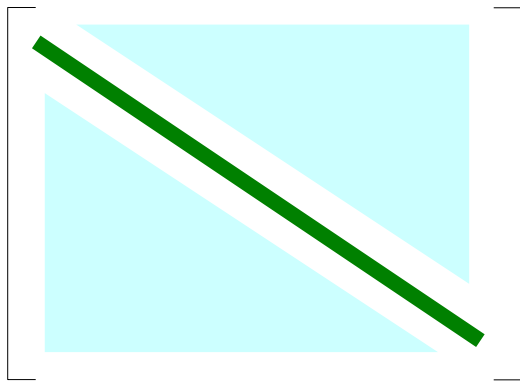
$A$

IDFT

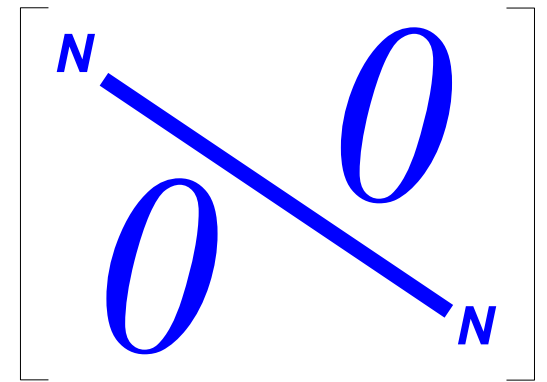
$B$

=

$C$



=



# Unitary Matrix

$$C = \begin{bmatrix} N & & 0 \\ & \ddots & \\ 0 & & N \end{bmatrix}$$

$$C = A \cdot B = A \cdot A^H = A \cdot A^* = N I$$

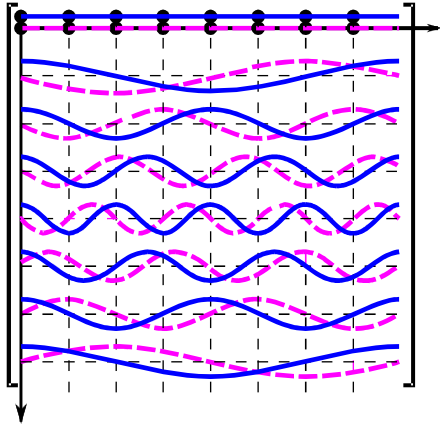
$$U \cdot U^H = I$$

**Unitary Matrix**



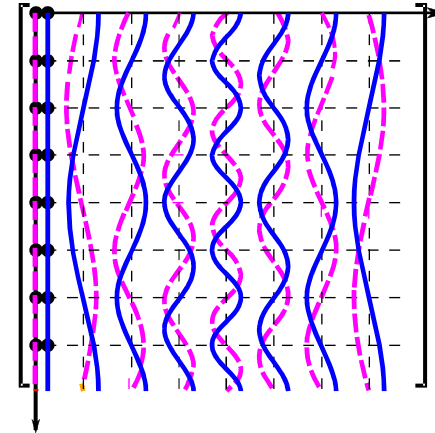
# Symmetric Matrices

*DFT Matrix in the row-wise view*

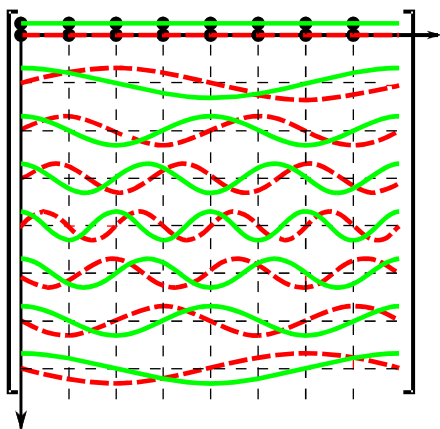


$$\mathbf{A} = \mathbf{A}^T$$

*DFT Matrix in the column-wise view*

