Euclidean Vector Space (1A)
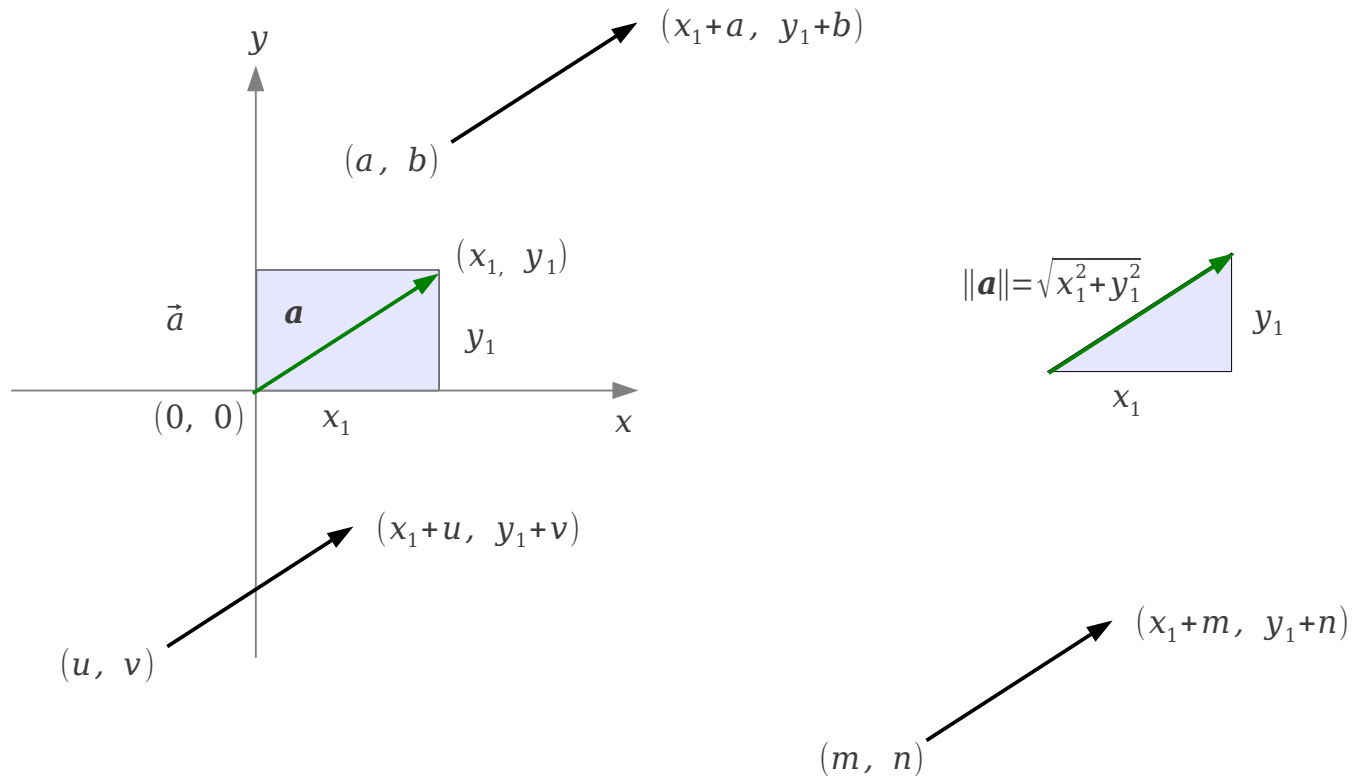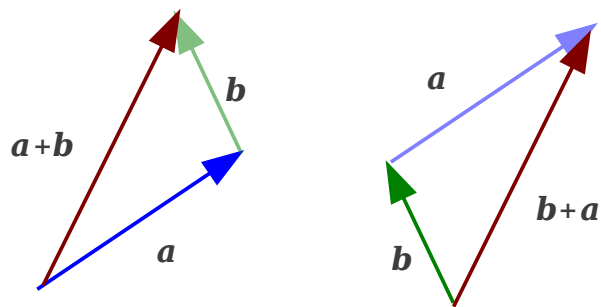
Young Won Lim
10/20/12

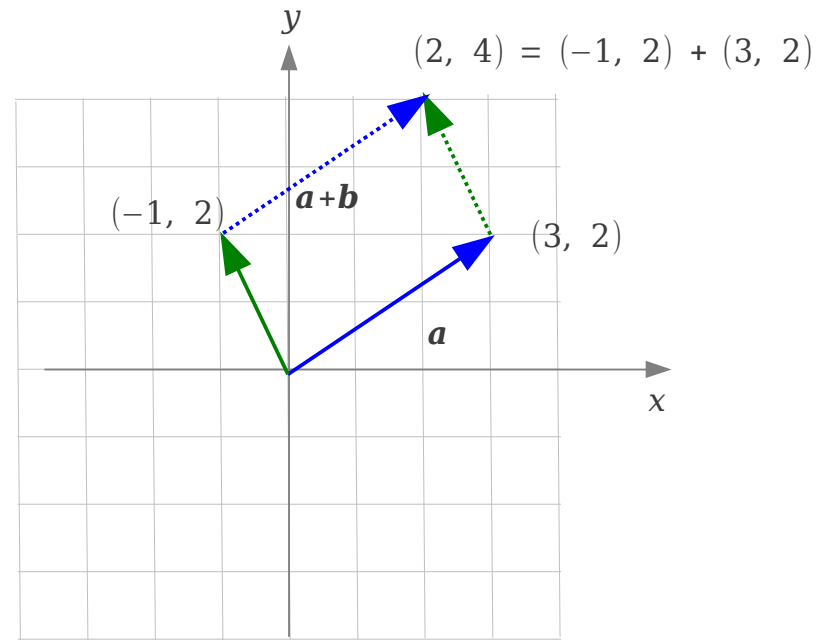Please send corrections (or suggestions) to youngwlim@hotmail.com.

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Vectors

$(x_1+a,\ y_1+b)$

$(a,\ b)$

y

$(x_{1,}\ y_1)$

$\vec{a}$

$\boldsymbol{a}$

y_1

$(0,\ 0)$

x_1

x

$(x_1+u,\ y_1+v)$

$(u,\ v)$

$\|\boldsymbol{a}\|=\sqrt{x_1^2+y_1^2}$

y_1

x_1

$(x_1+m,\ y_1+n)$

$(m,\ n)$

Vector Addition

$(2, 4) = (-1, 2) + (3, 2)$
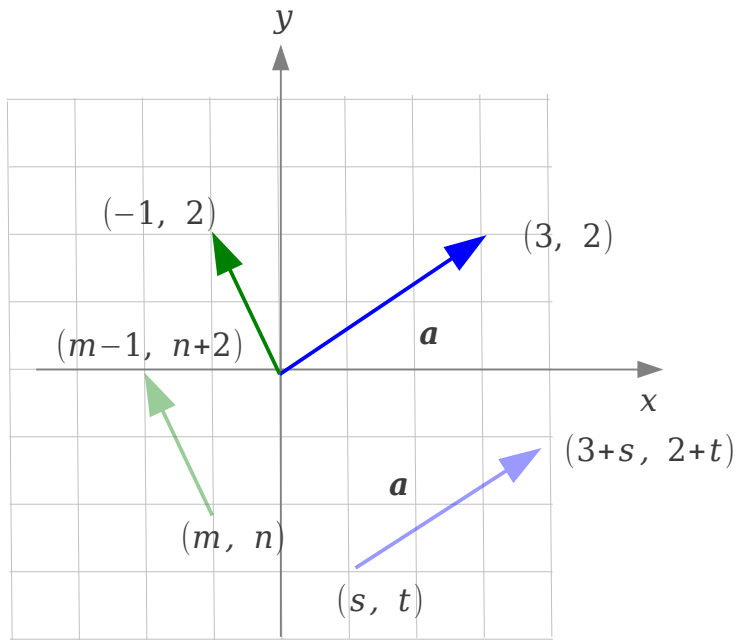
$a + b = b + a$
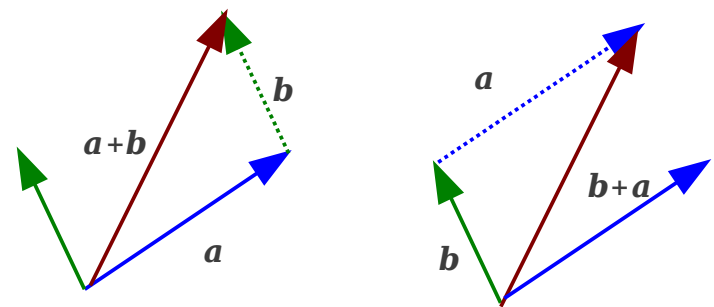
Vector Addition
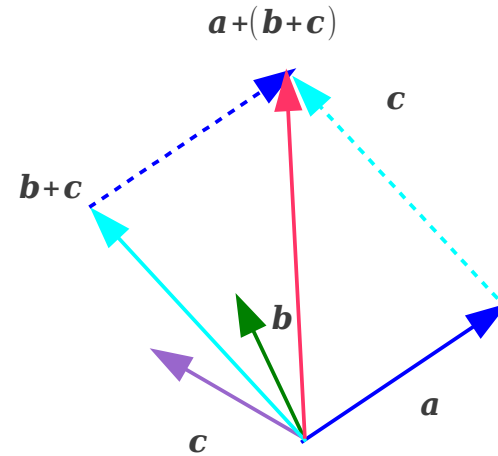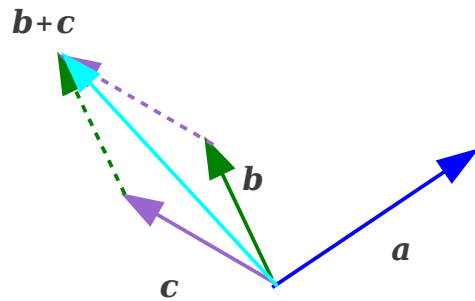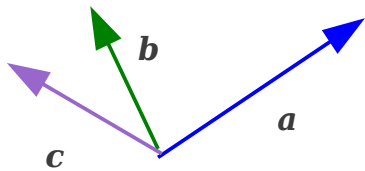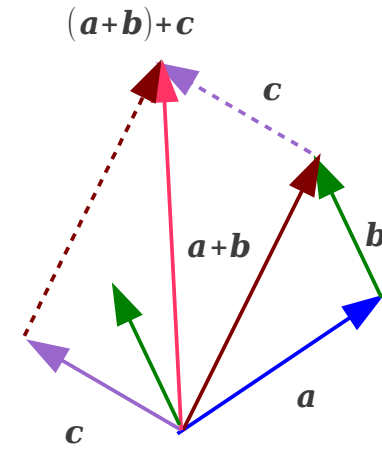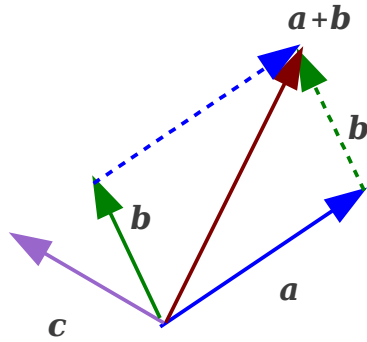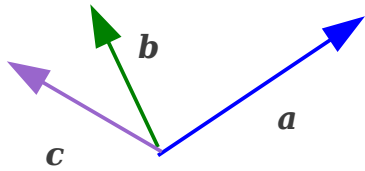
$$(a+b)+c = a+(b+c)$$

Vector Subtraction

$(−1,\ 2)$
$(3,\ 2)$
a
b
$−b$
$(4,\ 0)\ =\ (3,\ 2)\ −\ (−1,\ 2)$

$a−b$
b
a
$(−1,\ 2)$
$(3,\ 2)$

b
$−b$

a
$−b$
$a−b$
$−b$
a
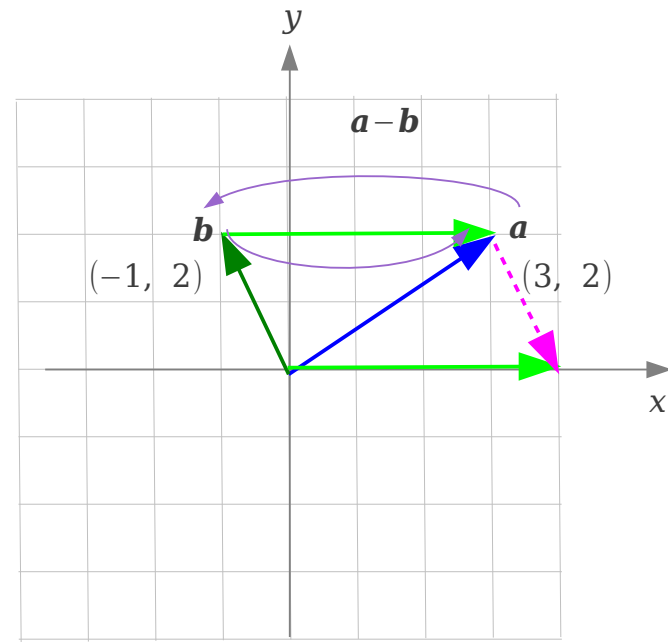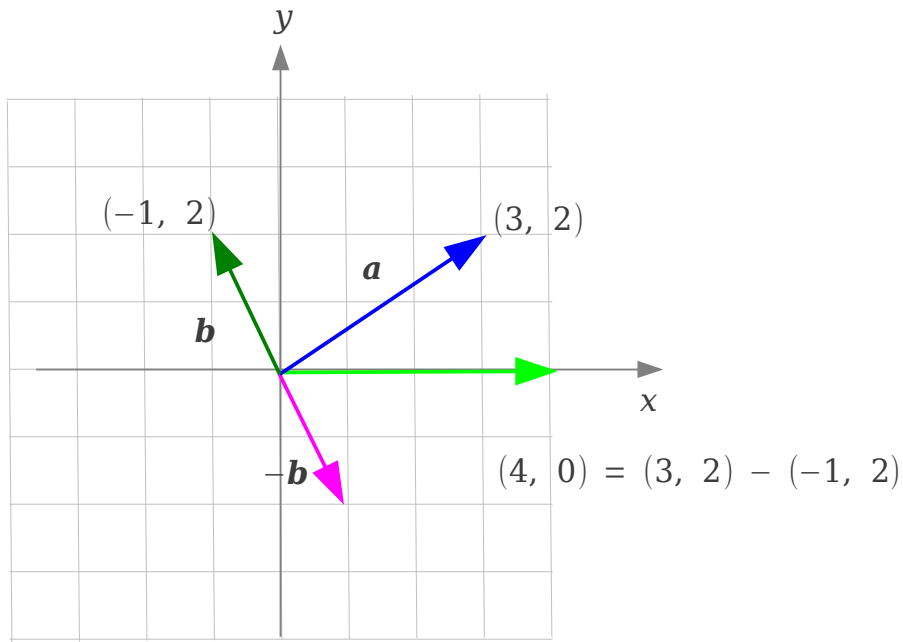
$a−b$

