CTFS (1A)

- Continuous Time Fourier Series

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3/30/11

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Fourier Series

$$f(x) = a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

$$a_0 = \frac{1}{2\pi} \int_{-\pi}^{+\pi} f(x)\, dx$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{+\pi} f(x) \cos nx\, dx$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{+\pi} f(x) \sin nx\, dx$$

$$n = 1,\ 2,\ 3,\ ...$$

Trigonometric Orthogonality

$$f(x) = a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

$$a_0 = \frac{1}{2\pi} \int_{-\pi}^{+\pi} f(x)\, dx$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{+\pi} f(x) \cos nx\, dx$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{+\pi} f(x) \sin nx\, dx$$

$$n = 1, 2, 3, \ldots$$

$$\int_{-\pi}^{+\pi} \cos nx \cos mx\, dx = 0 \quad (n \neq m)$$

$$\int_{-\pi}^{+\pi} \sin nx \sin mx\, dx = 0 \quad (n \neq m)$$

$$\int_{-\pi}^{+\pi} \sin nx \cos mx\, dx = 0$$

$$a_m \leftarrow f(x) \cos mx = a_0 \cos mx + \sum_{n=1}^{\infty} (a_n \cos nx \cos mx + b_n \sin nx \cos mx)$$

$$b_m \leftarrow f(x) \sin mx = a_0 \sin mx + \sum_{n=1}^{\infty} (a_n \cos nx \sin mx + b_n \sin nx \sin mx)$$

Any Period *p = 2L*

$$g(v) = a_0 + \sum_{n=1}^{\infty} (a_n \cos nv + b_n \sin nv)$$

$$f(x) = a_0 + \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi}{L} x + b_n \sin \frac{n\pi}{L} x \right)$$

$$a_0 = \frac{1}{2\pi} \int_{-\pi}^{+\pi} g(v) \, dv$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{+\pi} g(v) \cos nv \, dv$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{+\pi} g(v) \sin nv \, dv$$
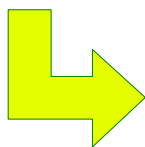
$$n = 1, \ 2, \ ...$$

$$a_0 = \frac{1}{2L} \int_{-L}^{+L} f(x) \, dx$$

$$a_n = \frac{1}{L} \int_{-L}^{+L} f(x) \cos \frac{n\pi x}{L} \, dx$$

$$b_n = \frac{1}{L} \int_{-L}^{+L} f(x) \sin \frac{n\pi x}{L} \, dx$$

$$n = 1, \ 2, \ 3, \ ...$$

$$v : \left[-\pi, \ +\pi \right]$$

$$x : \left[-L, \ +L \right]$$

$$v = \frac{\pi}{L} x$$

$$dv = \frac{\pi}{L} dx$$

Time and Frequency

$$f(x) = a_0 + \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi}{L} x + b_n \sin \frac{n\pi}{L} x \right)$$

$$x(t) = a_0 + \sum_{n=1}^{\infty} \left(a_n \cos \frac{2\pi n}{T} t + b_n \sin \frac{2\pi n}{T} t \right)$$

$$a_0 = \frac{1}{2L} \int_{-L}^{+L} f(x)\, dx$$

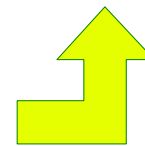$$a_n = \frac{1}{L} \int_{-L}^{+L} f(x) \cos \frac{n\pi x}{L}\, dx$$

$$b_n = \frac{1}{L} \int_{-L}^{+L} f(x) \sin \frac{n\pi x}{L}\, dx$$
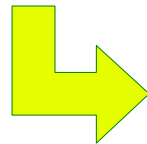
$$n = 1, 2, 3, \ldots$$

$$a_0 = \frac{1}{T} \int_0^T x(t)\, dt$$

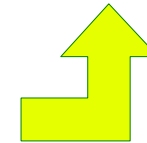$$a_n = \frac{2}{T} \int_0^T x(t) \cos \frac{2\pi n t}{T}\, dt$$

$$b_n = \frac{2}{T} \int_0^T x(t) \sin \frac{2\pi n t}{T}\, dt$$

$$n = 1, 2, \ldots$$

$$x: [-L, +L]$$

$$t: [0, T]$$

$$2L = T$$

Continuous Time Periodic Signal $x(t)$

Harmonic Frquency

$$x(t) = a_0 + \sum_{n=1}^{\infty}\left(a_n\cos\frac{2\pi n}{T}t + b_n\sin\frac{2\pi n}{T}t\right)$$

$$x(t) = a_0 + \sum_{n=1}^{\infty}\left(a_n\cos(n2\pi f_0 t) + b_n\sin(n2\pi f_0 t)\right)$$

$$a_0 = \frac{1}{T}\int_0^T x(t)\,dt$$

$$a_0 = \frac{1}{T}\int_0^T x(t)\,dt$$

$$a_n = \frac{2}{T}\int_0^T x(t)\cos\frac{2\pi nt}{T}\,dt$$

$$a_n = \frac{2}{T}\int_0^T x(t)\cos(n2\pi f_0 t)\,dt \quad n = 1,\ 2,\ ...$$

$$b_n = \frac{2}{T}\int_0^T x(t)\sin\frac{2\pi nt}{T}\,dt$$

$$b_n = \frac{2}{T}\int_0^T x(t)\sin(n2\pi f_0 t)\,dt \quad n = 1,\ 2,\ ...$$

$$n = 1,\ 2,\ ...$$

$$t:\ [0,\ T]$$

$$t:\ [0,\ T]$$

resolution frequency $\qquad f_0 = \dfrac{1}{T}$

n-th harmonic frequency $\qquad f_n = \boldsymbol{n}\,f_0 = \boldsymbol{n}\,\dfrac{1}{T}$

