LMS Background (1A)

- Linear Regression
- Polynomial Regression
- Multiple Regression
- General Multiple Regression
- Least Squares
- Linear Least Squares

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7/12/11

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Regression

Linear Regression

$$S_r = \sum_{i=1}^{n} \epsilon_i^2 = \sum_{i=1}^{n} \left(y_i - \left(a_0 + a_1 x_i\right)\right)^2$$

Polynomial Regression

$$S_r = \sum_{i=1}^{n} \epsilon_i^2 = \sum_{i=1}^{n} \left(y_i - \left(a_0 + a_1 x_i + a_2 x_i^2\right)\right)^2$$

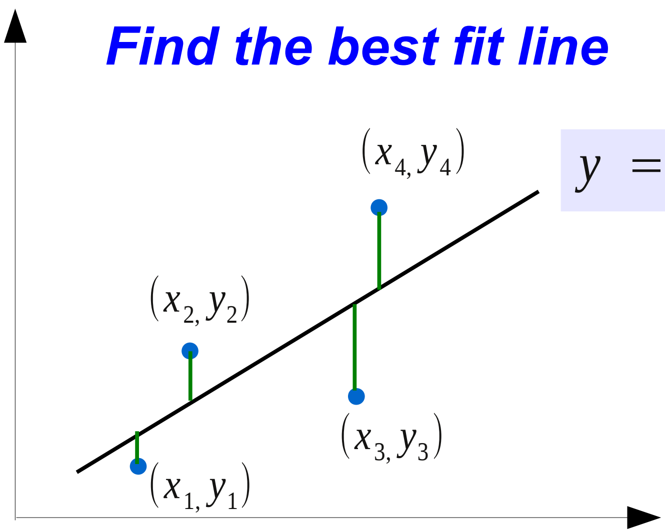
Multiple Linear Regression

$$S_r = \sum_{i=1}^{n} \epsilon_i^2 = \sum_{i=1}^{n} \left(y_i - \left(a_0 + a_1 x_{i,\mathbf{1}} + a_2 x_{i,\mathbf{2}}\right)\right)^2$$

General Multiple Linear Regression

$$S_r = \sum_{i=1}^{n} \epsilon_i^2 = \sum_{i=1}^{n} \left(y_i - \left(\beta_{\mathbf{0}} + \sum_{\mathbf{j}=1}^{m} x_{i\mathbf{j}} \beta_{\mathbf{j}}\right)\right)^2$$

Linear Regression (1)



$$y = a_0 + a_1 x$$

$a_0,\ a_1$ unknowns

$(x_i,\ y_i)$ measured data

random

$$S_r = \sum_{i=1}^{n} \epsilon_i^2 = \sum_{i=1}^{n} \left(y_i - (a_0 + a_1 x_i)\right)^2$$

Linear Regression (2)

Sum of the square of the residuals

$$S_r = \sum_{i=1}^{n} \epsilon_i^2 = \sum_{i=1}^{n} \left(y_i - (a_0 + a_1 x_i)\right)^2$$

$a_{0,}\ a_1$ unknowns

$(x_i,\ y_i)$ measured data

random

Minimum Condition

$$\frac{\partial S_r}{\partial a_0} = 2\sum_{i=1}^{n} (y_i - a_0 - a_1 x_i)(-1) = 0 \quad \Rightarrow \quad \sum_{i=1}^{n} a_0 + \sum_{i=1}^{n} a_1 x_i = \sum_{i=1}^{n} y_i$$

$$\frac{\partial S_r}{\partial a_1} = 2\sum_{i=1}^{n} (y_i - a_0 - a_1 x_i)(-x_i) = 0 \quad \Rightarrow \quad \sum_{i=1}^{n} a_0 x_i + \sum_{i=1}^{n} a_1 x_i^2 = \sum_{i=1}^{n} y_i x_i$$

$$\begin{bmatrix} \sum_{i=1}^{n} 1 & \sum_{i=1}^{n} x_i \\ \sum_{i=1}^{n} x_i & \sum_{i=1}^{n} x_i^2 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \end{bmatrix} = \begin{bmatrix} \sum_{i=1}^{n} y_i \\ \sum_{i=1}^{n} x_i y_i \end{bmatrix}$$