subtract **a** from **b**
arrow from **b** to **a**

Component Form
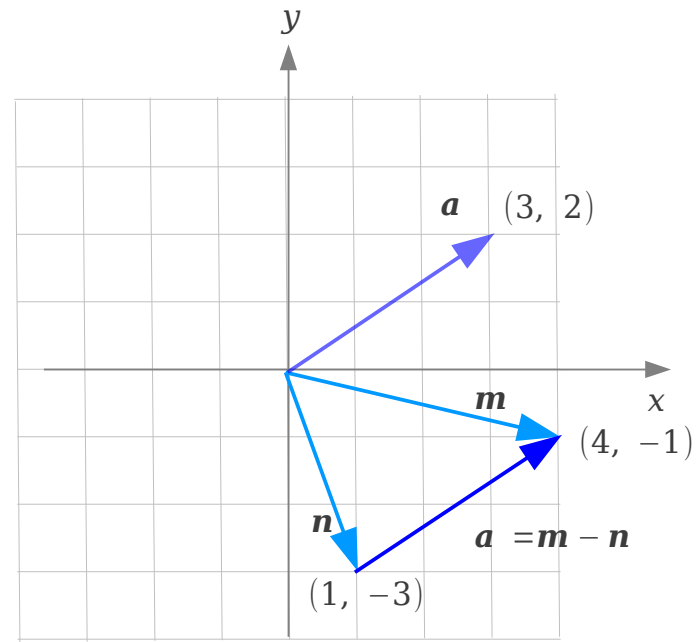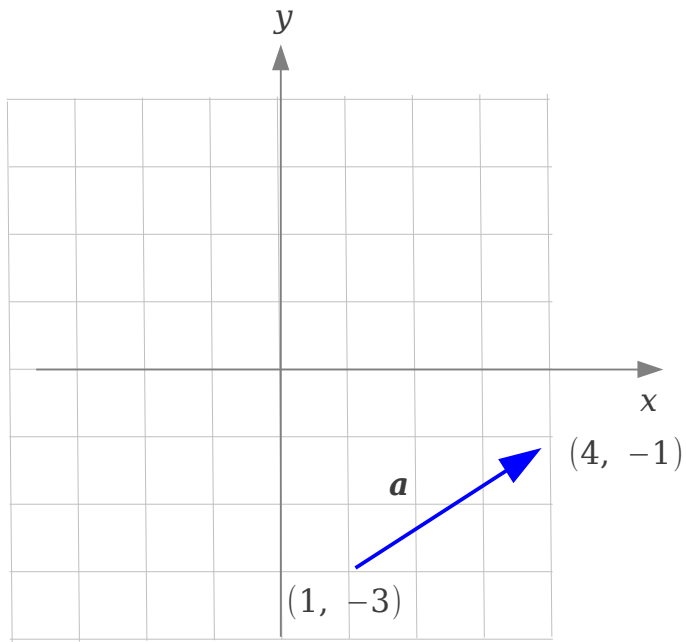
$$m = (4, -1)$$

$$n = (1, -3)$$

$$a = m - n = (4, -1) - (1, -3) = (3, 2)$$

Finding the **Component Form** of a vector

Scalar Multiplication (1)

$(-6,\ 12)$

$3(2\,\boldsymbol{b}) = 3(-2,\ 4)$

$2(3\,\boldsymbol{b}) = 2(-3,\ 6)$

$(-1,\ 2)$

$\boldsymbol{b}$

$-\boldsymbol{b}$

$(1,\ -2)$

$(-1)\boldsymbol{b} = -\boldsymbol{b}$

$(-2,\ 4)$

$\boldsymbol{b}$

$2\,\boldsymbol{b}$

$(-1,\ 2)$

$\boldsymbol{b}$

$(-3,\ 6)$

$\boldsymbol{b}$

$3\,\boldsymbol{b}$

$\boldsymbol{b}$

$(-1,\ 2)$

$2\,\boldsymbol{b}$

$2\,\boldsymbol{b}$

$6\,\boldsymbol{b}$

$3\,\boldsymbol{b}$

$2\,\boldsymbol{b}$

$2\,\boldsymbol{b}$

$3\,\boldsymbol{b}$

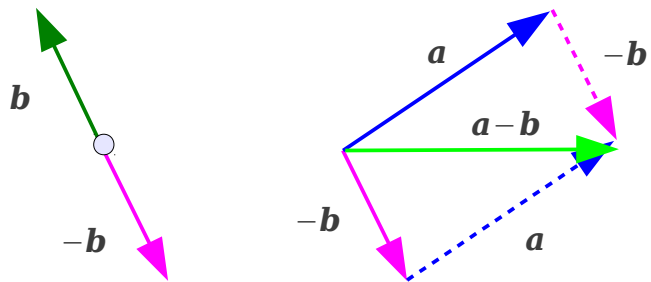$$\boldsymbol{u} + (-\boldsymbol{u}) = \boldsymbol{0}$$

$$k(\boldsymbol{u} + \boldsymbol{v}) = k\boldsymbol{u} + k\boldsymbol{v}$$

$$(k + m)\boldsymbol{u} = k\boldsymbol{u} + m\boldsymbol{u}$$

$$k(m\boldsymbol{u}) = (km)\boldsymbol{u}$$

$$1\boldsymbol{u} = \boldsymbol{u}$$

Scalar Multiplication (2)

(−2, 4)

(−3, 6)

5(−1, 2) = (−5, 10)

(−1, 2)

b

−**b**

(1, −2)

(−1)**b** = −**b**

b

2**b**

(−1, 2)

b

3(−1, 2)

b

3**b**

b

(−1, 2)

b

2(−1, 2)

2**b**

2(**a**+**b**)

2(3−1, 2+2)
= (4, 8)
= (6, 4) + (−2, 4)

(−2, 4)

2**b**

a+**b**

b

a+**b**

b

a

2**a**

(6, 4)

b

a

(3, 2)

$u + (-u) = 0$

$k(u + v) = ku + kv$

$(k + m)u = ku + mu$

$k(mu) = (km)u$

$1u = u$
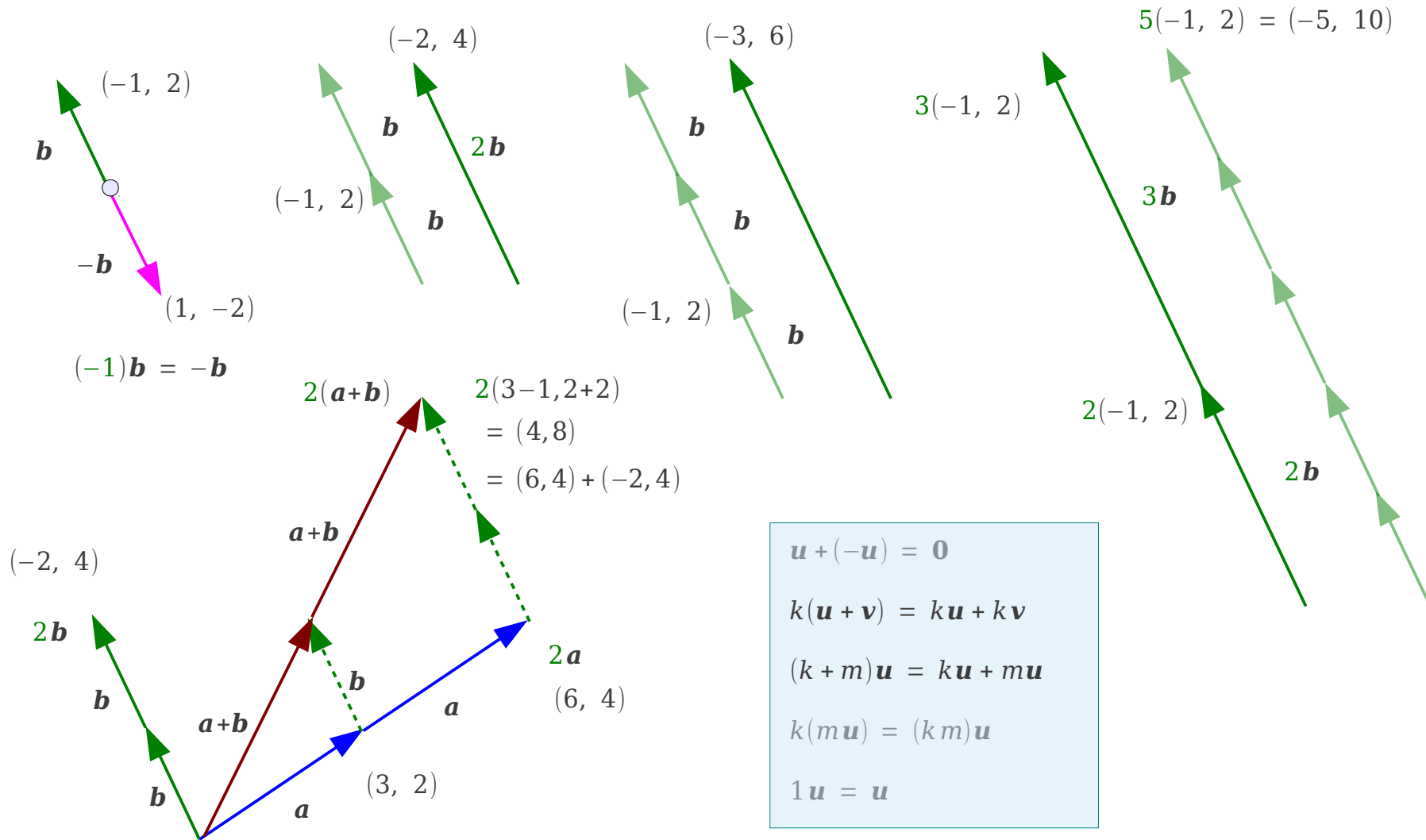
n-Space

Ordered 2-tuples $\quad (v_1,\ v_2) \qquad\qquad\qquad v_1,\ v_2 \in R$

$\qquad R^2 \quad\Longleftrightarrow\quad$ 2-space $\quad\Longleftrightarrow\quad \{\,all\ ordered\ 2-tuples\ (v_1,\ v_2)\,\}$

Ordered 3-tuples $\quad (v_1,\ v_2,\ v_3) \qquad\qquad v_1,\ v_2,\ v_3 \in R$

$\qquad R^3 \quad\Longleftrightarrow\quad$ 3-space $\quad\Longleftrightarrow\quad \{\,all\ ordered\ 3-tuples\ (v_1,\ v_2,\ v_3)\,\}$

Ordered n-tuples $\quad (v_1,\ v_2,\ \cdots,\ v_n) \qquad v_1,\ v_2,\ \cdots,\ v_n \in R$

$\qquad R^n \quad\Longleftrightarrow\quad$ n-space $\quad\Longleftrightarrow\quad \{\,all\ ordered\ n-tuples\ (v_1,\ v_2,\ \cdots,\ v_n)\,\}$

set

Properties of Real Vector Spaces

u, v, w vectors in R^n

k, m scalars

$$u + v = v + u$$

$$(u + v) + w = u + (v + w)$$

$$u + 0 = 0 + u = u$$

$$u + (-u) = 0$$

$$k(u + v) = ku + kv$$

$$(k + m)u = ku + mu$$

$$k(mu) = (km)u$$

$$1u = u$$

Linear Combination

$\boldsymbol{w}, \quad \boldsymbol{v}_1, \boldsymbol{v}_2, \cdots, \boldsymbol{v}_r \qquad$ vectors in $\quad R^n$

$\qquad k_1, k_2, \cdots, k_r \qquad$ scalars

$\boldsymbol{w}$ is a linear combination of vectors $\quad \boldsymbol{v}_1, \boldsymbol{v}_2, \cdots, \boldsymbol{v}_r$

$$\boldsymbol{w} = \quad k_1 \boldsymbol{v}_1 + k_2 \boldsymbol{v}_2 + \cdots + k_r \boldsymbol{v}_r$$

$$\begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = k_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + k_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$= m_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + m_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$= n_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + n_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} + n_3 \begin{bmatrix} 0 \\ -1 \end{bmatrix}$$

Vector Magnitude

- Norm
- Length $\quad\quad\quad \|\mathbf{v}\|$
- Magnitude

$R^2 \quad\quad \mathbf{v} = (v_1,\ v_2) \quad\quad\quad\quad\quad \|\mathbf{v}\| = \sqrt{v_1^2 + v_2^2} \geq 0$

$R^3 \quad\quad \mathbf{v} = (v_1,\ v_2,\ v_3) \quad\quad\quad \|\mathbf{v}\| = \sqrt{v_1^2 + v_2^2 + v_3^2} \geq 0$
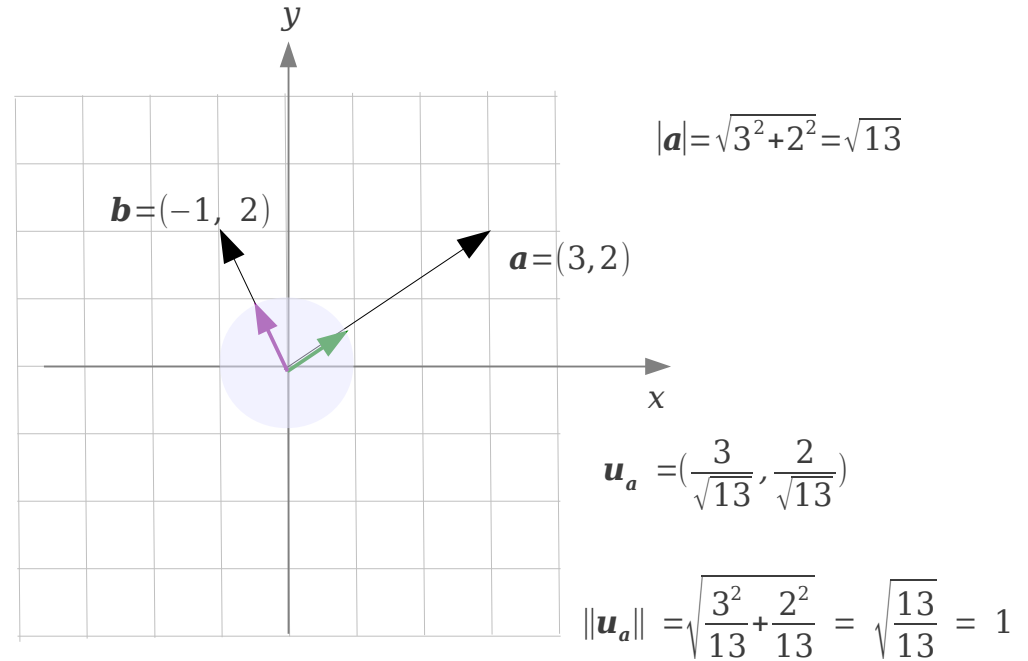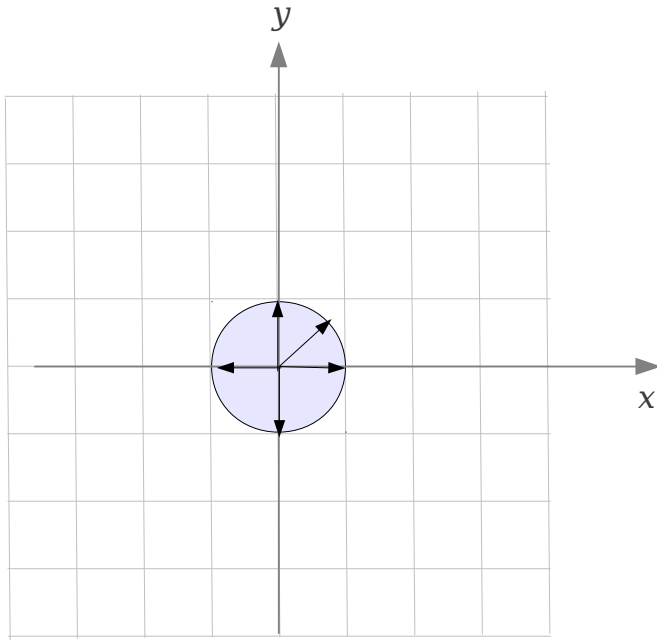
$R^n \quad\quad \mathbf{v} = (v_1,\ v_2,\ \cdots\ ,v_n) \quad \|\mathbf{v}\| = \sqrt{v_1^2 + v_2^2 + \cdots + v_n^2} \geq 0$