Radial Frequency

$$x(t) = a_0 + \sum_{n=1}^{\infty} \left(a_n \cos(n 2\pi f_0 t) + b_n \sin(n 2\pi f_0 t) \right)$$

$$x(t) = a_0 + \sum_{n=1}^{\infty} \left(a_n \cos(n \omega_0 t) + b_n \sin(n \omega_0 t) \right)$$

$$a_0 = \frac{1}{T} \int_0^T x(t)\, dt$$

$$a_n = \frac{2}{T} \int_0^T x(t) \cos(n 2\pi f_0 t)\, dt \quad n = 1, 2, \ldots$$

$$b_n = \frac{2}{T} \int_0^T x(t) \sin(n 2\pi f_0 t)\, dt \quad n = 1, 2, \ldots$$

$$a_0 = \frac{1}{T} \int_0^T x(t)\, dt$$

$$a_n = \frac{2}{T} \int_0^T x(t) \cos(n \omega_0 t)\, dt$$

$$b_n = \frac{2}{T} \int_0^T x(t) \sin(n \omega_0 t)\, dt$$

$$n = 1, 2, \ldots$$

$$t: [0, T]$$

$$t: [0, T]$$

linear frequency $\qquad f$

angular (radial) frequency $\qquad 2\pi f$

Euler Equation (1)

$$e^{j\omega t} = \cos \omega t + j \sin \omega t$$

$$e^{-j\omega t} = \cos \omega t - j \sin \omega t$$

$$\cos \omega t = \frac{e^{j\omega t} + e^{-j\omega t}}{2}$$

$$\sin \omega t = \frac{e^{j\omega t} - e^{-j\omega t}}{2j}$$

$$a_n \cos(n\omega_0 t) + b_n \sin(n\omega_0 t)$$

$$= a_n \frac{1}{2} (e^{jn\omega_0 t} + e^{-jn\omega_0 t}) + b_n \frac{1}{2j} (e^{jn\omega_0 t} - e^{-jn\omega_0 t})$$

$$= \frac{1}{2} (a_n - jb_n) e^{jn\omega_0 t} + \frac{1}{2} (a_n + jb_n) e^{-jn\omega_0 t}$$

$$= \frac{1}{2} (a_n - jb_n) e^{jn\omega_0 t} + \frac{1}{2} (a_n + jb_n) e^{-jn\omega_0 t}$$

$$= A_n e^{jn\omega_0 t} + B_n e^{-jn\omega_0 t}$$

$$x(t) = A_0 + \sum_{n=1}^{\infty} \left(A_n e^{jn\omega_0 t} + B_n e^{-jn\omega_0 t} \right)$$

Euler Equation (2)

$$a_n = \frac{2}{T} \int_0^T x(t) \cos(n\omega_0 t)\, dt$$

$$b_n = \frac{2}{T} \int_0^T x(t) \sin(n\omega_0 t)\, dt$$

$$A_n = \frac{1}{2}(a_n - jb_n)$$

$$B_n = \frac{1}{2}(a_n + jb_n)$$

$$A_n = \frac{1}{T} \int_0^T x(t)(\cos(n\omega_0 t) - j\sin(n\omega_0 t))\, dt$$

$$B_n = \frac{1}{T} \int_0^T x(t)(\cos(n\omega_0 t) + j\sin(n\omega_0 t))\, dt$$

$$A_n = \frac{1}{T} \int_0^T x(t)\, e^{-jn\omega_0 t}\, dt$$

$$B_n = \frac{1}{T} \int_0^T x(t)\, e^{+jn\omega_0 t}\, dt$$

$$x(t) = A_0 + \sum_{n=1}^{\infty}\left(A_n\, e^{+jn\omega_0 t} + B_n\, e^{-jn\omega_0 t}\right) = \sum_{n=0}^{\infty}\left(A_n\, e^{+jn\omega_0 t} + B_n\, e^{-jn\omega_0 t}\right)$$

Complex Fourier Series

$$x(t) = A_0 + \sum_{n=1}^{\infty} \left(A_n\, e^{+jn\omega_0 t} + B_n\, e^{-jn\omega_0 t} \right)$$

$$= \sum_{n=0}^{\infty} \left(A_n\, e^{+jn\omega_0 t} + B_n\, e^{-jn\omega_0 t} \right)$$

$$A_n = \frac{1}{T} \int_0^T x(t)\, e^{-jn\omega_0 t}\, dt$$

$$B_n = \frac{1}{T} \int_0^T x(t)\, e^{+jn\omega_0 t}\, dt$$

$$x(t) = \sum_{n=0}^{\infty} C_n\, e^{+jn\omega_0 t}$$

$$C_n = \frac{1}{T} \int_0^T x(t)\, e^{-jn\omega_0 t}\, dt$$

References

[1] http://en.wikipedia.org/
[2] J.H. McClellan, et al., Signal Processing First, Pearson Prentice Hall, 2003