

# Line Integrals (4A)

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- Line Integral
- Path Independence

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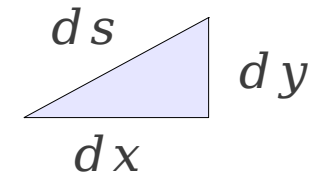
# Line Integral In the Plain

$$x = f(t) \quad \longrightarrow \quad \frac{dx}{dt} = f'(t) \quad \longrightarrow \quad dx = f'(t) dt$$

$$y = g(t) \quad \longrightarrow \quad \frac{dy}{dt} = g'(t) \quad \longrightarrow \quad dy = g'(t) dt$$

$$ds = \sqrt{[f'(t)]^2 + [g'(t)]^2} dt$$

Curve C  $a \leq t \leq b$



$$\int_C G(x, y) dx = \int_a^b G(f(t), g(t)) f'(t) dt$$

$$\int_C G(x, y) dy = \int_a^b G(f(t), g(t)) g'(t) dt$$

$$\int_C G(x, y) ds = \int_a^b G(f(t), g(t)) \sqrt{[f'(t)]^2 + [g'(t)]^2} dt$$

# Line Integral In Space

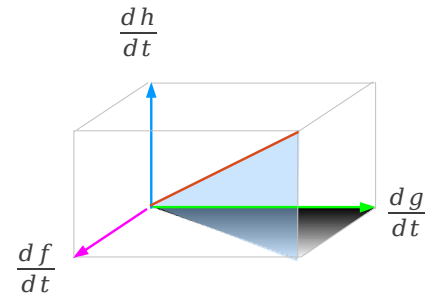
$$x = f(t) \quad \longrightarrow \quad \frac{dx}{dt} = f'(t) \quad \longrightarrow \quad dx = f'(t) dt$$

$$y = g(t) \quad \longrightarrow \quad \frac{dy}{dt} = g'(t) \quad \longrightarrow \quad dy = g'(t) dt$$

$$z = h(t) \quad \longrightarrow \quad \frac{dz}{dt} = h'(t) \quad \longrightarrow \quad dz = h'(t) dt$$

$$\text{Curve } C \quad a \leq t \leq b \quad ds = \sqrt{[f'(t)]^2 + [g'(t)]^2 + [h'(t)]^2} dt$$

$$\int_C G(x, y, z) dz = \int_a^b G(f(t), g(t), h(t)) h'(t) dt$$



$$\int_C G(x, y, z) ds = \int_a^b G(f(t), g(t), h(t)) \sqrt{[f'(t)]^2 + [g'(t)]^2 + [h'(t)]^2} dt$$

# Line Integral using $\mathbf{r}(t)$

Arc Length Parameter

$s$  increases in the direction of increasing  $t$

$$s(t) = \int_{t_0}^t |\mathbf{v}(\tau)| d\tau = \int_{t_0}^t \|\mathbf{r}'(\tau)\| d\tau = \int_{t_0}^t \sqrt{[f'(\tau)]^2 + [g'(\tau)]^2 + [h'(\tau)]^2} d\tau$$

$$ds = |\mathbf{v}(t)| dt$$

$$ds = \sqrt{[f'(t)]^2 + [g'(t)]^2 + [h'(t)]^2} dt$$

$$\begin{aligned} \int_C G(x, y, z) ds &= \int_a^b G(\mathbf{r}(t)) \|\mathbf{r}'(t)\| dt \\ &= \int_a^b G(f(t), g(t), h(t)) |\mathbf{v}(t)| dt \\ &= \int_a^b G(f(t), g(t), h(t)) \sqrt{[f'(t)]^2 + [g'(t)]^2 + [h'(t)]^2} dt \end{aligned}$$

