

# Complex Numbers (1A)

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- Complex Numbers
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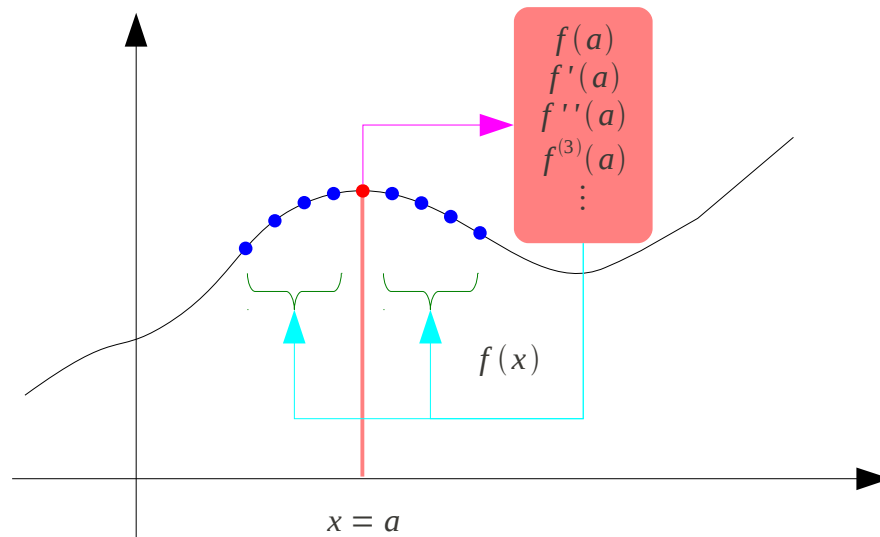
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# Taylor Series

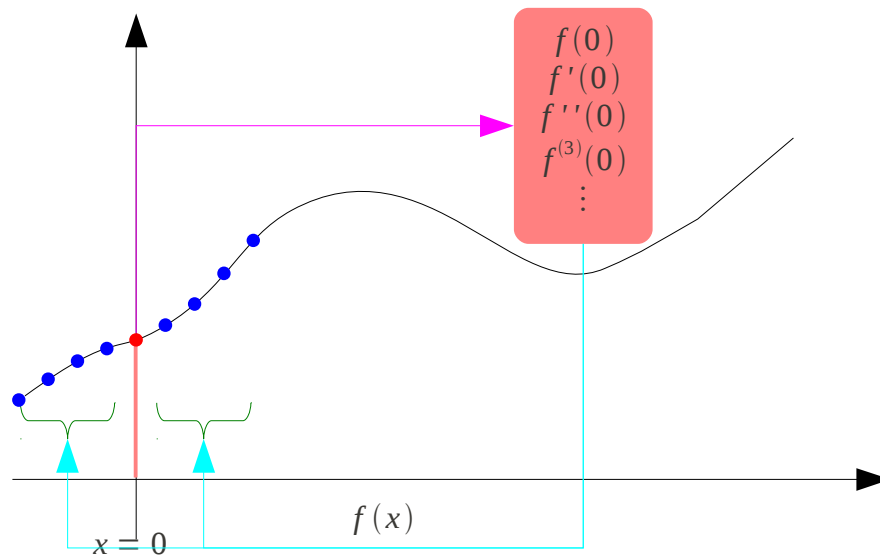
$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n + \cdots$$



# Maclaurin Series

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n + \cdots$$

$$f(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \cdots + \frac{f^{(n)}(0)}{n!}x^n + \cdots$$



# Power Series Expansion

$$f(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \cdots + \frac{f^{(n)}(0)}{n!}x^n + \cdots$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$$