$\|\mathbf{v}\| \geq 0$

$\|\mathbf{v}\| = 0 \quad\quad\quad\quad \mathbf{v} = \mathbf{0}$

$\|k\,\mathbf{v}\| = |k|\,\|\mathbf{v}\| \geq 0$

Unit Vector

$$|a| = \sqrt{3^2 + 2^2} = \sqrt{13}$$

$$b = (-1, 2)$$

$$a = (3, 2)$$

$$u_a = (\frac{3}{\sqrt{13}}, \frac{2}{\sqrt{13}})$$

$$\|u_a\| = \sqrt{\frac{3^2}{13} + \frac{2^2}{13}} = \sqrt{\frac{13}{13}} = 1$$

$$u = \frac{v}{\|v\|}$$

$$\|u\| = \left\|\frac{v}{\|v\|}\right\|$$

$$k = \|v\| \geq 0$$

$$= \left\|\frac{v}{k}\right\| = \frac{\|v\|}{k}$$

$$= \frac{\|v\|}{\|v\|} = 1$$

Standard Unit Vectors

$R^2 \iff$ 2-space $\iff \{\ all\ ordered\ 2-tuples\ (v_1,\ v_2)\ \}$

$$\boldsymbol{i} = (1,\ 0)$$
$$\boldsymbol{j} = (0,\ 1)$$

$$\boldsymbol{v} = (v_1,\ v_2)$$

$$= v_1(1,\ 0)$$
$$+ v_2(0,\ 1)$$

$R^3 \iff$ 3-space $\iff \{\ all\ ordered\ 3-tuples\ (v_1,\ v_2,\ v_3)\ \}$

$$\boldsymbol{i} = (1,\ 0,\ 0)$$
$$\boldsymbol{j} = (0,\ 1,\ 0)$$
$$\boldsymbol{k} = (0,\ 0,\ 1)$$

$$\boldsymbol{v} = (v_1,\ v_2,\ v_3)$$

$$= v_1(1,\ 0,\ 0)$$
$$+ v_2(0,\ 1,\ 0)$$
$$+ v_3(0,\ 0,\ 1)$$

$R^n \iff$ n-space $\iff \{\ all\ ordered\ n-tuples\ (v_1,\ v_2,\ \cdots,\ v_n)\ \}$

$$\boldsymbol{e_1} = (1,\ 0,\ \cdots,\ 0)$$
$$\boldsymbol{e_2} = (0,\ 1,\ \cdots,\ 0)$$
$$\cdots$$
$$\boldsymbol{e_n} = (0,\ 0,\ \cdots,\ 1)$$

$$\boldsymbol{v} = (v_1,\ v_2,\ \cdots,\ v_n)$$

$$= v_1(1,\ 0,\ \cdots,\ 0)$$
$$+ v_2(0,\ 1,\ \cdots,\ 0)$$
$$+ \cdots$$
$$+ v_n(0,\ 0,\ \cdots,\ 1)$$

Distance between Vectors

$\|\boldsymbol{b}\| = \sqrt{-1^2 + 2^2} = \sqrt{5}$

$\|\boldsymbol{a}\| = \sqrt{3^2 + 2^2} = \sqrt{13}$

$\boldsymbol{b} = (-1, 2)$ $\boldsymbol{a} - \boldsymbol{b}$ $\boldsymbol{a} = (3, 2)$
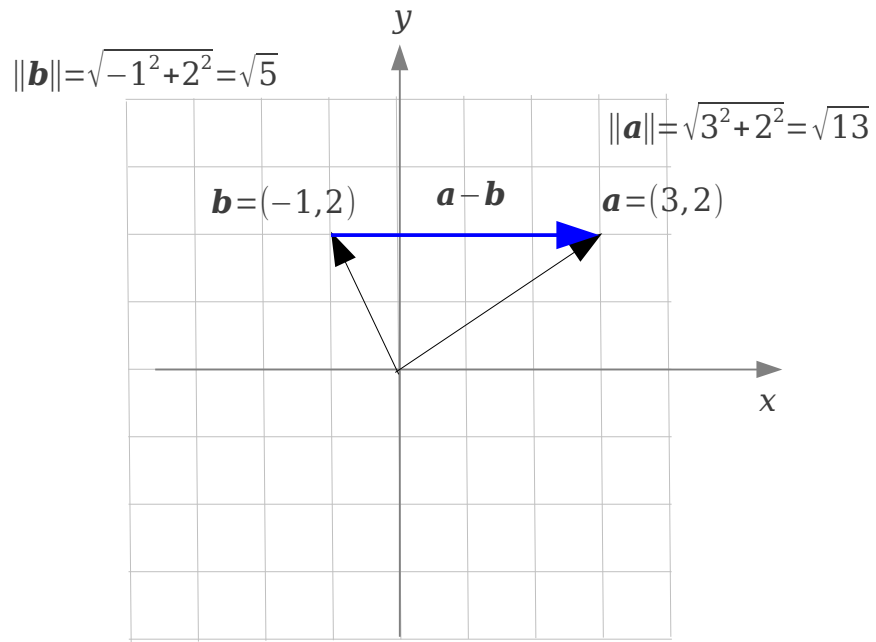
R^2 $d(\boldsymbol{a}, \boldsymbol{b}) = \|\boldsymbol{a} - \boldsymbol{b}\|$

$$= \sqrt{(a_1 - b_1)^2 + (a_2 - b_2)^2} \geq 0$$

R^3 $d(\boldsymbol{a}, \boldsymbol{b}) = \|\boldsymbol{a} - \boldsymbol{b}\|$

$$= \sqrt{(a_1 - b_1)^2 + (a_2 - b_2)^2 + (a_3 - b_3)^2} \geq 0$$

R^n $d(\boldsymbol{a}, \boldsymbol{b}) = \|\boldsymbol{a} - \boldsymbol{b}\|$

$$= \sqrt{(a_1 - b_1)^2 + (a_2 - b_2)^2 + \cdots + (a_n - b_n)^2} \geq 0$$

Law of Cosine

$$\|\boldsymbol{a}-\boldsymbol{b}\| = \sqrt{4^2+0^2} = 4$$

$b=(-1,2)$ $\boldsymbol{a}-\boldsymbol{b} = (4,0)$ $\boldsymbol{a}=(3,2)$

$$\|\boldsymbol{b}\|=\sqrt{-1^2+2^2}=\sqrt{5} \qquad \theta$$

$$\|\boldsymbol{a}\|=\sqrt{3^2+2^2}=\sqrt{13}$$
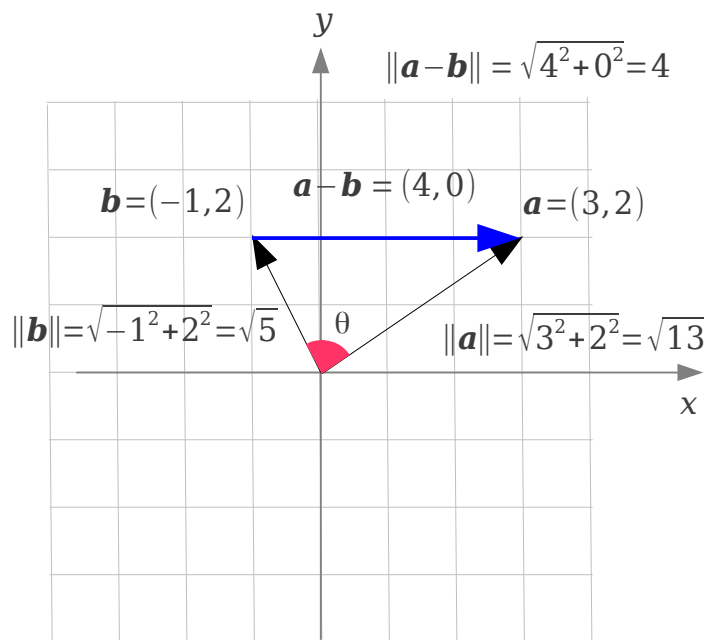
$$4^2=\sqrt{13}^2+\sqrt{5}^2-2\sqrt{13}\sqrt{5}\cos\theta$$

$$2=2\sqrt{13}\sqrt{5}\cos\theta \qquad \cos\theta = \frac{1}{\sqrt{65}}$$

$$\boldsymbol{a}\cdot\boldsymbol{b}=3\cdot-1+2\cdot2=-3+4=1$$

Inner Product

$$\boldsymbol{a}\cdot\boldsymbol{b} = a_1 b_1 + a_2 b_2 + \cdots + a_n b_n$$

$$R^2 \quad d(\boldsymbol{a},\boldsymbol{b}) = \|\boldsymbol{a}-\boldsymbol{b}\| = \sqrt{(a_1-b_1)^2 + (a_2-b_2)^2} \geq 0$$

Law of Cosine

$$\|\boldsymbol{a}-\boldsymbol{b}\|^2 = \|\boldsymbol{a}\|^2 + \|\boldsymbol{b}\|^2 - 2\|\boldsymbol{a}\|\,\|\boldsymbol{b}\|\cos\theta$$

$\boldsymbol{a}-\boldsymbol{b}$ $\boldsymbol{a} = (a_1,\ a_2)$

$= (a_1-b_1,\ a_2-b_2)$ $\boldsymbol{b} = (b_1,\ b_2)$

$(a_1-b_1)^2 + (a_2-b_2)^2$ -2

$a_1^2 + a_2^2$ $a_1^2 + a_2^2$ $\sqrt{a_1^2 + a_2^2}$

$b_1^2 + b_2^2$ $b_1^2 + b_2^2$ $\sqrt{b_1^2 + b_2^2}$

$-2(a_1 b_1 + a_2 b_2)$ $\cos\theta$

$$\boldsymbol{a}\cdot\boldsymbol{b} = \|\boldsymbol{a}\|\,\|\boldsymbol{b}\|\cos\theta$$

Dot Product

Dot Product
Euclidean Inner Product

$\boldsymbol{a}, \boldsymbol{b}$ Vectors in R^n

θ Angle between $\boldsymbol{a}, \boldsymbol{b}$

$\boldsymbol{a} \cdot \boldsymbol{b} = \|\boldsymbol{a}\| \, \|\boldsymbol{b}\| \cos\theta$

$\boldsymbol{b} \leftarrow \boldsymbol{a}$

$\boldsymbol{a} \cdot \boldsymbol{a} = \|\boldsymbol{a}\| \, \|\boldsymbol{a}\| \cos 0°$

$\boldsymbol{a} \cdot \boldsymbol{a} = \|\boldsymbol{a}\|^2$

$\|\boldsymbol{a}\| = \sqrt{\boldsymbol{a} \cdot \boldsymbol{a}}$

$R^2 \qquad \boldsymbol{a} \cdot \boldsymbol{b} = \|\boldsymbol{a}\| \, \|\boldsymbol{b}\| \cos\theta$

$$= a_1 b_1 + a_2 b_2$$

$R^3 \qquad \boldsymbol{a} \cdot \boldsymbol{b} = \|\boldsymbol{a}\| \, \|\boldsymbol{b}\| \cos\theta$

$$= a_1 b_1 + a_2 b_2 + a_3 b_3$$

$R^n \qquad \boldsymbol{a} \cdot \boldsymbol{b} = \|\boldsymbol{a}\| \, \|\boldsymbol{b}\| \cos\theta$

$$= a_1 b_1 + a_2 b_2 + \cdots a_n b_n$$

Properties of Dot Products

$$\boldsymbol{a} \cdot \boldsymbol{b} = \boldsymbol{b} \cdot \boldsymbol{a}$$

$$\boldsymbol{a} \cdot (\boldsymbol{b} + \boldsymbol{c}) = \boldsymbol{a} \cdot \boldsymbol{b} + \boldsymbol{a} \cdot \boldsymbol{c}$$

$$k(\boldsymbol{a} \cdot \boldsymbol{b}) = (k\,\boldsymbol{a}) \cdot \boldsymbol{b}$$

$$\boldsymbol{a} \cdot \boldsymbol{a} \geq 0 \qquad \boldsymbol{a} \cdot \boldsymbol{a} = 0 \iff \boldsymbol{a} = \boldsymbol{0}$$

$$
\begin{aligned}
R^n \qquad \boldsymbol{a} \cdot \boldsymbol{b} &= \|\boldsymbol{a}\|\,\|\boldsymbol{b}\|\cos\theta \\
&= a_1 b_1 + a_2 b_2 + \cdots a_n b_n
\end{aligned}
$$

$$\boldsymbol{0} \cdot \boldsymbol{a} = \boldsymbol{a} \cdot \boldsymbol{0} = 0$$

$$(\boldsymbol{a} + \boldsymbol{b}) \cdot \boldsymbol{c} = \boldsymbol{a} \cdot \boldsymbol{c} + \boldsymbol{b} \cdot \boldsymbol{c}$$

$$\boldsymbol{a} \cdot (\boldsymbol{b} - \boldsymbol{c}) = \boldsymbol{a} \cdot \boldsymbol{b} - \boldsymbol{a} \cdot \boldsymbol{c}$$

$$(\boldsymbol{a} - \boldsymbol{b}) \cdot \boldsymbol{c} = \boldsymbol{a} \cdot \boldsymbol{c} - \boldsymbol{b} \cdot \boldsymbol{c}$$

$$k(\boldsymbol{a} \cdot \boldsymbol{b}) = \boldsymbol{a} \cdot (k\,\boldsymbol{v})$$

Cauchy-Schwarz Inequality (1)

$$\cos\theta = \frac{a \cdot b}{\|a\| \, \|b\|}$$

$$R^n \qquad a \cdot b = \|a\| \, \|b\| \cos\theta$$

$$= a_1 b_1 + a_2 b_2 + \cdots \ a_n b_n$$

$$-1 \le \cos\theta \le +1$$

$$-1 \le \frac{a \cdot b}{\|a\| \, \|b\|} \le +1$$

$$\frac{|a \cdot b|}{\|a\| \, \|b\|} \le 1$$

$$a = (a_1, a_2, \cdots, a_n)$$

$$b = (b_1, b_2, \cdots, b_n)$$

$$a \cdot b = a_1 b_1 + a_2 b_2 + \cdots + a_n b_n$$

$$\|a\| = \sqrt{a_1^2 + a_2^2 + \cdots + a_n^2}$$

$$|a \cdot b| \le \|a\| \, \|b\|$$

$$\|b\| = \sqrt{b_1^2 + b_2^2 + \cdots + b_n^2}$$

$$a_1 b_1 + a_2 b_2 + \cdots + a_n b_n \le \sqrt{a_1^2 + a_2^2 + \cdots + a_n^2} \sqrt{b_1^2 + b_2^2 + \cdots + b_n^2}$$