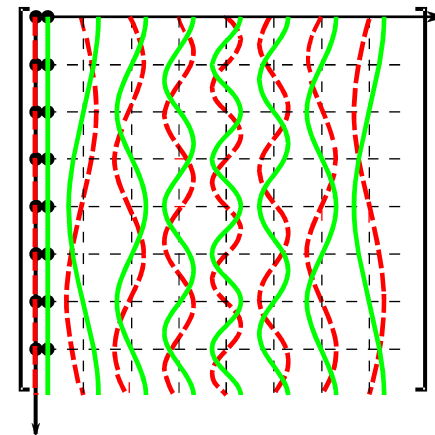


*IDFT Matrix in the row-wise view*



$$\mathbf{B} = \mathbf{B}^T$$

*IDFT Matrix in the column-wise view*

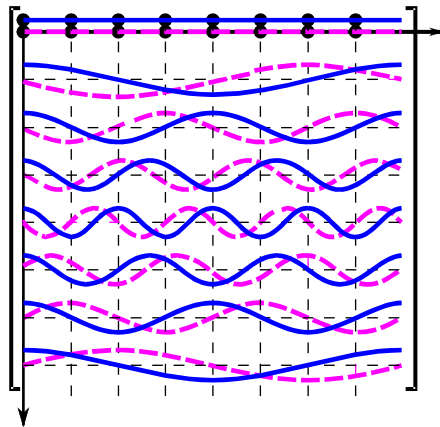


# Conjugate Transpose Matrices

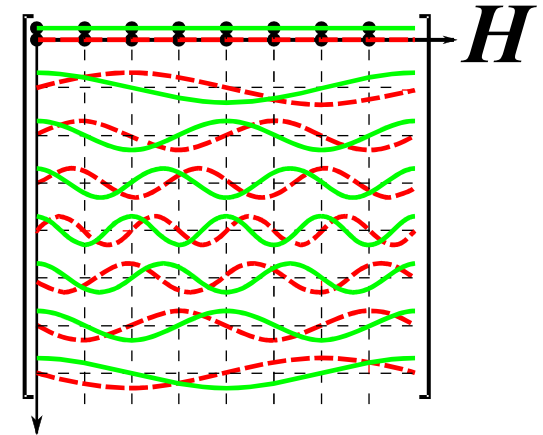
$$A = B^H \quad \rightarrow \quad A = B^* \quad \text{— Real} \quad \text{- - Imaginary}$$

DFT

$$A =$$



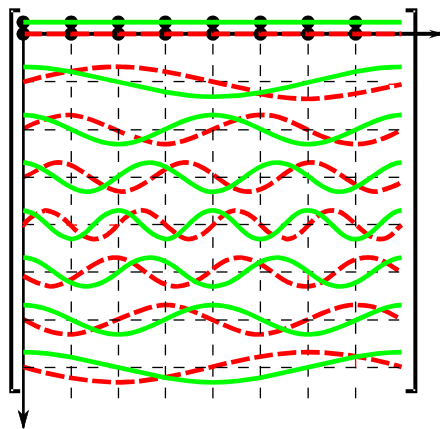
=



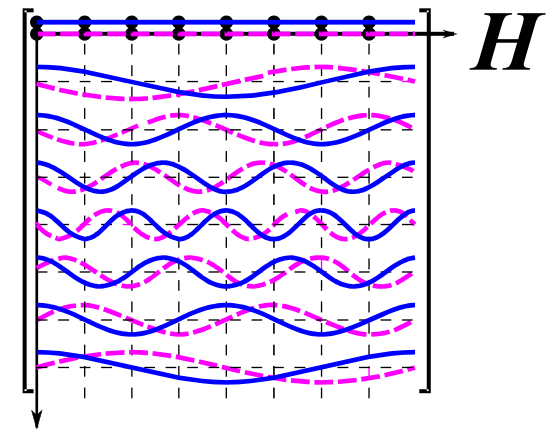
$$B = A^H \quad \rightarrow \quad B = A^* \quad \text{— Real} \quad \text{- - Imaginary}$$