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$$

# The Euler Constant $e$

$$\frac{d}{dx} a^x = \lim_{h \rightarrow 0} \frac{a^{(x+h)} - a^x}{h} = a^x \lim_{h \rightarrow 0} \frac{a^h - 1}{h}$$

$$\lim_{h \rightarrow 0} \frac{a^h - 1}{h} = 1 \quad \text{iif } a = e$$

$$\frac{d}{dx} e^x = e^x$$

$$e = 2.71828\dots$$

$$f(x) = e^x \quad \Rightarrow \quad f'(x) = e^x \quad \Rightarrow \quad f''(x) = e^x \quad \dots$$

$$f'(0) = 1$$

$$\lim_{h \rightarrow 0} \frac{e^h - e^0}{h - 0} = 1$$

# Complex Number

$$i = \sqrt{-1}$$

$$i^2 = -1$$

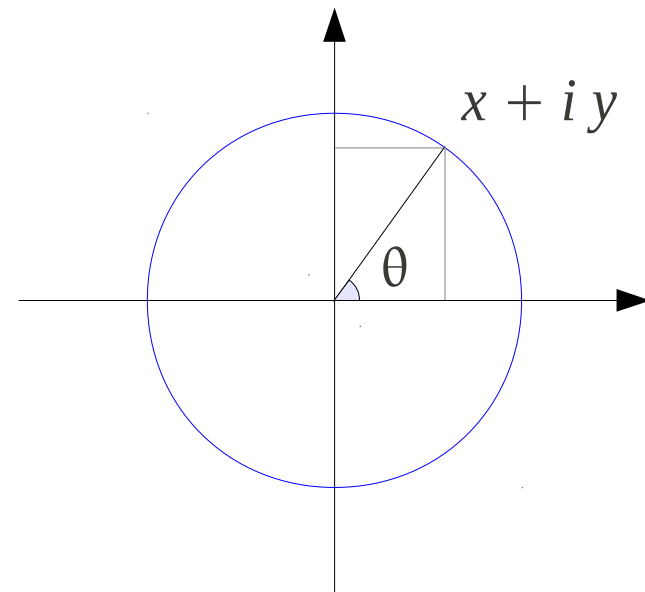
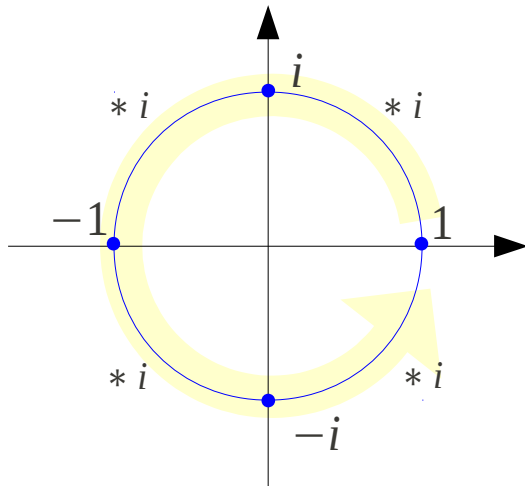
$$i^3 = -i$$

$$i^4 = +1$$

$$\begin{aligned}x + iy &= r \cos \theta + ir \sin \theta \\ &= r(\cos \theta + i \sin \theta)\end{aligned}$$

$$x = r \cos \theta$$

$$y = r \sin \theta$$



# Complex Power Series

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \dots$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots$$

$$e^{i\theta} = 1 + (i\theta) + \frac{(i\theta)^2}{2!} + \frac{(i\theta)^3}{3!} + \frac{(i\theta)^4}{4!} + \frac{(i\theta)^5}{5!} + \dots$$

$$= 1 + i\theta - \frac{\theta^2}{2!} - i\frac{\theta^3}{3!} + \frac{\theta^4}{4!} + i\frac{\theta^5}{5!} + \dots$$

$$= \left( 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} + \dots \right) + i \left( \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} + \dots \right)$$