Linear Regression (3)

$$\sum_{i=1}^{n} a_0 + \sum_{i=1}^{n} a_1 x_i = \sum_{i=1}^{n} y_i$$

$$n \cdot a_0 + \sum_{i=1}^{n} a_1 x_i = \sum_{i=1}^{n} y_i$$

$$a_0 = \frac{1}{n}\sum_{i=1}^{n} y_i - \frac{1}{n}\sum_{i=1}^{n} a_1 x_i$$

$$\sum_{i=1}^{n} a_0 x_i + \sum_{i=1}^{n} a_1 x_i^2 = \sum_{i=1}^{n} y_i x_i$$

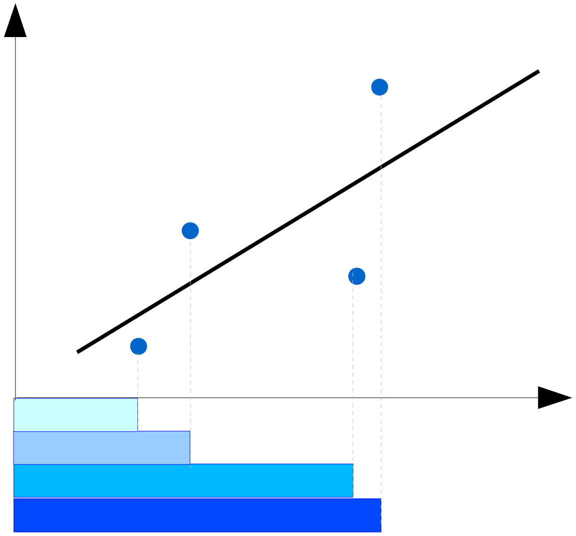
$$\left(\sum_{i=1}^{n} x_i\right)\left(\frac{1}{n}\sum_{i=1}^{n} y_i - \frac{1}{n}\sum_{i=1}^{n} a_1 x_i\right) + \sum_{i=1}^{n} a_1 x_i^2 = \sum_{i=1}^{n} y_i x_i$$

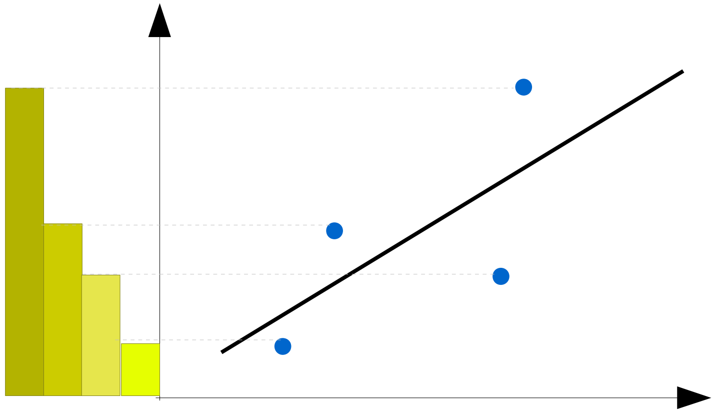
$$\frac{1}{n}\left(\sum_{i=1}^{n} x_i\right)\left(\sum_{i=1}^{n} y_i\right) - \frac{1}{n}\left(\sum_{i=1}^{n} x_i\right)^2 a_1 + \left(\sum_{i=1}^{n} x_i^2\right) a_1 = \left(\sum_{i=1}^{n} y_i x_i\right)$$

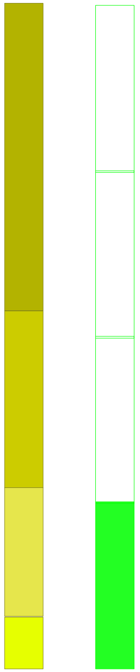
$$n\left(\sum_{i=1}^{n} x_i^2\right) a_1 - \left(\sum_{i=1}^{n} x_i\right)^2 a_1 = n\left(\sum_{i=1}^{n} y_i x_i\right) - \left(\sum_{i=1}^{n} x_i\right)\left(\sum_{i=1}^{n} y_i\right)$$

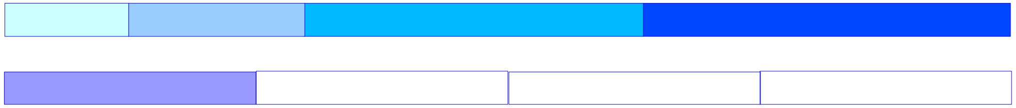
$$a_1 = \frac{n\left(\sum_{i=1}^{n} y_i x_i\right) - \left(\sum_{i=1}^{n} x_i\right)\left(\sum_{i=1}^{n} y_i\right)}{n\left(\sum_{i=1}^{n} x_i^2\right) - \left(\sum_{i=1}^{n} x_i\right)^2}$$