IDFT

$$B =$$

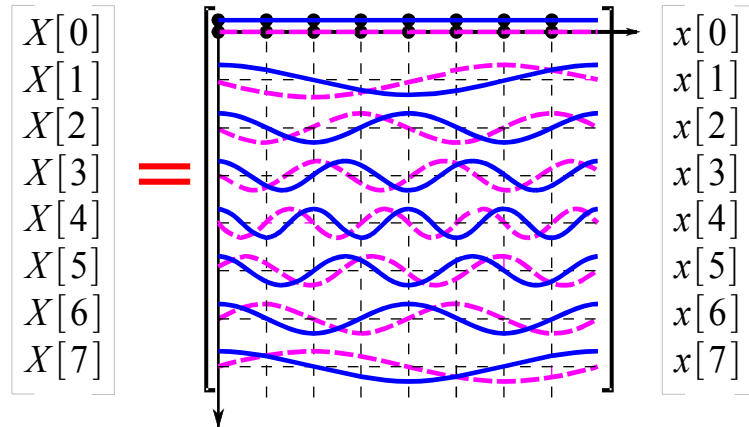


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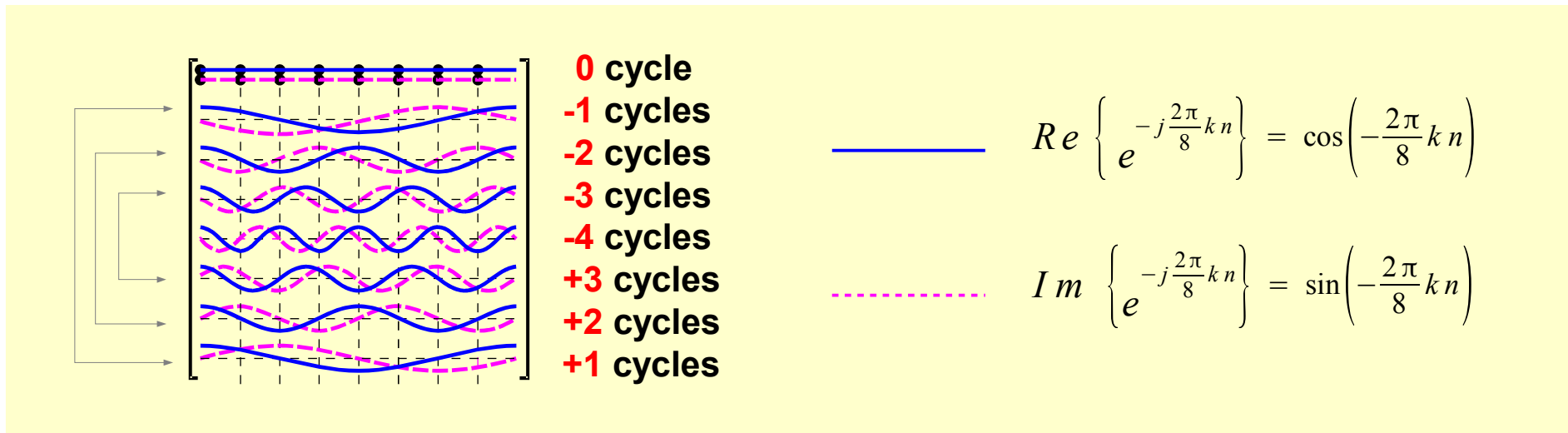




# X[k] measures frequency



**X[0]** measures “0 cycle” component in  $x$   
**X[1]** measures “+1 cycle” component in  $x$   
**X[2]** measures “+2 cycle” component in  $x$   
**X[3]** measures “+3 cycle” component in  $x$   
**X[4]** measures “+4 cycle” component in  $x$   
**X[5]** measures “-3 cycle” component in  $x$   
**X[6]** measures “-2 cycle” component in  $x$   
**X[7]** measures “-1 cycle” component in  $x$





## References

- [1] <http://en.wikipedia.org/>
- [2] J.H. McClellan, et al., Signal Processing First, Pearson Prentice Hall, 2003
- [3] A “graphical interpretation” of the DFT and FFT, by Steve Mann