$$e^{i\theta} = \cos \theta + i \sin \theta$$



# Euler Series (1)

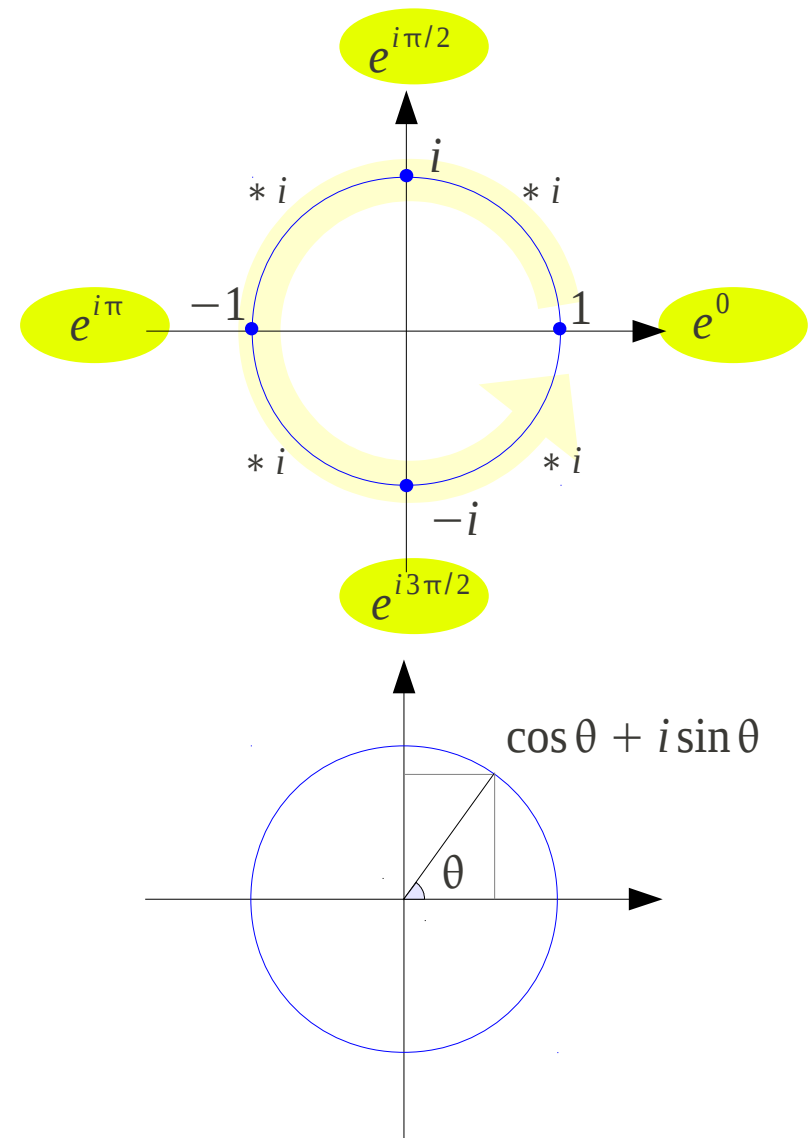
$$e^{i\theta} = \cos \theta + i \sin \theta$$

$$\Re\{e^{i\theta}\} = \cos \theta$$

$$\Im\{e^{i\theta}\} = \sin \theta$$

$$|e^{i\theta}| = \sqrt{\cos^2 \theta + \sin^2 \theta} = 1$$

$$\arg(e^{i\theta}) = \tan^{-1}\left(\frac{\sin \theta}{\cos \theta}\right) = \theta$$



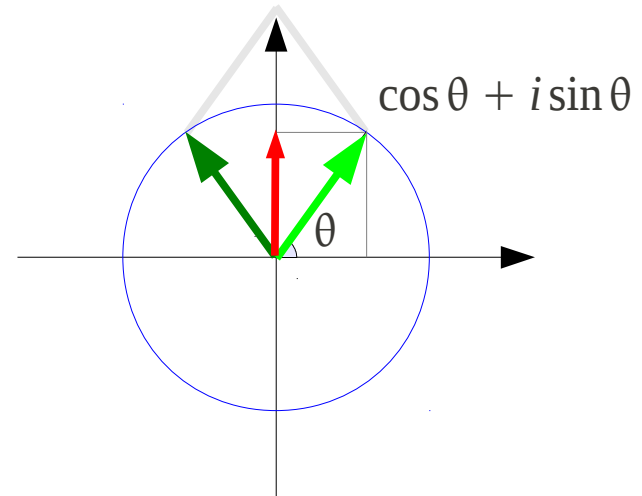
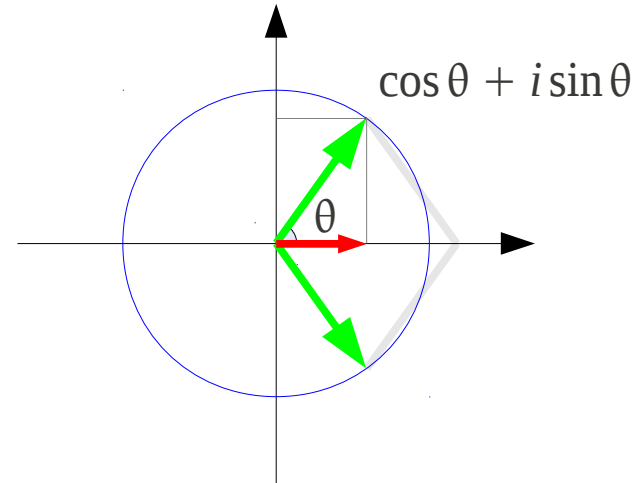
# Euler Series (2)