Mean Values of x_i, y_i



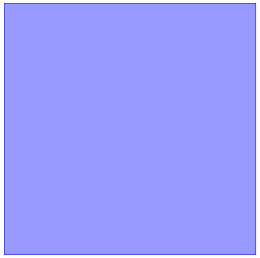






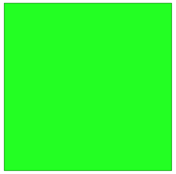
$$\frac{1}{n}\sum_{i=1}^{n} x_i$$

$$\frac{1}{n}\sum_{i=1}^{n} y_i$$

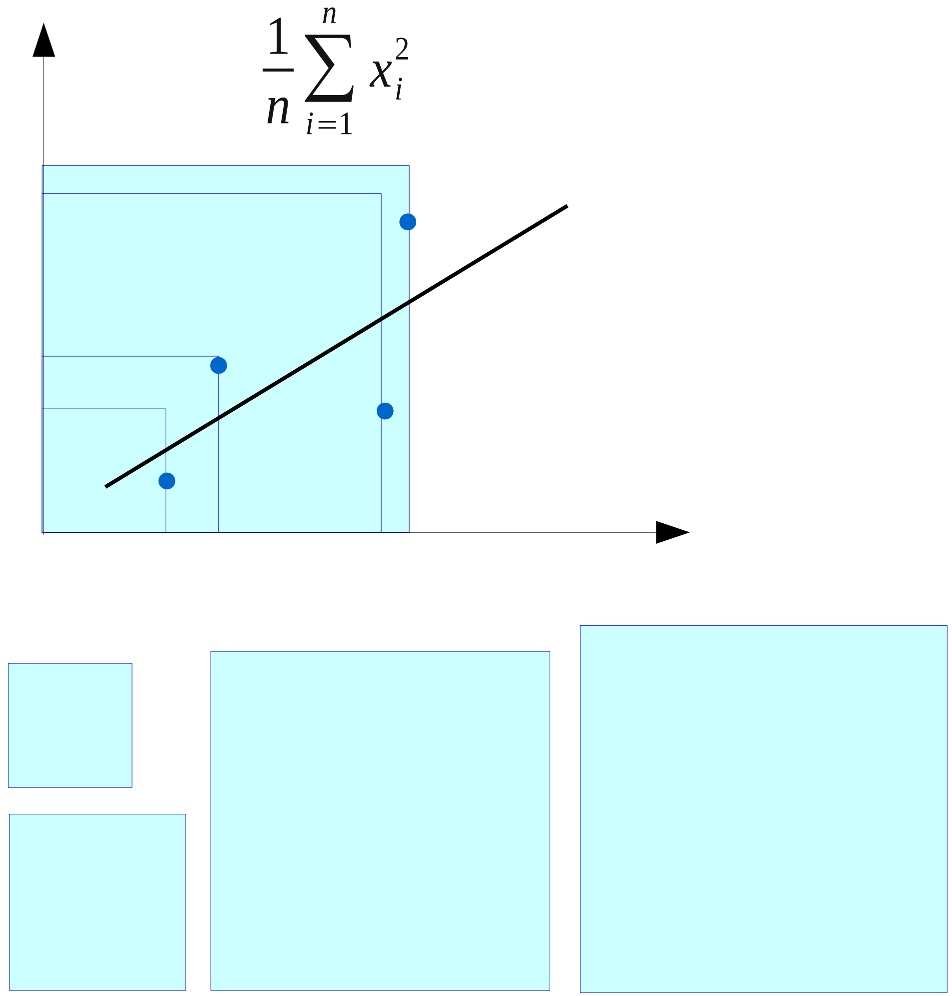


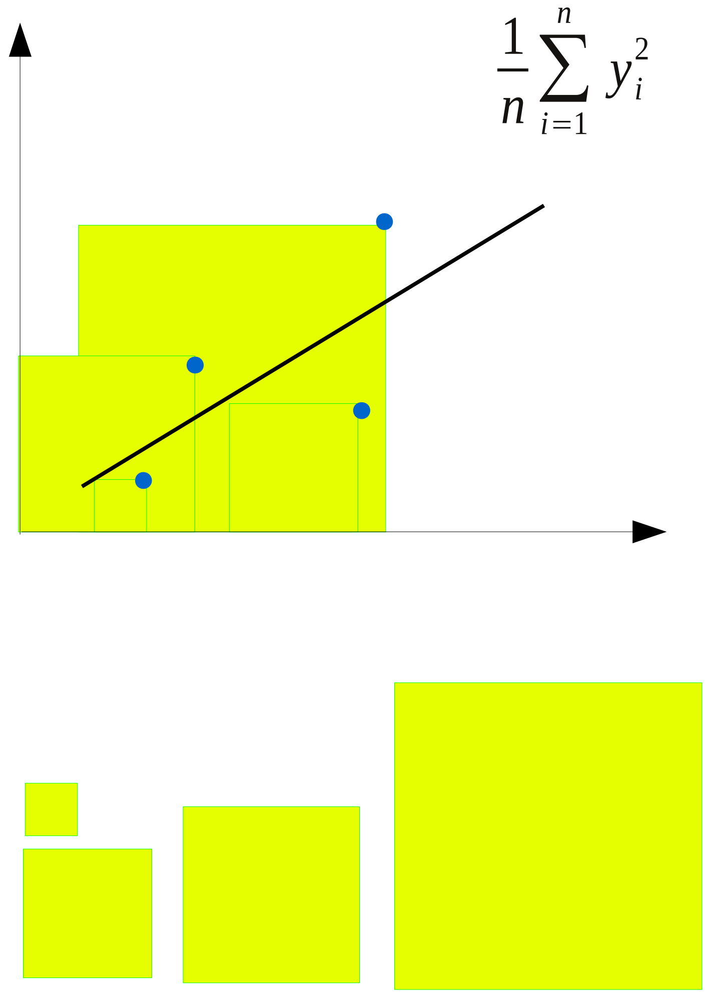
$$\left(\frac{1}{n}\sum_{i=1}^{n} x_i\right)^2$$