# Line Integral Notation

In many applications

$$\begin{aligned}\int_C G(x, y) ds &= \int_C P(x, y) dx + \int_C Q(x, y) dy \\ &= \int_C P(x, y) dx + Q(x, y) dy \\ &= \int_C P dx + Q dy\end{aligned}$$

$$\int_C G(x, y, z) ds = \int_C P(x, y, z) dx + Q(x, y, z) dy + R(x, y, z) dz$$

# Line Integral over a 2-D Vector Field (1)

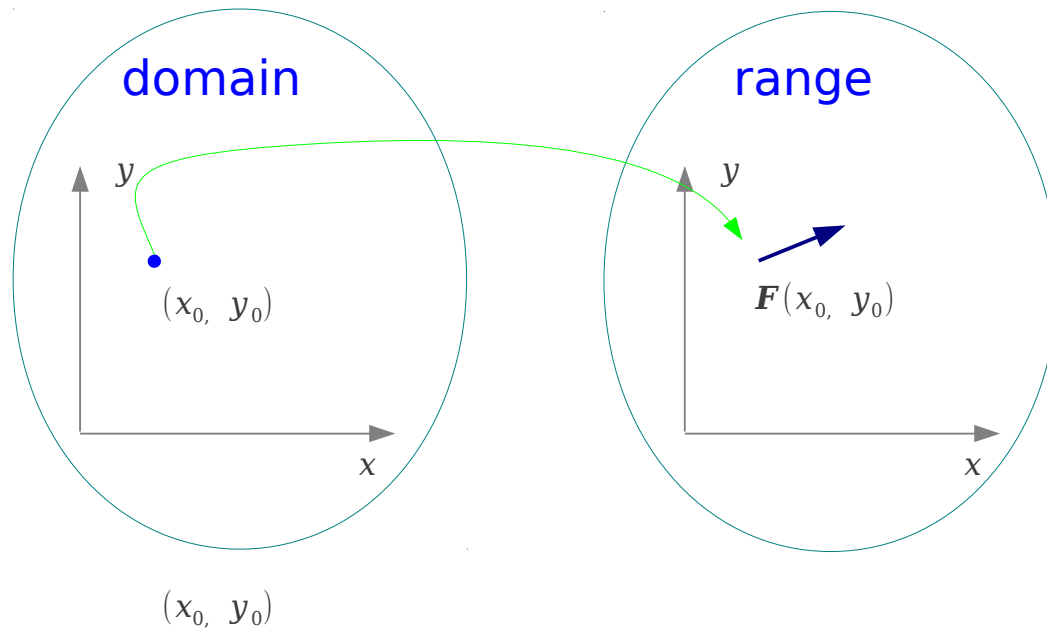
a given point in a 2-d space



A vector

$$(x_0, y_0)$$

$$\langle P(x_0, y_0), Q(x_0, y_0) \rangle$$



2 functions

$$(x_0, y_0) \longrightarrow P(x_0, y_0)$$

$$(x_0, y_0) \longrightarrow Q(x_0, y_0)$$

only points that are  
on the curve

$$\mathbf{r}(t) = f(t)\mathbf{i} + g(t)\mathbf{j} \longrightarrow \mathbf{F}(x_0, y_0) = P(x_0, y_0)\mathbf{i} + Q(x_0, y_0)\mathbf{j}$$

$$x = f(t) \quad y = g(t) \quad a \leq t \leq b$$

# Line Integral over a 3-D Vector Field (2)

$$\mathbf{r}(t) = f(t) \mathbf{i} + g(t) \mathbf{j}$$

$$\frac{d\mathbf{r}}{dt} = f'(t) \mathbf{i} + g'(t) \mathbf{j} = \frac{dx}{dt} \mathbf{i} + \frac{dy}{dt} \mathbf{j}$$

$$\frac{d\mathbf{r}}{dt} dt = \left( \frac{dx}{dt} \mathbf{i} + \frac{dy}{dt} \mathbf{j} \right) dt = dx \mathbf{i} + dy \mathbf{j}$$