Cauchy-Schwarz Inequality (2)

Assume $0 \leq \theta \leq 90°$

$R^n \qquad \boldsymbol{a} \cdot \boldsymbol{b} = \|\boldsymbol{a}\| \|\boldsymbol{b}\| \cos\theta$

$$= a_1 b_1 + a_2 b_2 + \cdots a_n b_n$$

$\boldsymbol{a} = (a_1, a_2, \cdots, a_n)$

$\boldsymbol{b} = (b_1, b_2, \cdots, b_n)$

$\boldsymbol{a} \cdot \boldsymbol{b} = a_1 b_1 + a_2 b_2 + \cdots + a_n b_n$

$\|\boldsymbol{a}\| = \sqrt{a_1^2 + a_2^2 + \cdots + a_n^2}$

$\|\boldsymbol{b}\| = \sqrt{b_1^2 + b_2^2 + \cdots + b_n^2}$

$|\boldsymbol{a} \cdot \boldsymbol{b}| \leq \|\boldsymbol{a}\| \|\boldsymbol{b}\| \quad for \ 0 \leq \theta \leq 180°$

Triangular Inequalities
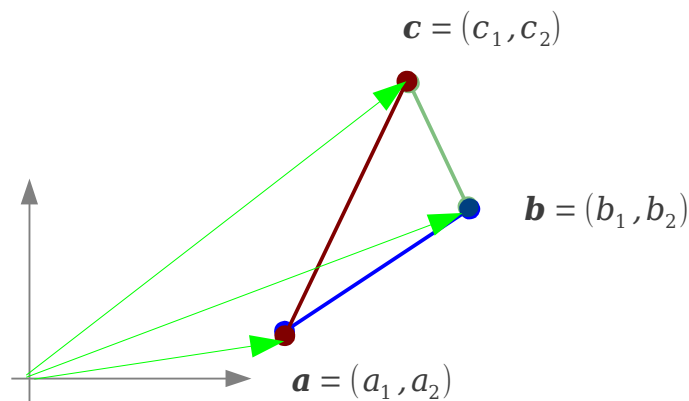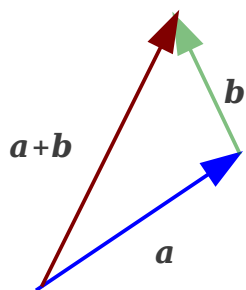
$$\|a+b\| \le \|a\| \, \|b\|$$

$$\|a+b\|^2 = (a+b)\cdot(a+b)$$
$$= (a\cdot a) + 2(a\cdot b) + (b\cdot b)$$
$$\le \|a\|^2 + 2|a\cdot b| + \|b\|^2$$
$$\le \|a\|^2 + 2\|a\|\,\|b\| + \|b\|^2$$
$$\le (\|a\| + \|b\|)^2$$

$$a\cdot a = \|a\|^2$$
$$b\cdot b = \|b\|^2$$
$$a\cdot b \le |a\cdot b|$$
$$|a\cdot b| \le \|a\|\,\|b\|$$

$$d(a,b) \le d(a,c) + d(c,b)$$

$$d(a,b) = \|a-b\|$$
$$= \|a-c+c-b\|$$
$$= \|(a-c) + (c-b)\|$$
$$\le \|a-c\| + \|c-b\|$$
$$\le d(a,c) + d(c,b)$$

Projection (1)

u

$w_\perp$ u

w
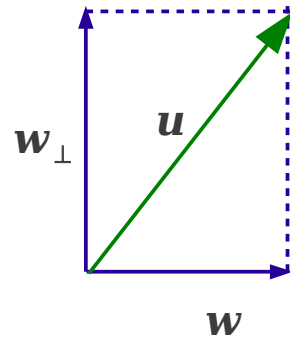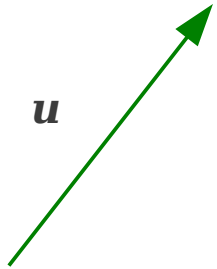
$$u = w + w_\perp$$

$$u - w = w_\perp$$

unit vectors

$\dfrac{w_\perp}{\|w_\perp\|}$

$\dfrac{w}{\|w\|}$

Various Decomposition

$w_\perp$

w

$w_\perp$

w

$w_\perp$

w

$w_\perp$

w

Projection (2)

u

$w_\perp$ u

w

$$u = w + w_\perp$$

$$u - w = w_\perp$$

$$\frac{w_\perp}{\|w_\perp\|}$$

$$\frac{w}{\|w\|}$$

u

w

1

$$u \cdot \frac{w}{\|w\|} = \|u\| \cos\theta$$

$$w = k\,a$$

w a

unit vectors

$$\boxed{\frac{w}{\|w\|} = \frac{a}{\|a\|}}$$

u

w a
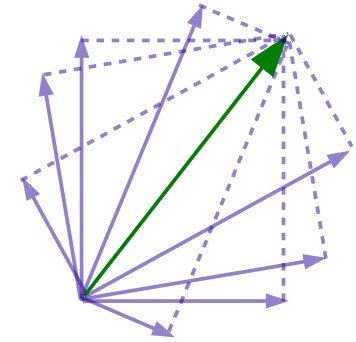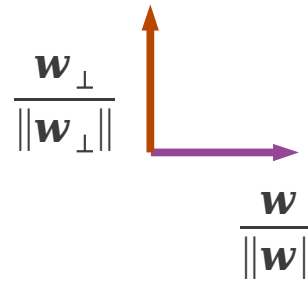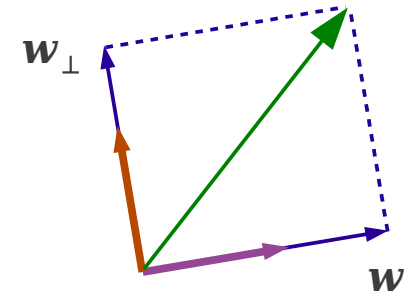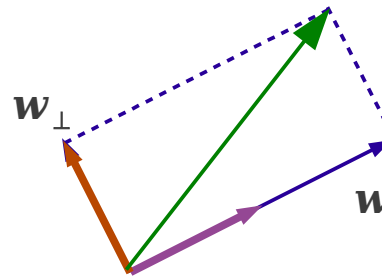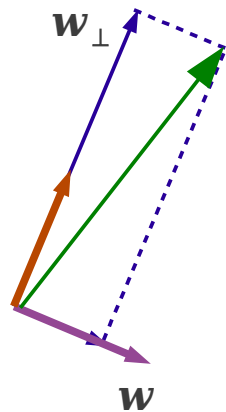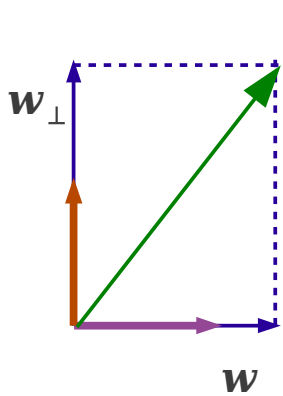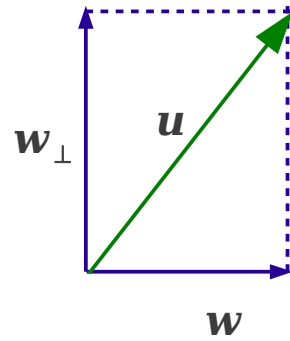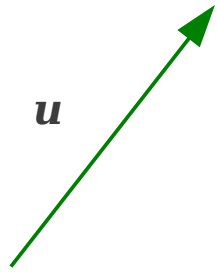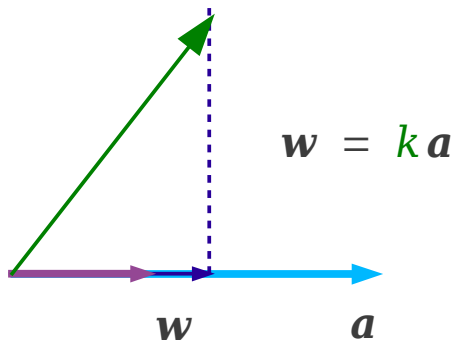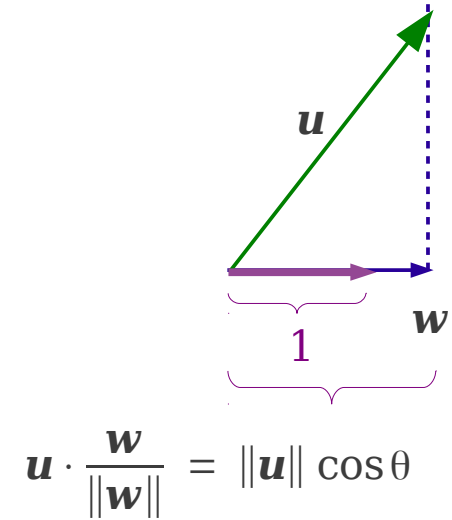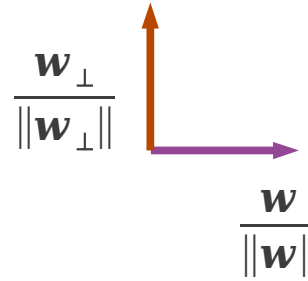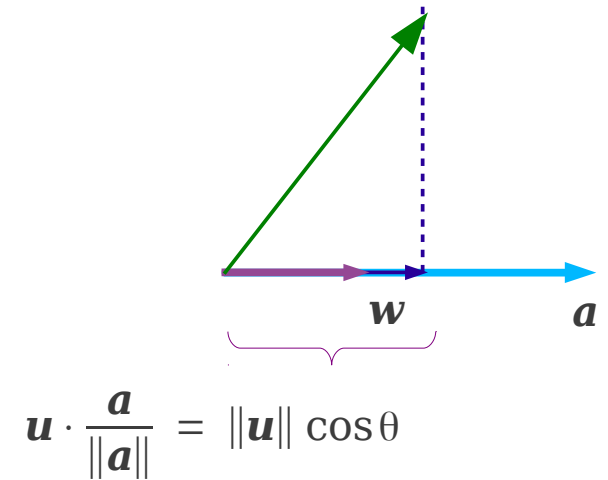
$$u \cdot \frac{a}{\|a\|} = \|u\| \cos\theta$$

Projection (3)
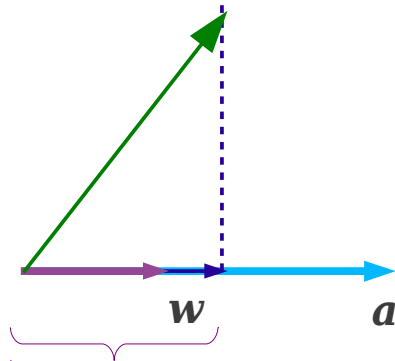
$$u \cdot \frac{a}{\|a\|} = \|u\| \cos\theta$$
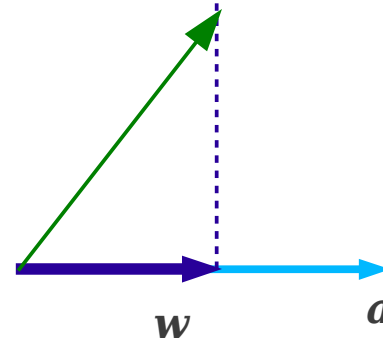
$$\left(u \cdot \frac{a}{\|a\|}\right) \frac{a}{\|a\|} = \left(\frac{u \cdot a}{\|a\|}\right) \frac{a}{\|a\|} = \frac{u \cdot a}{\|a\|^2} a$$

scale factor   unit vectors

another scale factor

$$u = w + w_\perp$$

$$u - w = w_\perp$$
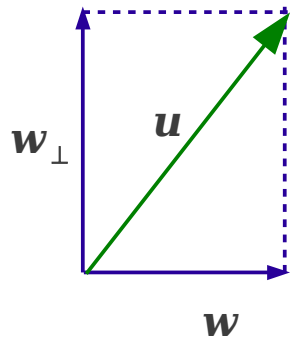
$$w = k\,a \qquad = \frac{u \cdot a}{\|a\|^2} a$$

$$w_\perp = u - w \qquad = u - \frac{u \cdot a}{\|a\|^2} a$$

Projection (4)
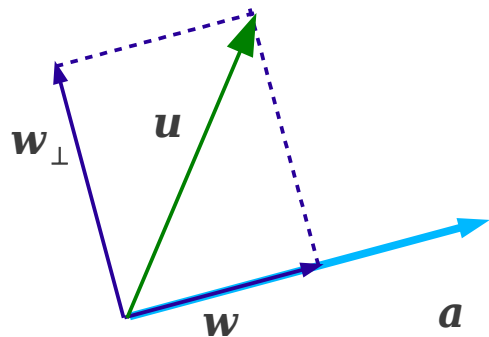
vector component of **u** along **a**

$$w = k\,a$$

$$proj_a u = \frac{u \cdot a}{\|a\|^2}\, a$$

vector component of **u** orthogonal **a**

$$w_\perp = u - w$$

$$u - proj_a u = u - \frac{u \cdot a}{\|a\|^2}\, a$$

$$\| proj_a u \| = \frac{|u \cdot a|}{\|a\|} = \|u\| \cos\theta$$

Orthogonality

$$\cos\theta \;=\; \frac{\boldsymbol{a}\cdot\boldsymbol{b}}{\|\boldsymbol{a}\|\,\|\boldsymbol{b}\|}$$

$$R^n \qquad \boldsymbol{a}\cdot\boldsymbol{b} \;=\; \|\boldsymbol{a}\|\,\|\boldsymbol{b}\|\cos\theta$$

$$= a_1 b_1 + a_2 b_2 + \cdots a_n b_n$$

$$-1 \;\leq\; \cos\theta \;\leq\; +1$$

$$-1 \;\leq\; \frac{\boldsymbol{a}\cdot\boldsymbol{b}}{\|\boldsymbol{a}\|\,\|\boldsymbol{b}\|} \;\leq\; +1$$

$$\frac{|\boldsymbol{a}\cdot\boldsymbol{b}|}{\|\boldsymbol{a}\|\,\|\boldsymbol{b}\|} \;\leq\; 1$$

$$\cos 90^{\circ} \;=\; 0$$

$$\frac{\boldsymbol{a}\cdot\boldsymbol{b}}{\|\boldsymbol{a}\|\,\|\boldsymbol{b}\|} \;=\; 0$$

$$\boldsymbol{a}\cdot\boldsymbol{b} \;=\; 0$$

Point-Normal Equation (1)

R^2

$\boldsymbol{n} = (a, b)$ $\boldsymbol{x} = (x, y)$

z

y

R^3

z

$\boldsymbol{n} = (a, b, c)$

y

$\boldsymbol{x} = (x, y, z)$

x
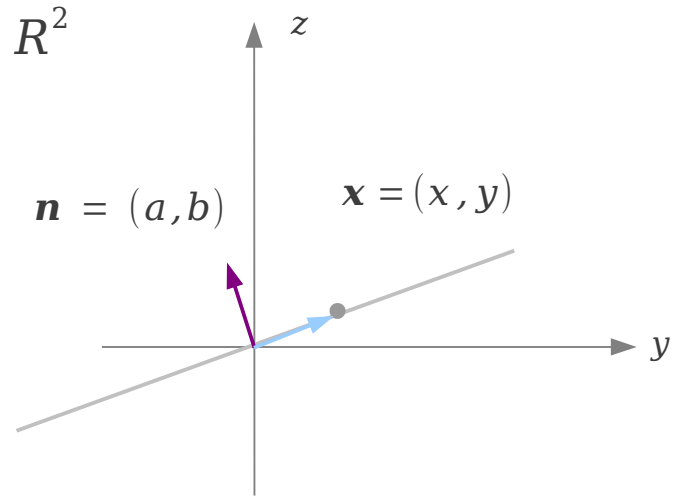
$\boldsymbol{n} \cdot \boldsymbol{x} = 0$