$$\left(\frac{1}{n}\sum_{i=1}^{n} y_i\right)^2$$

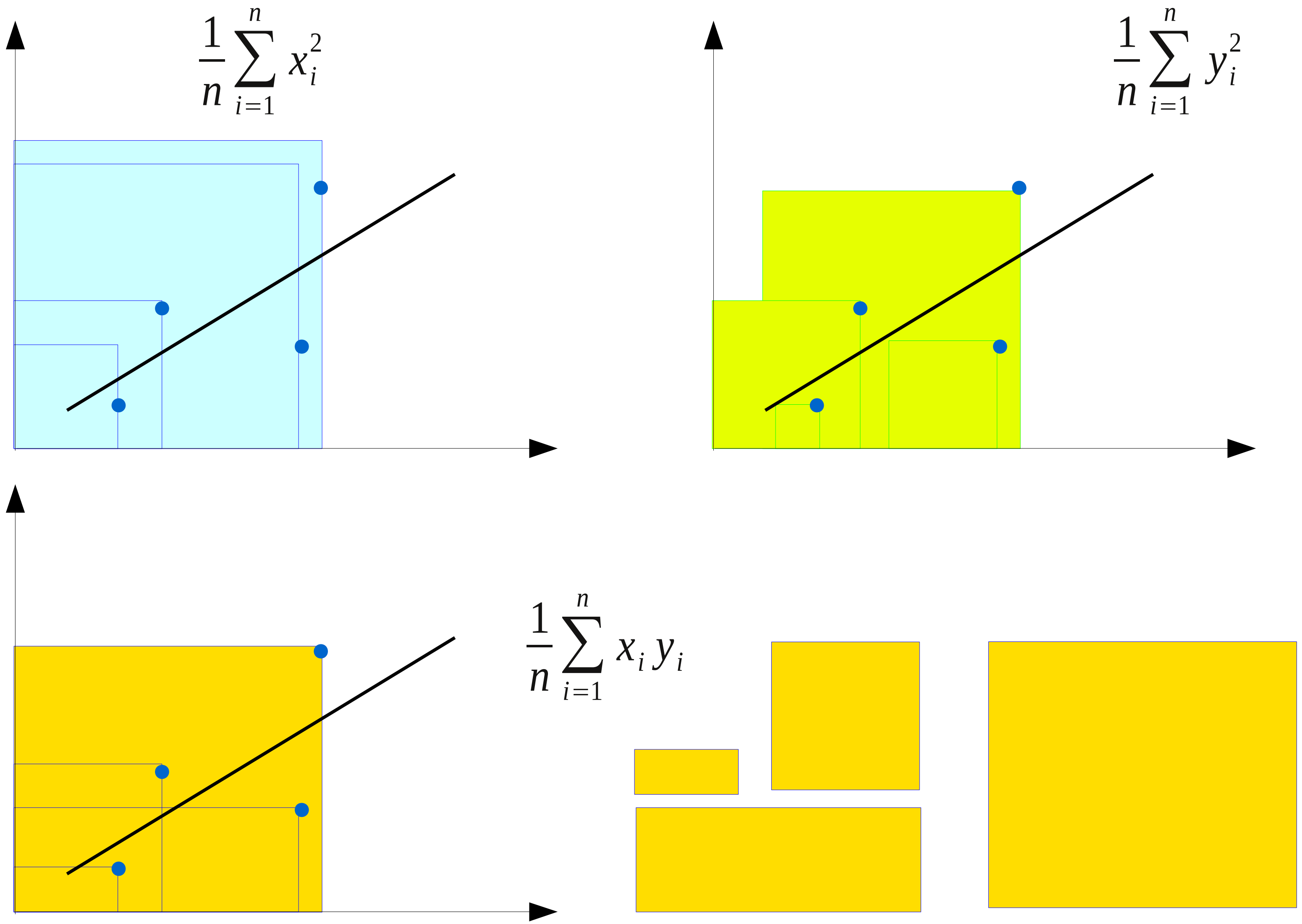


Mean Values of x_i^2, y_i^2, $x_i y_i$ (1)



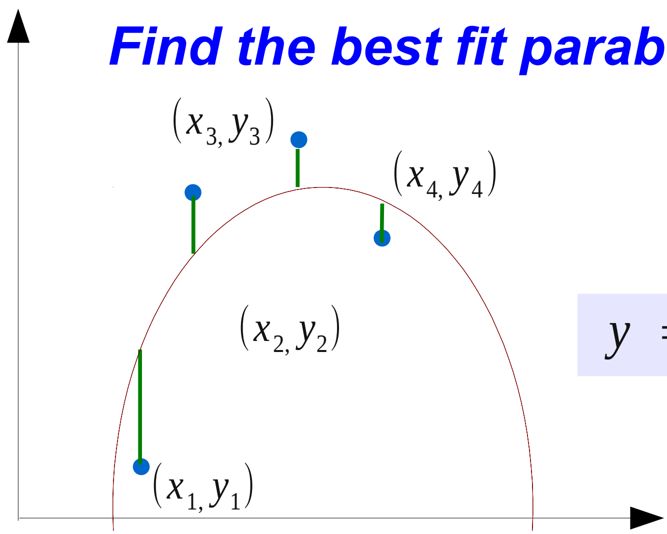


Mean Values of x_i^2, y_i^2, $x_i y_i$ (2)



Polynomial Regression (1)

Find the best fit parabola



$$y = a_0 + a_1 x + a_2 x$$

$a_0,\ a_1,\ a_2$ *unknowns*

$(x_i,\ y_i)$ *measured data*

random

$$S_r = \sum_{i=1}^{n} \epsilon_i^2 = \sum_{i=1}^{n} \left(y_i - \left(a_0 + a_1 x_i + a_2 x_i^2\right)\right)^2$$

Polynomial Regression (2)

Sum of the square of the residuals

$$S_r = \sum_{i=1}^{n} \epsilon_i^2 = \sum_{i=1}^{n} \left(y_i - \left(a_0 + a_1 x_i + a_2 x_i^2\right)\right)^2$$

$a_{0,}\ \ a_{1,}\ \ a_2$ unknowns

$(x_i,\ y_i)$ measured data

random

Minimum Condition

$$\frac{\partial S_r}{\partial a_0} = 2\sum_{i=1}^{n} \left(y_i - a_0 - a_1 x_i - a_2 x_i^2\right)(-1) = 0$$

$$\frac{\partial S_r}{\partial a_1} = 2\sum_{i=1}^{n} \left(y_i - a_0 - a_1 x_i - a_2 x_i^2\right)(-x_i) = 0$$

$$\frac{\partial S_r}{\partial a_2} = 2\sum_{i=1}^{n} \left(y_i - a_0 - a_1 x_i - a_2 x_i^2\right)(-x_i^2) = 0$$