$$e^{i\theta} = \cos \theta + i \sin \theta$$

$$e^{-i\theta} = \cos \theta - i \sin \theta$$

$$\Re\{e^{i\theta}\} = \cos \theta = \frac{e^{i\theta} + e^{-i\theta}}{2}$$

$$\Im\{e^{i\theta}\} = \sin \theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}$$



# DeMoivre's Theorem

$$e^{i\theta} = \cos \theta + i \sin \theta$$

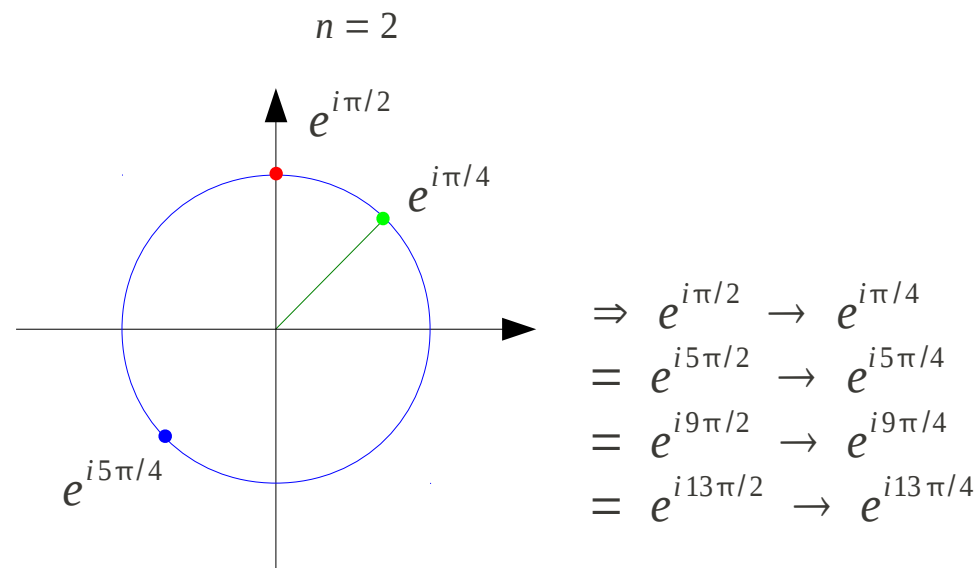
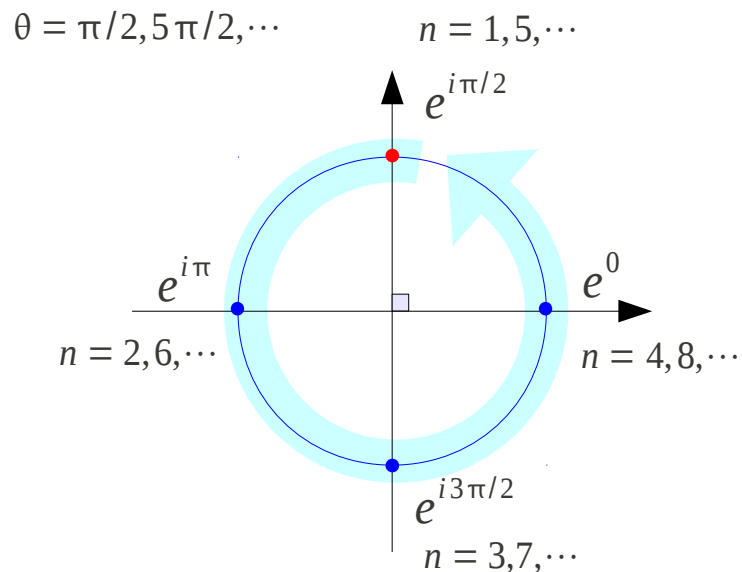
$$(e^{i\theta})^n = (\cos \theta + i \sin \theta)^n$$