$a x + b y = 0$

$\boldsymbol{n} \cdot \boldsymbol{x} = 0$

$a x + b y + c z = 0$

Point-Normal Equation (2)

R^2

$\boldsymbol{x} = (x, y)$

$\boldsymbol{x_0} = (x_0, y_0)$

$\boldsymbol{n} = (a, b)$

R^3

$\boldsymbol{x} = (x, y, z)$

$\boldsymbol{x} = (x_0, y_0, z_0)$

$\boldsymbol{n} = (a, b, c)$
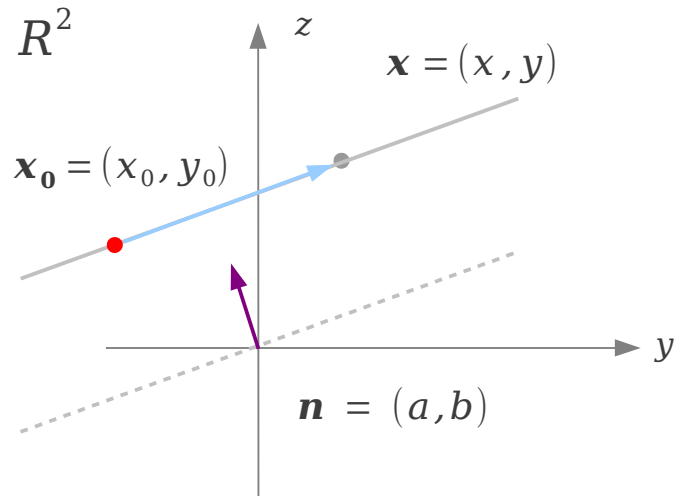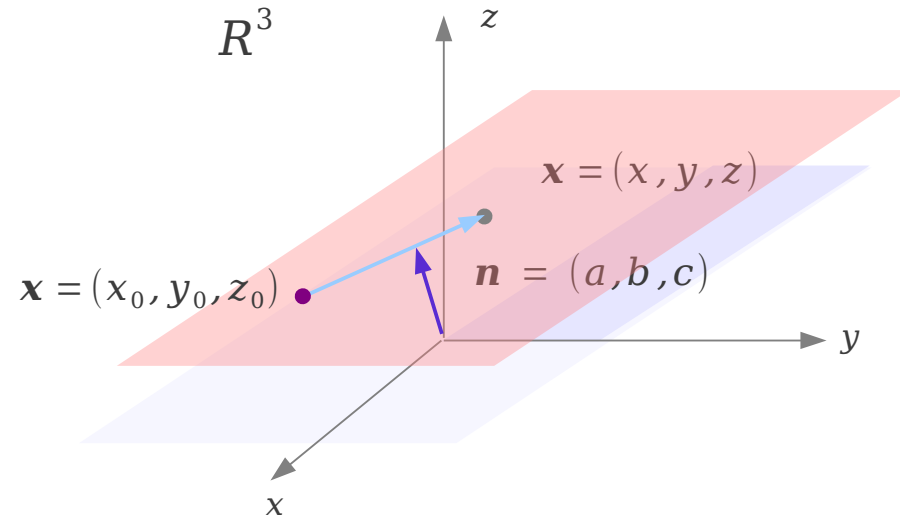
$$\boldsymbol{n} \cdot (\boldsymbol{x} - \boldsymbol{x_0}) = 0$$

$$a(x - x_0) + b(y - y_0) = 0$$

$$\boldsymbol{n} \cdot (\boldsymbol{x} - \boldsymbol{x_0}) = 0$$

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$$

Parametric Equation (1)

R^2

R^1

$\boldsymbol{x} = (x, y)$

$\boldsymbol{v} = (a, b)$

z

y

R^3

R^2

$(a_2, b_2, c_2) = \boldsymbol{v_2}$

$(a_1, b_1, c_1) = \boldsymbol{v_1}$

$t_2 \boldsymbol{v_2}$

$t_1 \boldsymbol{v_1}$

$\boldsymbol{x} = (x, y, z)$

z

y

x
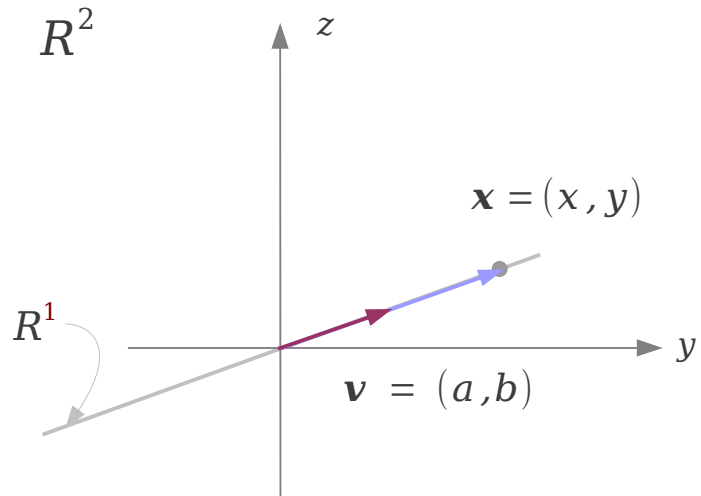
$$\boldsymbol{x} = t\,\boldsymbol{v}$$

$$x = t\,a$$
$$y = t\,b$$
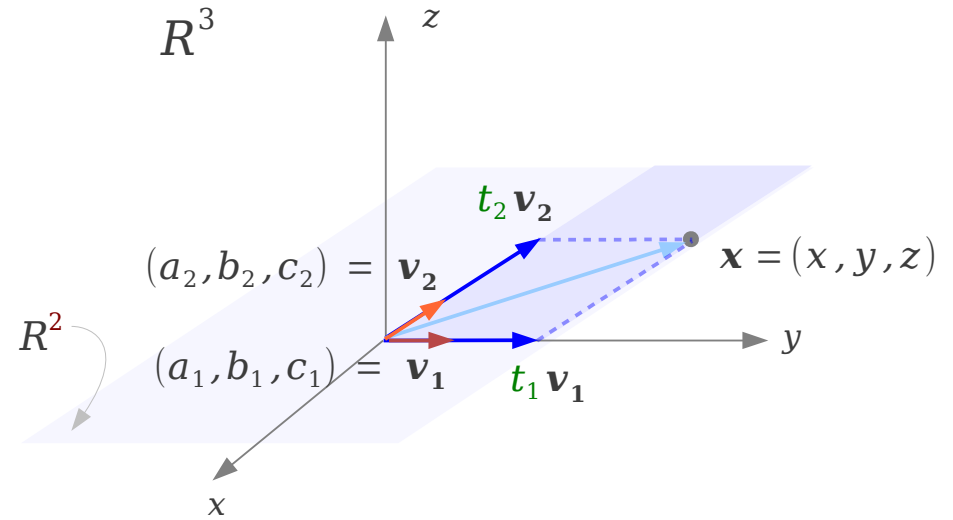
$$\boldsymbol{x} = t_1 \boldsymbol{v_1} + t_2 \boldsymbol{v_2}$$

$$x = t_1 a_1 + t_2 a_2$$
$$y = t_1 b_1 + t_2 b_2$$
$$z = t_1 c_1 + t_2 c_2$$

Parametric Equation (2)

R^2

$$\boldsymbol{x} = \boldsymbol{x_0} + t\,\boldsymbol{v}$$

$$x = x_0 + t\,a$$
$$y = y_0 + t\,b$$

R^3
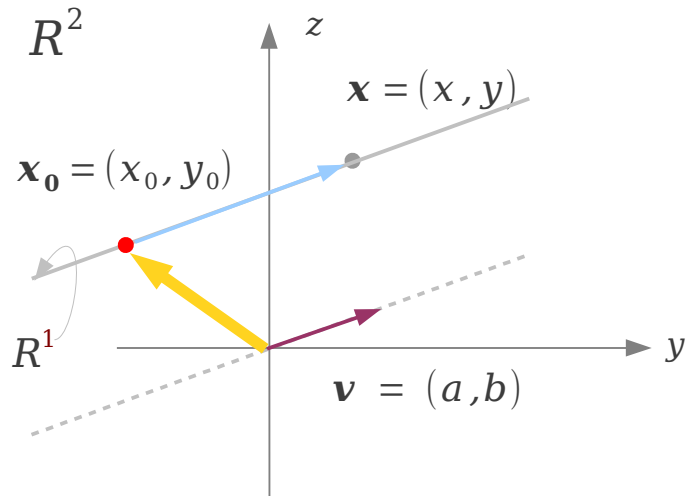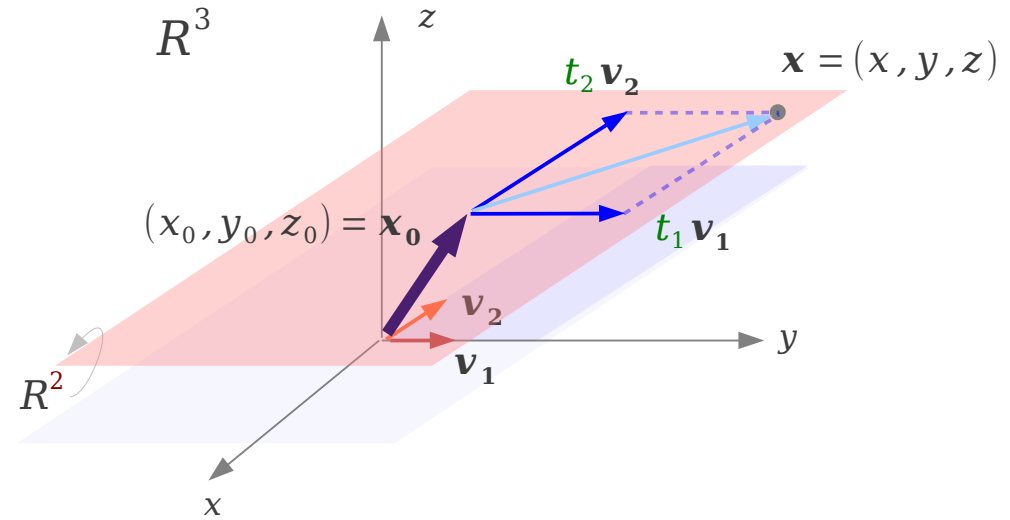
$$\boldsymbol{x} = \boldsymbol{x_0} + t_1\,\boldsymbol{v_1} + t_2\,\boldsymbol{v_2}$$

$$x = x_0 + t_1 a_1 + t_2 a_2$$
$$y = y_0 + t_1 b_1 + t_2 b_2$$
$$z = z_0 + t_1 c_1 + t_2 c_2$$

Parametric Equation (3)

Line Equation in R^n

x_0 , v vectors in R^n

$$x = x_0 + t\,v$$

x_0 , x_1 vectors in R^n

$$v = (x_1 - x_0)$$

$$x = x_0 + t(x_1 - x_0)$$

$$x = (1 - t)\,x_0 + t\,x_1$$

Plane Equation in R^n

x_0 , v_1 , v_2 vectors in R^n

$$x = x_0 + t_1\,v_1 + t_2\,v_2$$

Linear System (1)

Linear Equations

Corresponding Homogeneous Equation
$$a_1 x_1 + a_2 x_2 + \cdots + a_n x_n = 0$$

$$\boldsymbol{a} = (a_1, a_2, \cdots, a_n)$$

normal vector $\qquad \boldsymbol{a} \cdot \boldsymbol{x} = b$

$$\boldsymbol{x} = (x_1, x_2, \cdots, x_n)$$

$$\boldsymbol{a} \cdot \boldsymbol{x} = 0$$

> each **solution** vector $\boldsymbol{x}$ of a **homogeneous** equation
>
> **orthogonal** to the coefficient vector $\boldsymbol{a}$

Homogeneous Linear System

$$a_{11} x_1 + a_{12} x_2 + \cdots + a_{1n} x_n = 0 \qquad\qquad \boldsymbol{r_1} \cdot \boldsymbol{x} = 0$$
$$a_{21} x_1 + a_{22} x_2 + \cdots + a_{2n} x_n = 0 \qquad\qquad \boldsymbol{r_2} \cdot \boldsymbol{x} = 0$$
$$\cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \qquad\qquad \cdots$$
$$a_{m1} x_1 + a_{m2} x_2 + \cdots + a_{mn} x_n = 0 \qquad\qquad \boldsymbol{r_m} \cdot \boldsymbol{x} = 0$$

Linear System (2)

Homogeneous Linear System

$$a_{11} x_1 + a_{12} x_2 + \cdots + a_{1n} x_n = 0 \qquad\qquad r_1 \cdot x = 0$$

$$a_{21} x_1 + a_{22} x_2 + \cdots + a_{2n} x_n = 0 \qquad\qquad r_2 \cdot x = 0$$

$$\cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \qquad\qquad\qquad \cdots$$

$$a_{m1} x_1 + a_{m2} x_2 + \cdots + a_{mn} x_n = 0 \qquad\qquad r_m \cdot x = 0$$

> each **solution** vector x of a **homogeneous** equation
>
> **orthogonal** to the row vector r_i of the coefficient matrix

Homogeneous Linear System $\qquad A \cdot x = 0 \qquad\qquad A : m \times n$

solution set consists of all vectors in R^n

that are **orthogonal** to every row vector of A

Linear System (3)

Non-Homogeneous Linear System $\qquad A \cdot x = b$ $\qquad A : m \times n$

Homogeneous Linear System $\qquad\qquad A \cdot x = 0$

a particular solution

$$A \cdot x = b$$

> **solution set** consists of all vectors in R^n
>
> that are **orthogonal** to every row vector of A