$$d\mathbf{r} = dx \mathbf{i} + dy \mathbf{j}$$

$$\mathbf{F}(x, y) = P(x, y) \mathbf{i} + Q(x, y) \mathbf{j}$$

$$\mathbf{F} \cdot d\mathbf{r} = P(x, y) dx + Q(x, y) dy$$

$$\int_c \mathbf{F} \cdot d\mathbf{r} = \int_c P(x, y) dx + Q(x, y) dy$$



# Line Integral over a 3-D Vector Field (1)

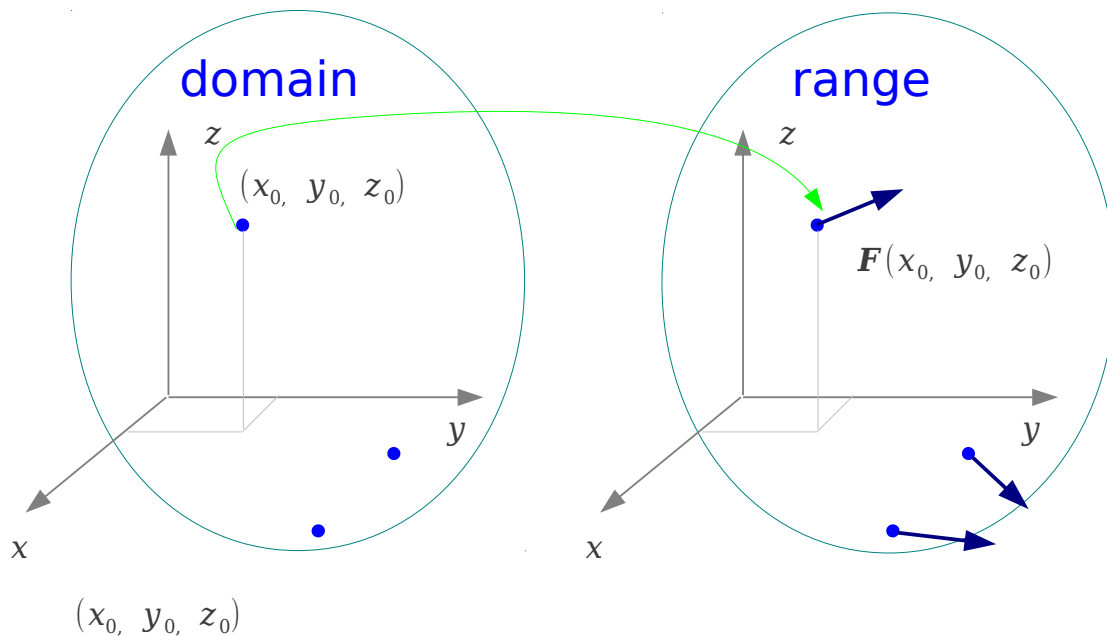
A given point in a 3-d space



A vector

$$(x_0, y_0, z_0)$$

$$\langle P(x_0, y_0, z_0), Q(x_0, y_0, z_0), R(x_0, y_0, z_0) \rangle$$



3 functions

$$(x_0, y_0, z_0) \longrightarrow P(x_0, y_0, z_0)$$

$$(x_0, y_0, z_0) \longrightarrow Q(x_0, y_0, z_0)$$

$$(x_0, y_0, z_0) \longrightarrow R(x_0, y_0, z_0)$$

only points that are  
on the curve

$$\longrightarrow \mathbf{F}(x_0, y_0, z_0) = P(x_0, y_0, z_0)\mathbf{i} + Q(x_0, y_0, z_0)\mathbf{j} + R(x_0, y_0, z_0)\mathbf{k}$$

$$\mathbf{r}(t) = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k}$$

$$x = f(t) \quad y = g(t) \quad z = h(t) \quad a \leq t \leq b$$

# Line Integral over a 3-D Vector Field (2)

$$\mathbf{r}(t) = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k}$$

$$\frac{d\mathbf{r}}{dt} = f'(t)\mathbf{i} + g'(t)\mathbf{j} + h'(t)\mathbf{k} = \frac{dx}{dt}\mathbf{i} + \frac{dy}{dt}\mathbf{j} + \frac{dz}{dt}\mathbf{k}$$

$$\frac{d\mathbf{r}}{dt} dt = \left( \frac{dx}{dt}\mathbf{i} + \frac{dy}{dt}\mathbf{j} + \frac{dz}{dt}\mathbf{k} \right) dt = dx\mathbf{i} + dy\mathbf{j} + dz\mathbf{k}$$

$$d\mathbf{r} = dx\mathbf{i} + dy\mathbf{j} + dz\mathbf{k}$$

$$\mathbf{F}(x, y, z) = P(x, y, z)\mathbf{i} + Q(x, y, z)\mathbf{j} + R(x, y, z)\mathbf{k}$$

$$\mathbf{F} \cdot d\mathbf{r} = P(x, y, z) dx + Q(x, y, z) dy + R(x, y, z) dz$$

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C P(x, y, z) dx + Q(x, y, z) dy + R(x, y, z) dz$$