$$e^{in\theta} = \cos n\theta + i \sin n\theta$$

$$(e^{i\theta})^{1/n} = (\cos \theta + i \sin \theta)^{1/n}$$

$$(e^{i\theta})^{1/n} \Rightarrow \cos \frac{\theta}{n} + i \sin \frac{\theta}{n}$$

$$= \cos \left( \frac{\theta + 2k\pi}{n} \right) + i \sin \left( \frac{\theta + 2k\pi}{n} \right)$$

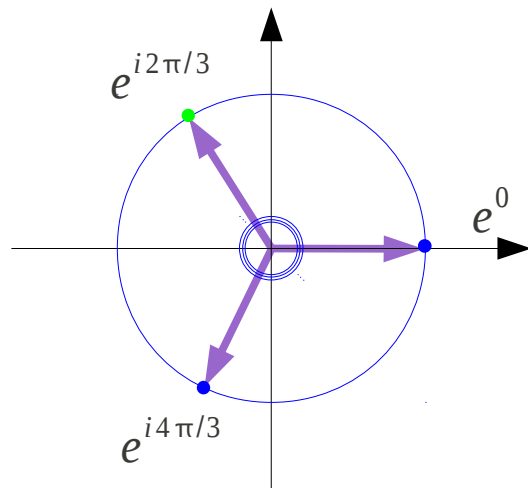


# Complex Roots

$$z = r e^{i\theta} \Rightarrow z^{1/n} = r^{1/n} (e^{i\theta})^{1/n} \Rightarrow \sqrt[n]{r} \left( \cos \frac{\theta}{n} + i \sin \frac{\theta}{n} \right)$$

$$= \sqrt[n]{r} \left( \cos \left( \frac{\theta + 2k\pi}{n} \right) + i \sin \left( \frac{\theta + 2k\pi}{n} \right) \right)$$

$$z^3 = 1$$



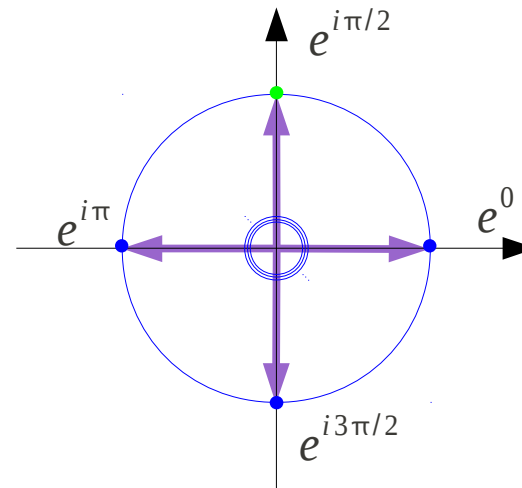
$$n = 3$$

$$\Rightarrow e^{2\pi} \rightarrow e^{2\pi/3}$$

$$= e^{4\pi} \rightarrow e^{4\pi/3}$$

$$= e^{6\pi} \rightarrow e^{6\pi/3}$$

$$z^4 = 1$$



$$n = 4$$

$$\Rightarrow e^{2\pi} \rightarrow e^{2\pi/4}$$

$$= e^{4\pi} \rightarrow e^{4\pi/4}$$

$$= e^{6\pi} \rightarrow e^{6\pi/4}$$

$$= e^{8\pi} \rightarrow e^{8\pi/4}$$

## References

- [1] <http://en.wikipedia.org/>
- [2] <http://planetmath.org/>
- [3] M.L. Boas, “Mathematical Methods in the Physical Sciences”
- [4] D.G. Zill, “Advanced Engineering Mathematics”