+

> **a <u>particular</u> solution** $\quad x_0 \qquad A \cdot x_0 = b$

R^3

$n = (a, b, c)$

$x = (x, y, z)$

R^3

$x = (x, y, z)$

$x = (x_0, y_0, z_0)$

$n = (a, b, c)$

Vector Addition

u
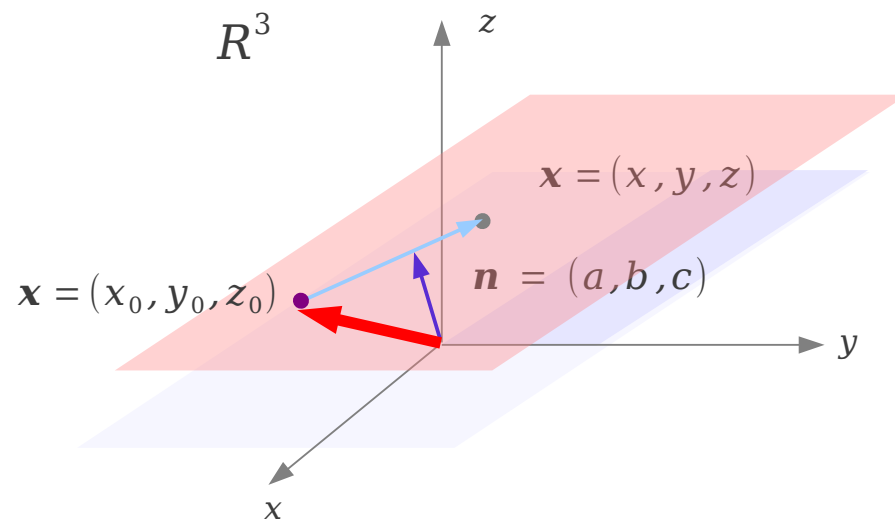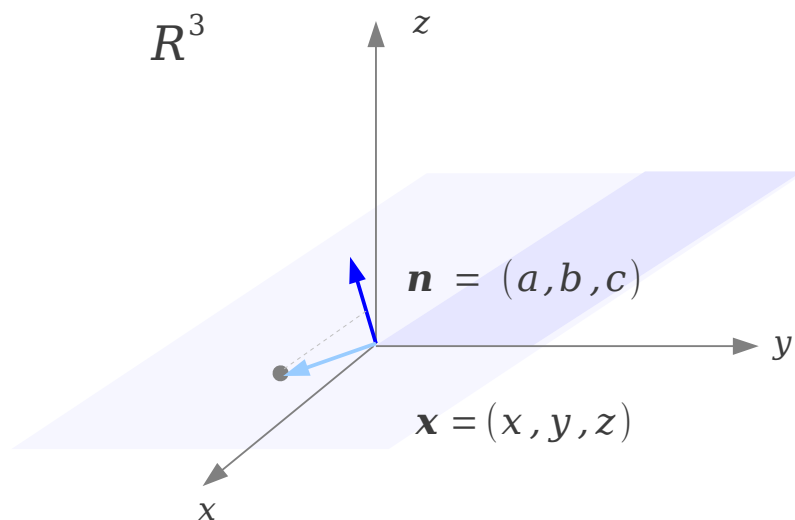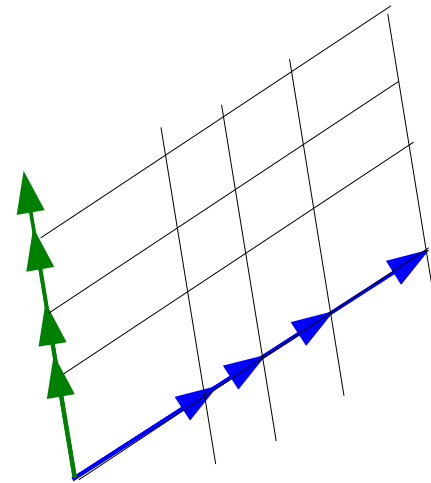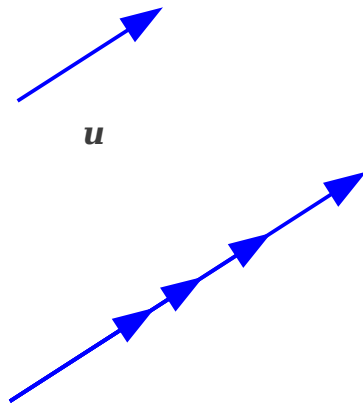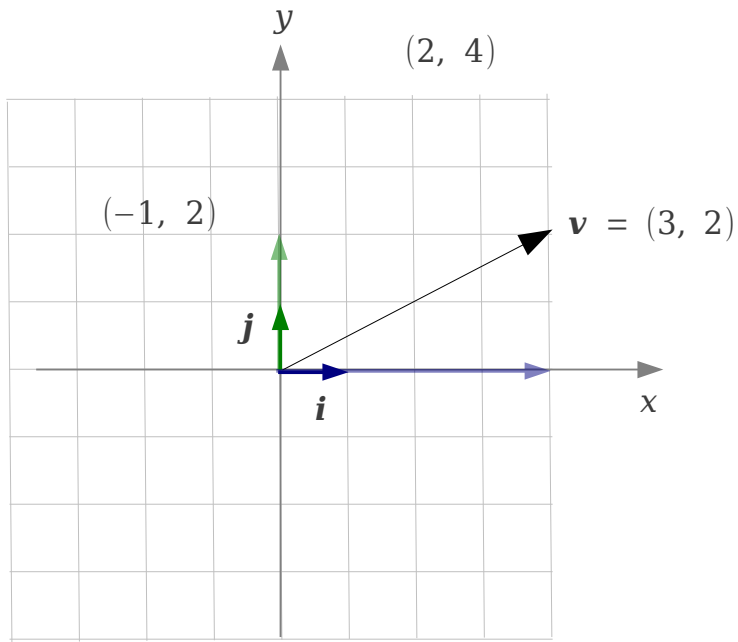
Basis

$$v = 3\,i + 2\,j$$

$$a\begin{bmatrix}1\\0\end{bmatrix} + b\begin{bmatrix}0\\1\end{bmatrix} = 0 \quad \Longrightarrow \quad a = b = 0$$

basis
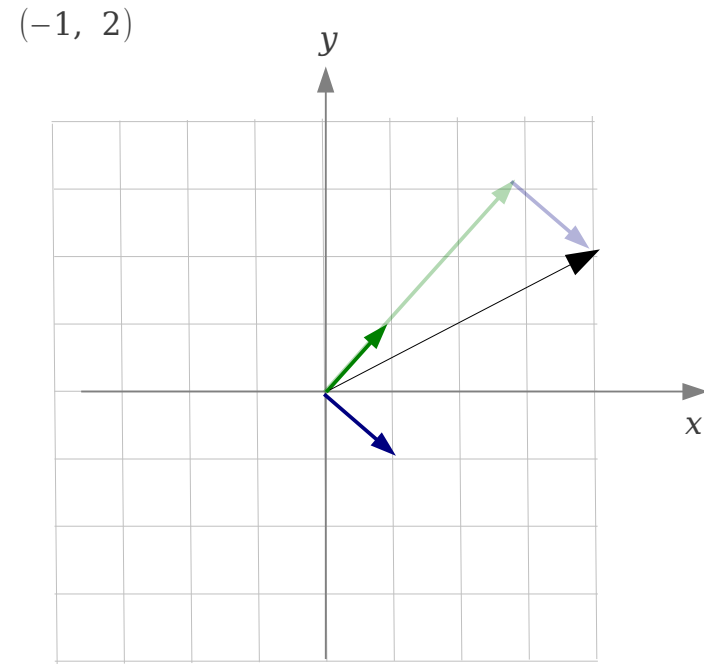
$$a\begin{bmatrix}1\\1\end{bmatrix} + b\begin{bmatrix}1\\-1\end{bmatrix} = 0 \quad \Longrightarrow \quad a = b = 0$$

basis

Cross Product (1)

Determinant of order 3

$$a = a_1 i + a_2 j + a_3 k = \langle a_1, a_2, a_3 \rangle$$

$$b = b_1 i + b_2 j + b_3 k = \langle b_1, b_2, b_3 \rangle$$

$$a \times b = \begin{vmatrix} i & j & k \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = i \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} - j \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} + k \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}$$

$$= (a_2 b_3 - a_3 b_2) i - (a_1 b_3 - a_3 b_1) i + (a_1 b_2 - a_2 b_1) i$$

Cross Product (2)

$$k = \langle 0,\ 0,\ 1 \rangle$$

$$j = \langle 0,\ 1,\ 0 \rangle$$

$$i = \langle 1,\ 0,\ 0 \rangle$$

$$a = \langle a_1,\ a_2,\ a_3 \rangle$$
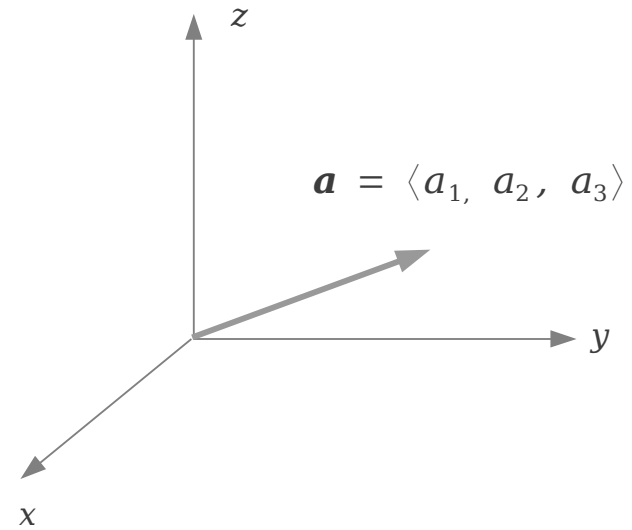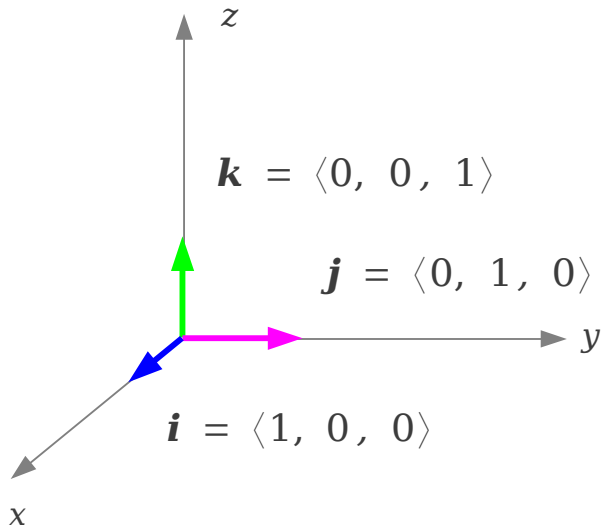
$$i \times j = \begin{vmatrix} i & j & k \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{vmatrix} = k \quad \Longleftarrow \quad \text{normal to } i \ \& \ j \quad \Longrightarrow \quad j \times i = \begin{vmatrix} i & j & k \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{vmatrix} = -k$$

$$j \times k = \begin{vmatrix} i & j & k \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = i \quad \Longleftarrow \quad \text{normal to } j \ \& \ k \quad \Longrightarrow \quad k \times j = \begin{vmatrix} i & j & k \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{vmatrix} = -i$$

$$k \times i = \begin{vmatrix} i & j & k \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{vmatrix} = j \quad \Longleftarrow \quad \text{normal to } k \ \& \ i \quad \Longrightarrow \quad i \times k = \begin{vmatrix} i & j & k \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{vmatrix} = -j$$

Right Hand Rule

$$a \times b = \begin{vmatrix} i & j & k \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

$$b \times a = \begin{vmatrix} i & j & k \\ b_1 & b_2 & b_3 \\ a_1 & a_2 & a_3 \end{vmatrix}$$

Normal direction n

n

a θ b

a θ b

n

$a \times b$

a θ b

Magnitude = $\|a \times b\| = \|a\| \|b\| \sin \theta$

a θ b

$b \times a$
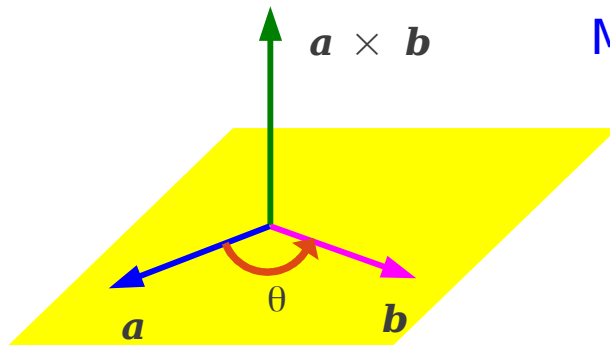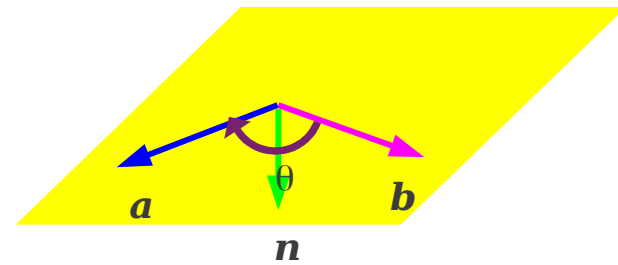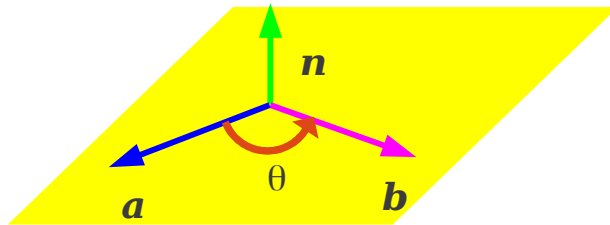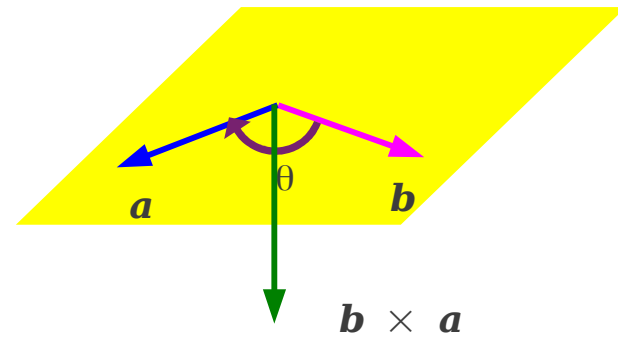
Line Equations (1)

Vector Equation

$$\boldsymbol{r} = \boldsymbol{r}_2 + t\,\boldsymbol{a}$$

Parameter

Direction Vector

$$\boldsymbol{r}_2 = \langle x_2, y_2, z_2 \rangle$$
$$\boldsymbol{a} = \langle a_1, a_2, a_3 \rangle$$

Parametric Equation

$$x = x_2 + t\,a_1$$
$$y = y_2 + t\,a_2$$
$$z = z_2 + t\,a_3$$

Component

$$t\,a_1 = x - x_2$$
$$t\,a_2 = y - y_2$$
$$t\,a_3 = z - z_2$$

Symmetric Equation

$$t = \frac{x - x_2}{a_1} = \frac{y - y_2}{a_2} = \frac{z - z_2}{a_3}$$

Elimination of parameter
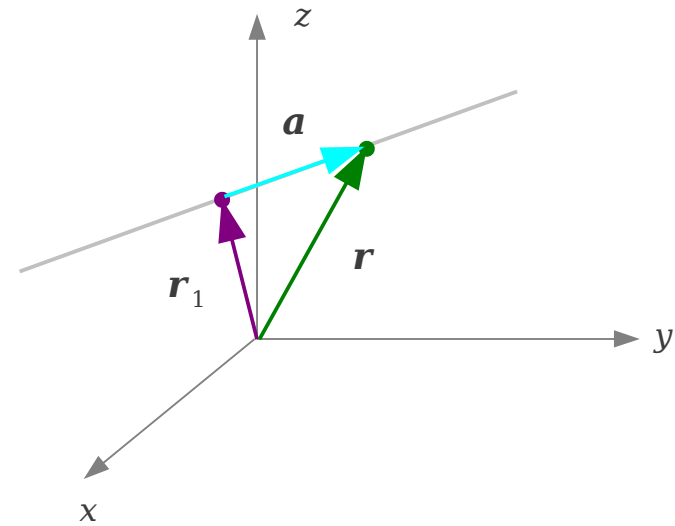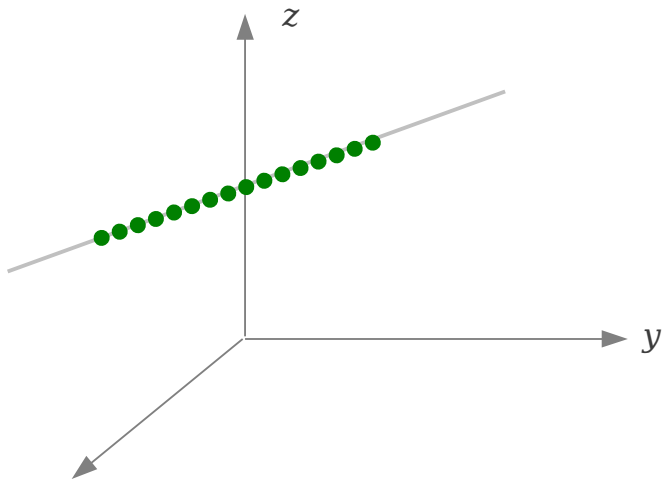
Line Equations (2)