# Line Integral in Vector Fields

$$\mathbf{r}(t) = f(t) \mathbf{i} + g(t) \mathbf{j}$$

$$d\mathbf{r} = dx \mathbf{i} + dy \mathbf{j}$$

$$\mathbf{F}(x, y, z) = P \mathbf{i} + Q \mathbf{j}$$

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C P dx + Q dy$$

$$\begin{cases} P(x, y) \\ Q(x, y) \end{cases}$$

$$\mathbf{r}(t) = f(t) \mathbf{i} + g(t) \mathbf{j} + h(t) \mathbf{k}$$

$$d\mathbf{r} = dx \mathbf{i} + dy \mathbf{j} + dz \mathbf{k}$$

$$\mathbf{F}(x, y, z) = P \mathbf{i} + Q \mathbf{j} + R \mathbf{k}$$

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C P dx + Q dy + R dz$$

$$\begin{cases} P(x, y, z) \\ Q(x, y, z) \\ R(x, y, z) \end{cases}$$

# Work (1)

$$W = \mathbf{F} \cdot \mathbf{d}$$

A force field  $\mathbf{F}(x, y) = P(x, y)\mathbf{i} + Q(x, y)\mathbf{j}$

A smooth curve  $C: x = f(t), y = g(t), a \leq t \leq b$

Work done by  $\mathbf{F}$  along  $C$  
$$W = \int_C \mathbf{F}(x, y) \cdot d\mathbf{r}$$
$$= \int_C P(x, y) dx + Q(x, y) dy$$

$$\frac{d\mathbf{r}}{dt} = \frac{d\mathbf{r}}{ds} \frac{ds}{dt} \quad d\mathbf{r} = \frac{d\mathbf{r}}{ds} ds \quad d\mathbf{r} = \mathbf{T} ds$$

$$W = \int_C \mathbf{F} \cdot d\mathbf{r} = \int_C \mathbf{F} \cdot \mathbf{T} ds$$

## Work (2)

$$\frac{d\mathbf{r}}{dt} = \frac{d\mathbf{r}}{ds} \frac{ds}{dt} \quad d\mathbf{r} = \frac{d\mathbf{r}}{ds} ds \quad d\mathbf{r} = \mathbf{T} ds$$

$$\begin{aligned} W &= \int_c \mathbf{F} \cdot d\mathbf{r} = \int_c \mathbf{F} \cdot \mathbf{T} ds \\ &= \int_{t_1}^{t_0} \mathbf{F} \cdot \frac{d\mathbf{r}}{dt} dt \\ &= \int_{t_0}^{t_1} \left( P \frac{df}{dt} + Q \frac{dg}{dt} + R \frac{dh}{dt} \right) dt \\ &= \int_{t_0}^{t_1} \left( P \frac{dx}{dt} + Q \frac{dy}{dt} + R \frac{dz}{dt} \right) dt \\ &= \int_{t_0}^{t_1} P dx + Q dy + R dz \end{aligned}$$

$$\begin{aligned} \mathbf{F}(x, y, z) &= P\mathbf{i} + Q\mathbf{j} + R\mathbf{k} \\ &= P(x, y, z)\mathbf{i} \\ &\quad + Q(x, y, z)\mathbf{j} \\ &\quad + R(x, y, z)\mathbf{k} \end{aligned}$$

$$\mathbf{r}(t) = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k}$$

$$x = f(t)$$

$$y = g(t)$$

$$z = h(t)$$


# Circulation


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A Simple Closed Curve  $C \rightarrow$  Circulation

$$\text{circulation} = \oint_C \mathbf{F} \cdot d\mathbf{r} = \oint_C \mathbf{F} \cdot \mathbf{T} ds$$

# Path Independence

$C_1 \neq C_2$    $\int_{C_1} \mathbf{F} \cdot d\mathbf{r} \neq \int_{C_2} \mathbf{F} \cdot d\mathbf{r}$  In general