Find the best fit parabola

Polynomial Regression (3)

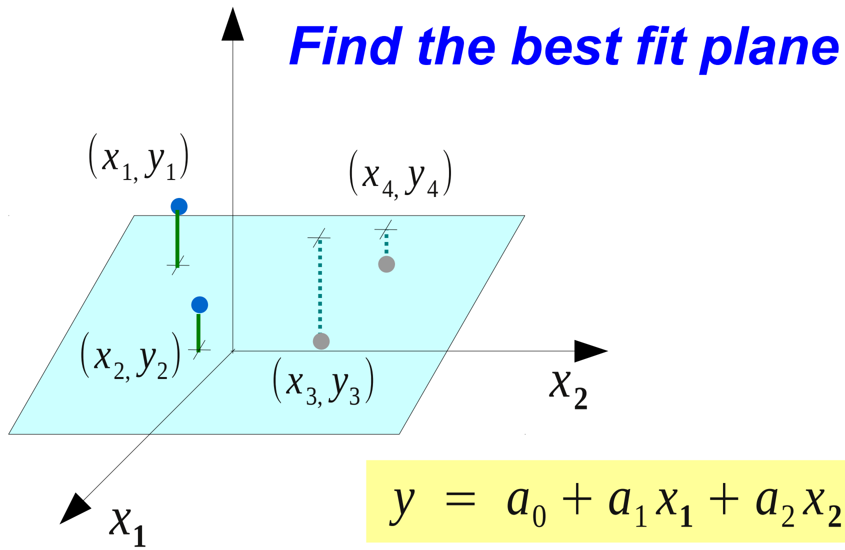
$$\left(\sum_{i=1}^{n} 1\right) \cdot a_0 + \left(\sum_{i=1}^{n} x_i\right) \cdot a_1 + \left(\sum_{i=1}^{n} x_i^2\right) \cdot a_2 = \left(\sum_{i=1}^{n} y_i\right)$$

$$\left(\sum_{i=1}^{n} x_i\right) \cdot a_0 + \left(\sum_{i=1}^{n} x_i^2\right) \cdot a_1 + \left(\sum_{i=1}^{n} x_i^3\right) \cdot a_2 = \left(\sum_{i=1}^{n} x_i y_i\right)$$

$$\left(\sum_{i=1}^{n} x_i^2\right) \cdot a_0 + \left(\sum_{i=1}^{n} x_i^3\right) \cdot a_1 + \left(\sum_{i=1}^{n} x_i^4\right) \cdot a_2 = \left(\sum_{i=1}^{n} x_i^2 y_i\right)$$

$$\begin{pmatrix} \left(\sum_{i=1}^{n} 1\right) & \left(\sum_{i=1}^{n} x_i\right) & \left(\sum_{i=1}^{n} x_i^2\right) \\ \left(\sum_{i=1}^{n} x_i\right) & \left(\sum_{i=1}^{n} x_i^2\right) & \left(\sum_{i=1}^{n} x_i^3\right) \\ \left(\sum_{i=1}^{n} x_i^2\right) & \left(\sum_{i=1}^{n} x_i^3\right) & \left(\sum_{i=1}^{n} x_i^4\right) \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} \left(\sum_{i=1}^{n} y_i\right) \\ \left(\sum_{i=1}^{n} x_i y_i\right) \\ \left(\sum_{i=1}^{n} x_i^2 y_i\right) \end{pmatrix}$$

Multiple Linear Regression (1)



$$y = a_0 + a_1 x_1 + a_2 x_2$$

$a_0, \; a_1, \; a_2$ *unknowns*

$(x_{i,1}, \; x_{i,2}, \; y_i)$ *measured data*

random

$$S_r = \sum_{i=1}^{n} \epsilon_i^2 = \sum_{i=1}^{n} \left(y_i - (a_0 + a_1 x_{i,1} + a_2 x_{i,2})\right)^2$$

Multiple Linear Regression (2)

Sum of the square of the residuals

$$S_r = \sum_{i=1}^{n} \epsilon_i^2 = \sum_{i=1}^{n} \left(y_i - \left(a_0 + a_1 x_{i,\mathbf{1}} + a_2 x_{i,\mathbf{2}}\right)\right)^2$$