Direction Vector

Direction Vector

$t = 1.2$ Parameter

$t = 1.7$ Parameter

Line Equations (3)

A Passing Point

r_2

Direction Vector

a

$t\,a$

Parameter

r_2
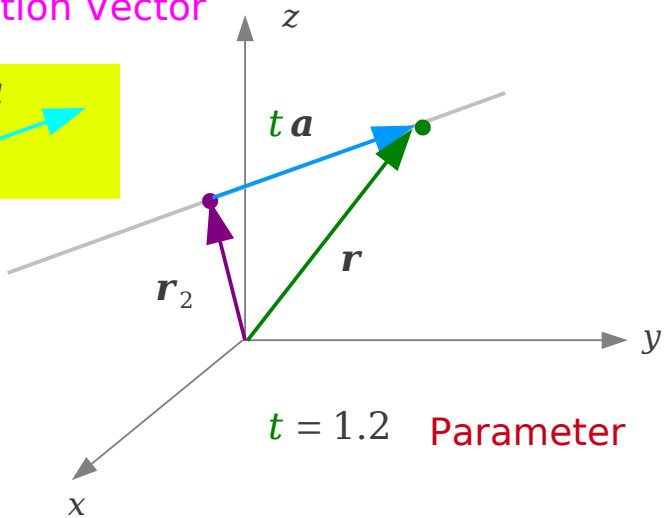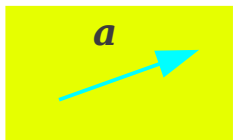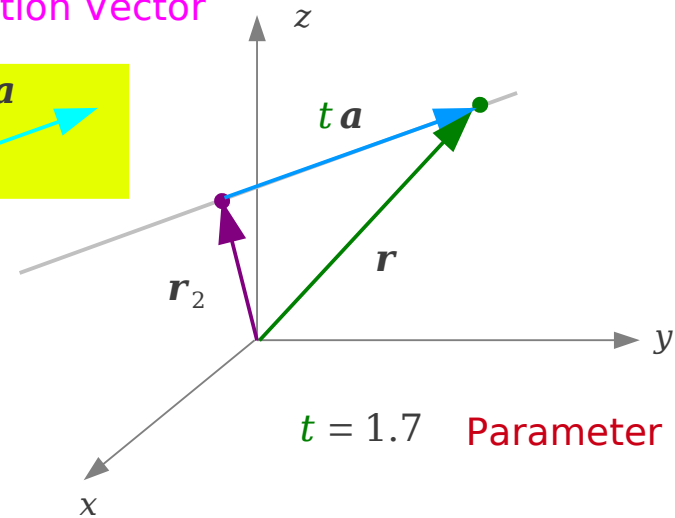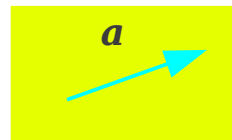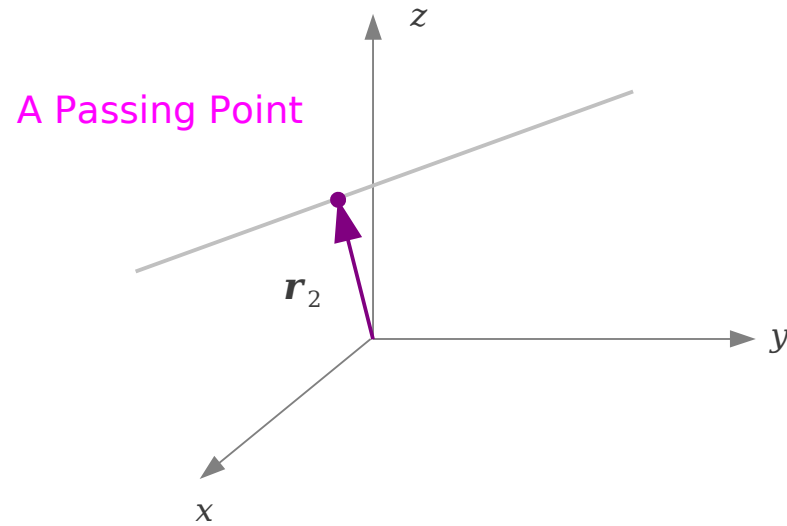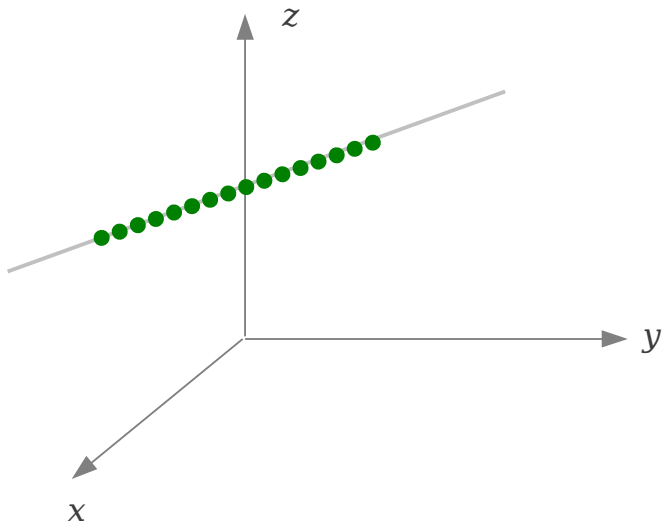
r
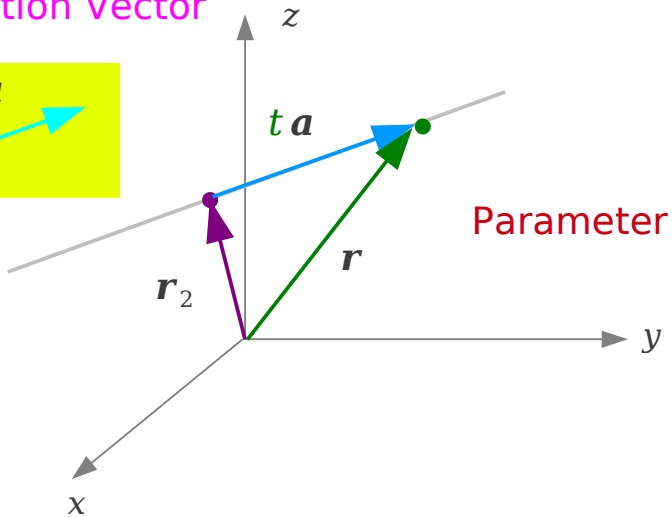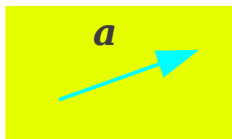
$$r = r_2 + t\,a$$

$$r_2 = \langle x_2,\ y_2,\ z_2 \rangle$$
$$a = \langle a_1,\ a_2,\ a_3 \rangle$$

Plane Equations (1)

Vector equation

$$\boldsymbol{n} \cdot (\boldsymbol{r} - \boldsymbol{r}_1) = 0$$

Normal Vector

$$\boldsymbol{r} = \langle x, y, z \rangle$$
$$\boldsymbol{r}_1 = \langle x_1, y_1, z_1 \rangle$$
$$\boldsymbol{n} = \langle a, b, c \rangle$$

$$\boldsymbol{r} - \boldsymbol{r}_1 = \langle x - x_1, \ y - y_1, \ z - z_1 \rangle$$
$$\boldsymbol{n} = \langle \quad a, \quad b, \quad c \rangle$$

Cartesian equation

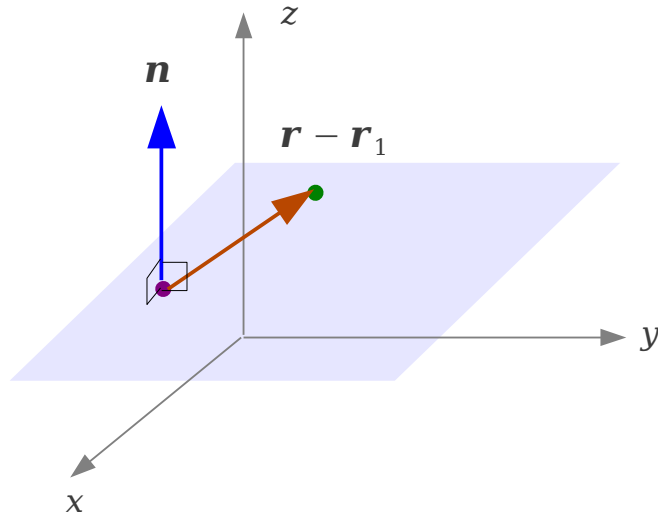$$a(x - x_1) + b(y - y_1) + c(z - z_1) = 0$$

Plane Equations (2)
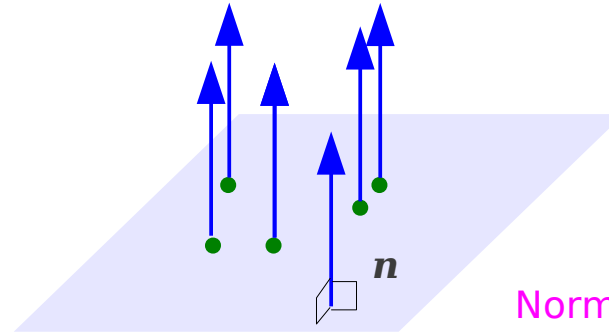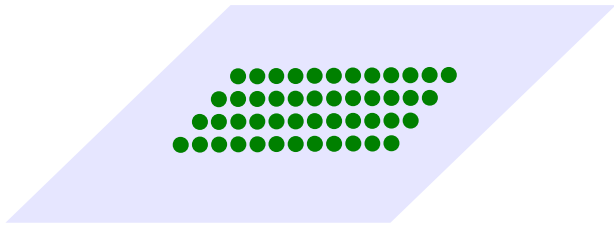
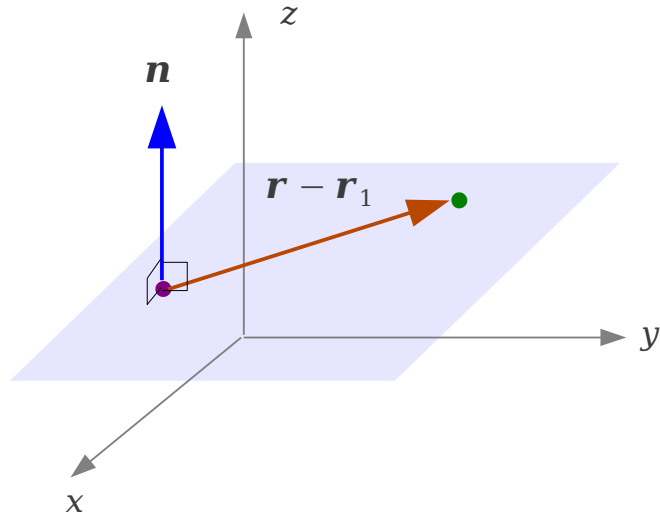Normal Vector

No Parameter

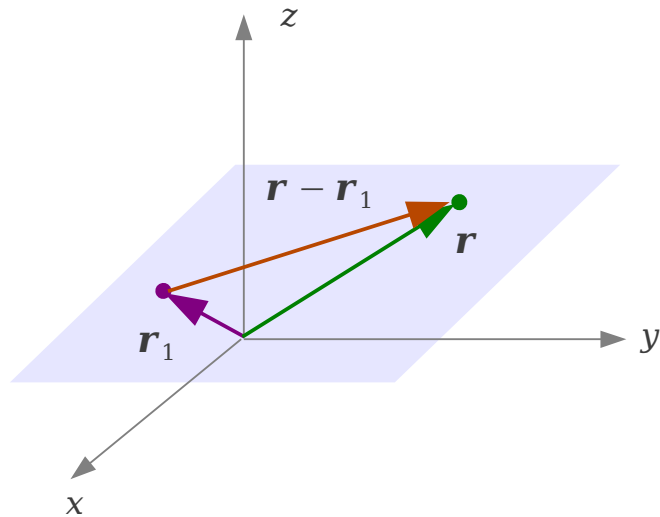$\boldsymbol{r} - \boldsymbol{r}_1$

$\boldsymbol{r}$

$\boldsymbol{r}_1$

$\boldsymbol{r} - \boldsymbol{r}_1$

$\boldsymbol{r}$

$\boldsymbol{r}_1$

$\boldsymbol{n}$

z

y

x
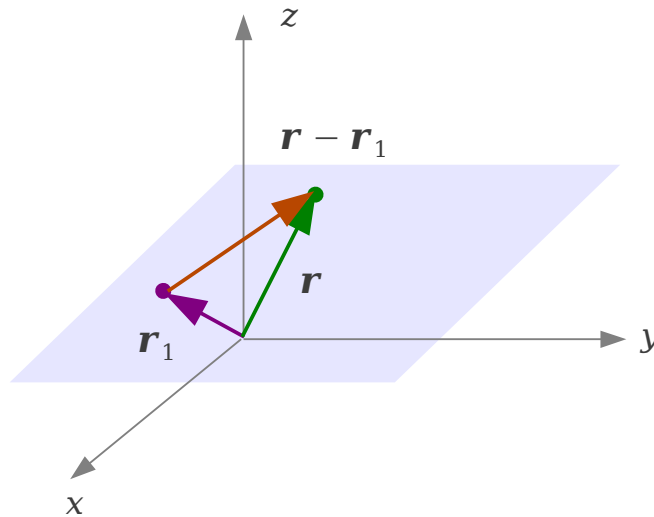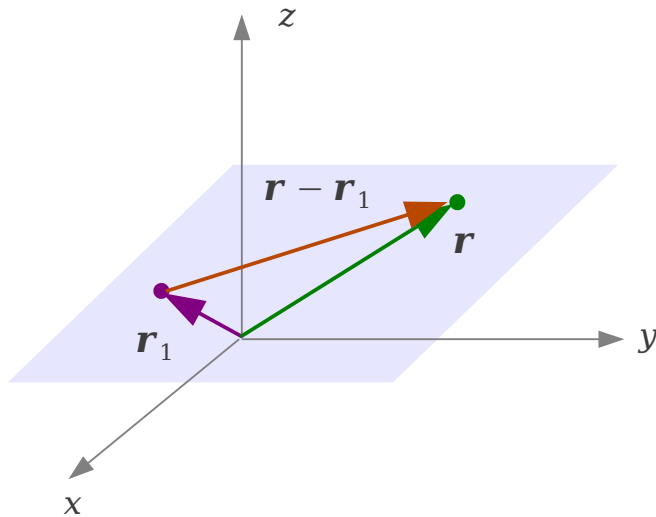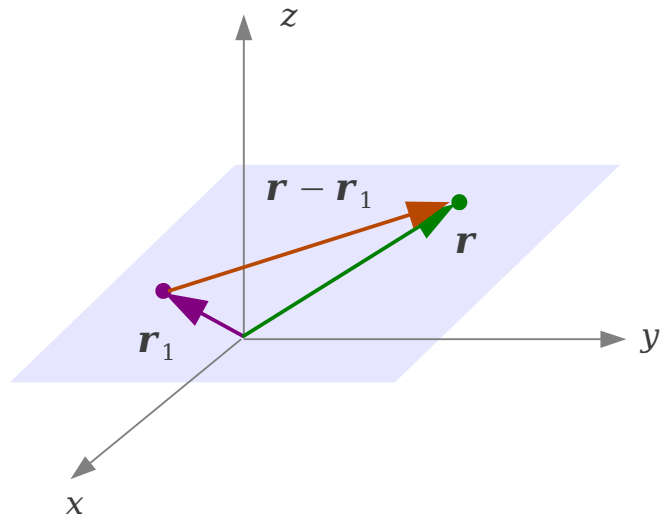
Plane Equations (3)