  $\int_{C_1} \mathbf{F} \cdot d\mathbf{r} = \int_{C_2} \mathbf{F} \cdot d\mathbf{r}$  Exceptional Case

## Path Independence

for a special kind of a vector field  $\mathbf{F}$



## Conservative Vector Field

If we can find a scalar function  $\Phi$

that satisfies  $\mathbf{F} = \nabla \Phi$

$\mathbf{F}$  is a gradient field of a scalar function  $\Phi$

# Conservative Vector Field

$\mathbf{F}$  can be written as the gradient of a scalar function  $\Phi$

$$\mathbf{F} = \nabla \Phi$$



A vector function  $\mathbf{F}$  in 2-d or 3-d space is conservative

$$\mathbf{F}(x, y) = P(x, y)\mathbf{i} + Q(x, y)\mathbf{j}$$

If  $\Phi(x, y)$  satisfies

$$\begin{aligned}\nabla \Phi(x, y) &= \frac{\partial \Phi}{\partial x}\mathbf{i} + \frac{\partial \Phi}{\partial y}\mathbf{j} \\ &= P(x, y)\mathbf{i} + Q(x, y)\mathbf{j}\end{aligned}$$

$$\begin{cases} \frac{\partial}{\partial x}\Phi(x, y) = P(x, y) \\ \frac{\partial}{\partial y}\Phi(x, y) = Q(x, y) \end{cases}$$

$$\mathbf{F}(x, y, z) = P(x, y, z)\mathbf{i} + Q(x, y, z)\mathbf{j} + R(x, y, z)\mathbf{k}$$

If  $\Phi(x, y, z)$  satisfies

$$\begin{aligned}\nabla \Phi(x, y, z) &= \frac{\partial \Phi}{\partial x}\mathbf{i} + \frac{\partial \Phi}{\partial y}\mathbf{j} + \frac{\partial \Phi}{\partial z}\mathbf{k} \\ &= P(x, y, z)\mathbf{i} + Q(x, y, z)\mathbf{j} + R(x, y, z)\mathbf{k}\end{aligned}$$

$$\begin{cases} \frac{\partial}{\partial x}\Phi(x, y, z) = P(x, y, z) \\ \frac{\partial}{\partial y}\Phi(x, y, z) = Q(x, y, z) \\ \frac{\partial}{\partial z}\Phi(x, y, z) = R(x, y, z) \end{cases}$$



# Fundamental Line Integral Theorem (1)

$\mathbf{F}$  can be written as the gradient of a scalar function  $\Phi$   $\mathbf{F} = \nabla \Phi$

➔ A vector function  $\mathbf{F}$  in 2-d or 3-d space is conservative

Conservative vector field  $\mathbf{F}(x, y) = P(x, y) \mathbf{i} + Q(x, y) \mathbf{j}$

➔  $\int_C \nabla \Phi \cdot d\mathbf{r} = \Phi(B) - \Phi(A)$        $A = (f(a), g(a)), B = (f(b), g(b))$

$$\int_a^b f'(x) dx = f(b) - f(a)$$

think as a differentiation

$$\mathbf{F} = \nabla \Phi \quad \Phi^\nabla(x, y)$$

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C \nabla \Phi \cdot d\mathbf{r} = \Phi(B) - \Phi(A)$$

# Fundamental Line Integral Theorem (2)

Conservative vector field  $\mathbf{F}(x, y) = P(x, y)\mathbf{i} + Q(x, y)\mathbf{j}$



$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C \nabla \Phi \cdot d\mathbf{r} = \Phi(B) - \Phi(A)$$

$$\begin{aligned} A &= (f(a), g(a)) \\ B &= (f(b), g(b)) \end{aligned}$$

$$\mathbf{F}(x, y) = \nabla \Phi(x, y) = \frac{\partial \Phi}{\partial x} \mathbf{i} + \frac{\partial \Phi}{\partial y} \mathbf{j}$$

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C \mathbf{F} \cdot \frac{d\mathbf{r}(t)}{dt} dt = \int_C \mathbf{F} \cdot \mathbf{r}'(t) dt$$

$$= \int_C \left( \frac{\partial \Phi}{\partial x} \mathbf{i} + \frac{\partial \Phi}{\partial y} \mathbf{j} \right) \cdot \left( \frac{dx}{dt} \mathbf{i} + \frac{dy}{dt} \mathbf{j} \right) dt = \int_C \left( \frac{\partial \Phi}{\partial x} \frac{dx}{dt} + \frac{\partial \Phi}{\partial y} \frac{dy}{dt} \right) dt$$

$$= \int_C \left( \frac{\partial \Phi}{\partial t} \right) dt$$

$$= [\Phi(x(t), y(t))]_a^b = \Phi(x(b), y(b)) - \Phi(x(a), y(a))$$

$$= \Phi(B) - \Phi(A)$$

# Connected Region (1)

## Connected

Every pair of points A and B in the region can be joined by a piecewise smooth curve that lies entirely in the region

## Simply Connected

Connected and every simple closed curve lying entirely within the region can be shrunk, or contracted, to a point without leaving the region

➡ The interior of the curve lies also entirely in the region

➡ No holes in the region

## Disconnected

Cannot be joined by a piecewise smooth curve that lies entirely in the region

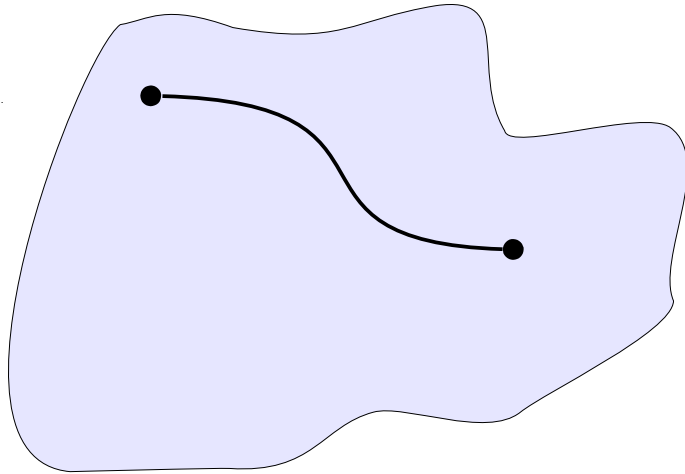
**Multiply Connected**      Many holes within the region

**Open Connected**          Contains no boundary points

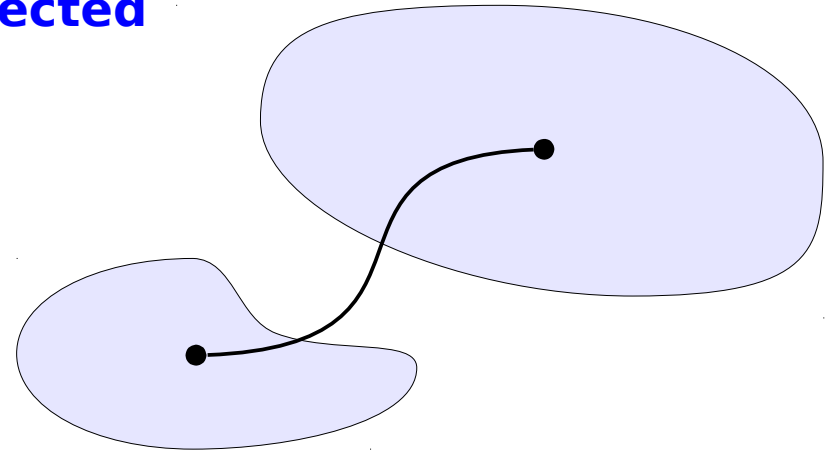
# Connected Region (2)