$$\boxed{\boldsymbol{n} \cdot (\boldsymbol{r} - \boldsymbol{r}_1) = 0}$$

$$\boldsymbol{r} = \langle x, y, z \rangle$$

$$\boldsymbol{r}_1 = \langle x_1, y_1, z_1 \rangle$$

$$\boldsymbol{n} = \langle a, b, c \rangle$$
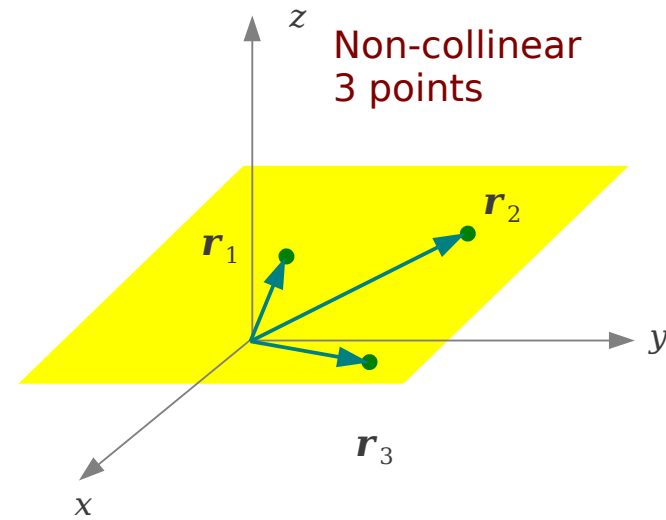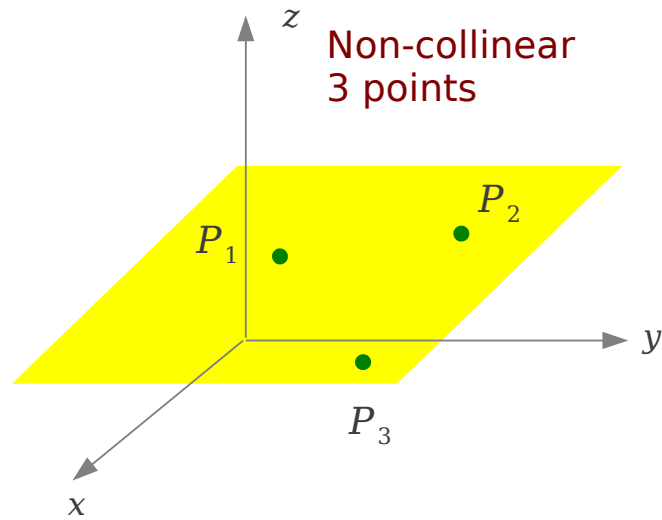
Plane Equations (3)

$$\boldsymbol{n} \cdot (\boldsymbol{r} - \boldsymbol{r}_1) = 0$$

$$\boldsymbol{r} = \langle x, y, z \rangle$$

$$\boldsymbol{r}_1 = \langle x_1, y_1, z_1 \rangle$$

$$\boldsymbol{n} = \langle a, b, c \rangle$$

Euclidean (1A)
Vector Space

Normal Vector & 3 Points

Non-collinear
3 points

P_2

P_1

P_3

z

y

x

Non-collinear
3 points
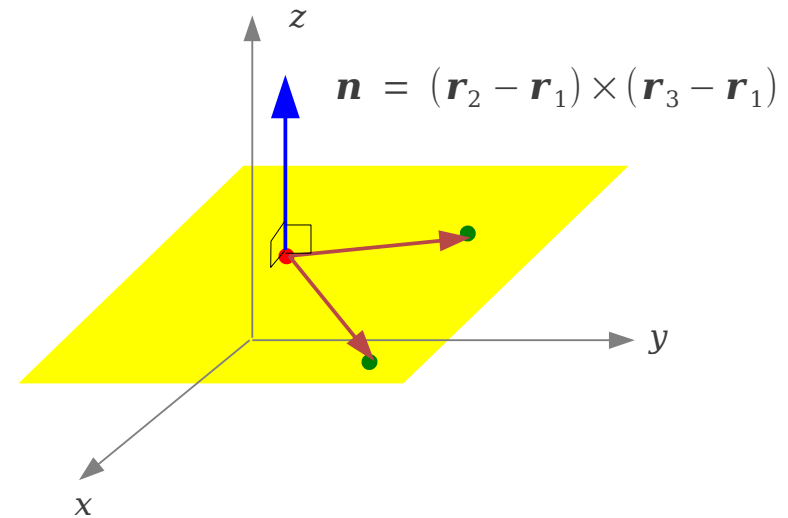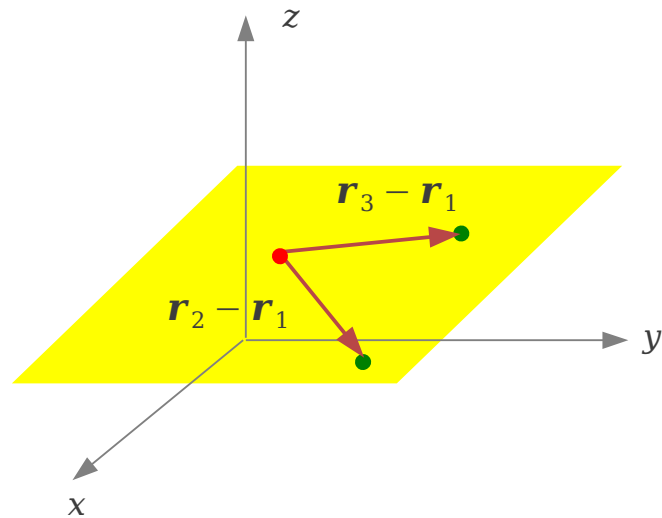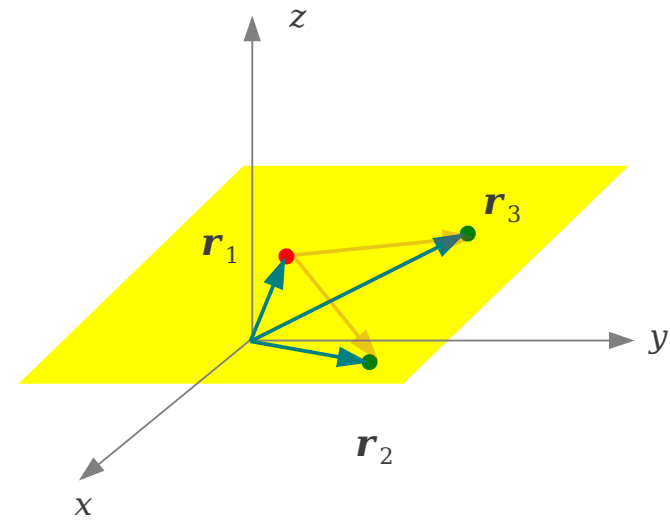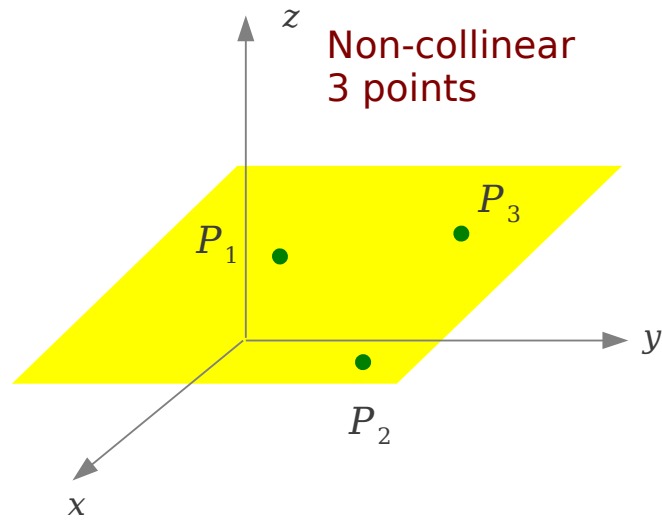
r_2

r_1

r_3

z

y

x
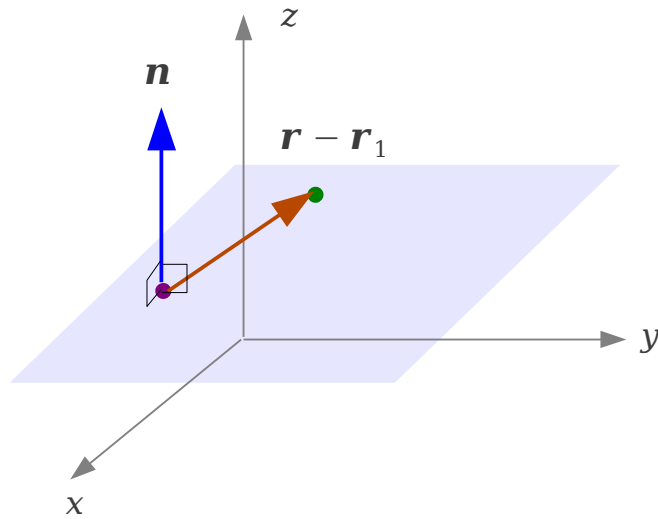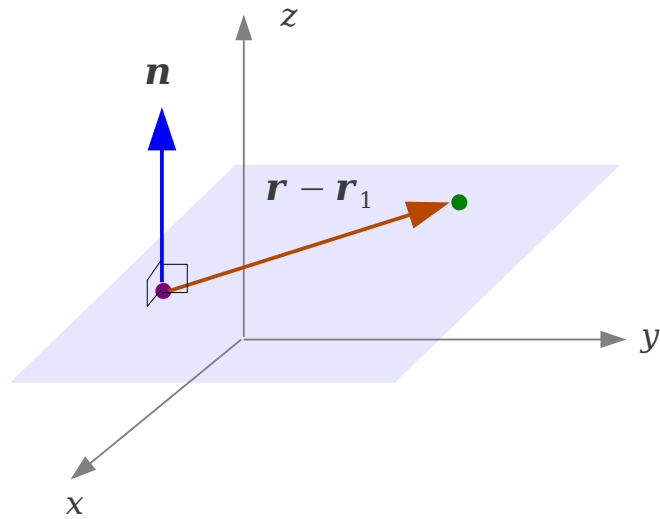
Graph of a plane

Line intersection of two planes

Point of intersection of a line and plane

Normal Vector & 3 Points

Non-collinear
3 points

P_1 P_3 P_2

r_1 r_3 r_2

$r_3 - r_1$

$r_2 - r_1$

$$n = (r_2 - r_1) \times (r_3 - r_1)$$

Vector Triple Product

$$a \times (b \times c)$$

⟹ Perpendicular to $b \times c$

⟹ Any vector perpendicular to $b \times c$
   lies in the plane perpendicular to $b \times c$

⟹ lies in the plane of b and c

Perpendicular to the plane of

$$a \times (b \times c) \implies mb + nc$$

$$(a \cdot c)b - (a \cdot b)c$$

$b \times c$

$mb + nc$

b θ c
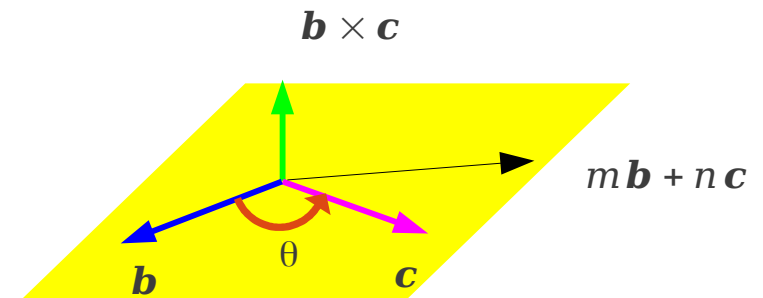
Scalar Triple Product

$$a \cdot (b \times c)$$

$$a = a_1 i + a_2 j + a_3 k$$
$$b = b_1 i + b_2 j + b_3 k$$
$$c = c_1 i + c_2 j + c_3 k$$

$$b \times c = \begin{vmatrix} i & j & k \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = i \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - j \begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + k \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}$$

$$a \cdot (b \times c) = (a_1 i + a_2 j + a_3 k) \cdot \left(i \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - j \begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + k \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix} \right)$$
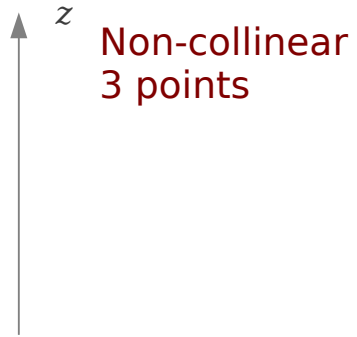
$$= \left(a_1 \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - a_2 \begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + a_3 \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix} \right) = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$
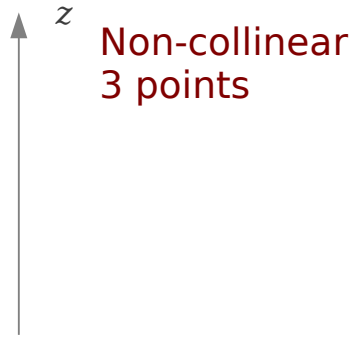
Normal Vector & 3 Points

Non-collinear
3 points

$$\boldsymbol{a} \times \boldsymbol{b} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = \boldsymbol{i} \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} - \boldsymbol{j} \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} + \boldsymbol{k} \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}$$

Normal Vector & 3 Points

Non-collinear
3 points

z

z

References

[1] http://en.wikipedia.org/
[2] http://planetmath.org/
[3] M.L. Boas, "Mathematical Methods in the Physical Sciences"
[4] D.G. Zill, "Advanced Engineering Mathematics"