**Connected**

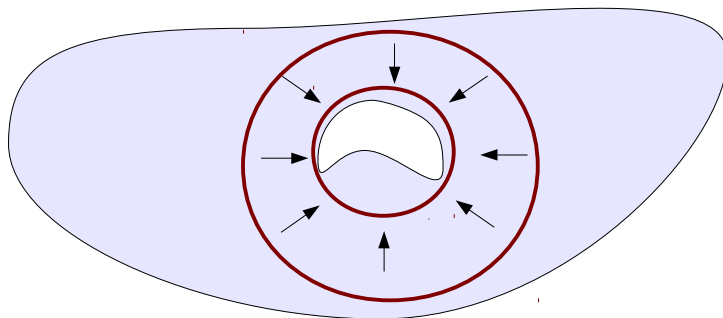
**Simply Connected**



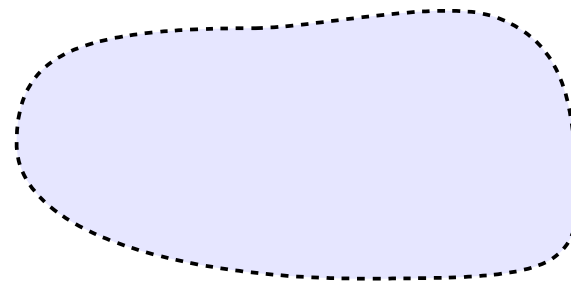
**Disconnected**



**Multiply Connected**



**Open Connected**



# Equivalence

In an open connected region

Path Independence  $\int_C \mathbf{F} \cdot d\mathbf{r}$



Conservative

$$\mathbf{F} = \nabla \Phi$$




Closed path C

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = 0$$

# Test for a Conservative Field

$\mathbf{F}(x, y) = P(x, y) \mathbf{i} + Q(x, y) \mathbf{j}$  : **conservative** vector field  
in an **open** region R

  $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$

$\mathbf{F}(x, y) = P(x, y) \mathbf{i} + Q(x, y) \mathbf{j}$  : **conservative** vector field in R

  $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$  for all points in a **simply connected** region R

$$\mathbf{F} = \nabla \Phi = \frac{\partial \Phi}{\partial x} \mathbf{i} + \frac{\partial \Phi}{\partial y} \mathbf{j} \quad \rightarrow \quad \frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$$
$$\left\{ \begin{array}{l} P = \frac{\partial \Phi}{\partial x} \\ Q = \frac{\partial \Phi}{\partial y} \end{array} \right. \quad \left\{ \begin{array}{l} \frac{\partial P}{\partial y} = \frac{\partial^2 \Phi}{\partial x \partial y} \\ \frac{\partial Q}{\partial x} = \frac{\partial^2 \Phi}{\partial x \partial y} \end{array} \right.$$

# Equivalence in 3-D

In an open connected region

Path Independence  $\int_C \mathbf{F} \cdot d\mathbf{r}$   $\longleftrightarrow$

Conservative  $\mathbf{F} = \nabla \Phi$   $\longleftrightarrow$

Closed path C  $\oint_C \mathbf{F} \cdot d\mathbf{r} = 0$   $\longleftrightarrow$

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x} \quad \frac{\partial P}{\partial z} = \frac{\partial R}{\partial x} \quad \frac{\partial Q}{\partial z} = \frac{\partial R}{\partial y}$$

$$\text{curl } \mathbf{F} = \left( \frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x} \right) \mathbf{i} + \left( \frac{\partial P}{\partial z} - \frac{\partial R}{\partial x} \right) \mathbf{j} + \left( \frac{\partial Q}{\partial z} - \frac{\partial R}{\partial y} \right) \mathbf{k}$$

$$\mathbf{F} = P \mathbf{i} + Q \mathbf{j} + R \mathbf{k} \quad \mathbf{F} = \nabla \Phi = \frac{\partial \Phi}{\partial x} \mathbf{i} + \frac{\partial \Phi}{\partial y} \mathbf{j} + \frac{\partial \Phi}{\partial z} \mathbf{k}$$

# 2-Divergence

Flux across rectangle boundary

$$\approx \left( \frac{\partial M}{\partial x} \Delta x \right) \Delta y + \left( \frac{\partial N}{\partial y} \Delta y \right) \Delta x = \left( \frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} \right) \Delta x \Delta y$$

Flux density

$$= \left( \frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} \right)$$

Divergence of  $\mathbf{F}$

Flux Density



## References

- [1] <http://en.wikipedia.org/>
- [2] <http://planetmath.org/>
- [3] M.L. Boas, “Mathematical Methods in the Physical Sciences”
- [4] D.G. Zill, “Advanced Engineering Mathematics”