$a_{0,}\ \ a_{1,}\ \ a_2$ *unknowns*

$(x_{i,\mathbf{1}},\ \ x_{i,\mathbf{2}},\ \ y_i)$ *measured data*

random

Minimum Condition

$$\frac{\partial S_r}{\partial a_0} = 2\sum_{i=1}^{n} (y_i - a_0 - a_1 x_{i,\mathbf{1}} - a_2 x_{i,\mathbf{2}})(-1) = 0$$

$$\frac{\partial S_r}{\partial a_1} = 2\sum_{i=1}^{n} (y_i - a_0 - a_1 x_{i,\mathbf{1}} - a_2 x_{i,\mathbf{2}})(-x_{i,\mathbf{1}}) = 0$$

$$\frac{\partial S_r}{\partial a_2} = 2\sum_{i=1}^{n} (y_i - a_0 - a_1 x_{i,\mathbf{1}} - a_2 x_{i,\mathbf{2}})(-x_{i,\mathbf{2}}) = 0$$

Multiple Linear Regression (3)

$$\left(\sum_{i=1}^{n} 1\right)\cdot a_0 + \left(\sum_{i=1}^{n} x_{i,\mathbf{1}}\right)\cdot a_1 + \left(\sum_{i=1}^{n} x_{i,\mathbf{2}}\right)\cdot a_2 = \left(\sum_{i=1}^{n} y_i\right)$$

$$\left(\sum_{i=1}^{n} x_{i,\mathbf{1}}\right)\cdot a_0 + \left(\sum_{i=1}^{n} x_{i,\mathbf{1}}^2\right)\cdot a_1 + \left(\sum_{i=1}^{n} x_{i,\mathbf{1}} x_{i,\mathbf{2}}\right)\cdot a_2 = \left(\sum_{i=1}^{n} x_{i,\mathbf{1}} y_i\right)$$

$$\left(\sum_{i=1}^{n} x_{i,\mathbf{2}}\right)\cdot a_0 + \left(\sum_{i=1}^{n} x_{i,\mathbf{1}} x_{i,\mathbf{2}}\right)\cdot a_1 + \left(\sum_{i=1}^{n} x_{i,\mathbf{2}}^2\right)\cdot a_2 = \left(\sum_{i=1}^{n} x_{i,\mathbf{2}} y_i\right)$$

$$\begin{pmatrix} \left(\sum_{i=1}^{n} 1\right) & \left(\sum_{i=1}^{n} x_{i\mathbf{1}}\right) & \left(\sum_{i=1}^{n} x_{i\mathbf{2}}\right) \\ \left(\sum_{i=1}^{n} x_{i\mathbf{1}}\right) & \left(\sum_{i=1}^{n} x_{i\mathbf{1}}^2\right) & \left(\sum_{i=1}^{n} x_{i\mathbf{1}} x_{i\mathbf{2}}\right) \\ \left(\sum_{i=1}^{n} x_{i\mathbf{2}}\right) & \left(\sum_{i=1}^{n} x_{i\mathbf{1}} x_{i\mathbf{2}}\right) & \left(\sum_{i=1}^{n} x_{i\mathbf{2}}^2\right) \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} \left(\sum_{i=1}^{n} y_i\right) \\ \left(\sum_{i=1}^{n} x_{i\mathbf{1}} y_i\right) \\ \left(\sum_{i=1}^{n} x_{i\mathbf{2}} y_i\right) \end{pmatrix}$$

Multiple Linear Regression – General (1)

Sum of the square of the residuals

$$S_r = \sum_{i=1}^{n} \epsilon_i^2 = \sum_{i=1}^{n} \left(y_i - \left(\beta_0 + \sum_{j=1}^{m} x_{ij}\beta_j \right) \right)^2$$

$\beta_0, \ \beta_1, \cdots, \ \beta_m$ *unknowns*

$(x_{i1}, \ x_{i2}, \cdots, x_{im}, \ \ y_i)$ *measured data*

random

$$y = \beta_0 + \sum_{j=1}^{m} x_j \beta_j = \beta_0 + x_1\beta_1 + x_2\beta_2 + \cdots + x_m\beta_m$$

Minimum Condition

$$\frac{\partial S_r}{\partial \beta_0} = 2\sum_{i=1}^{n} (y_i - \beta_0 - \beta_1 x_{i1} - \beta_2 x_{i2} - \cdots - \beta_m x_{im})(-1) = 0$$

$$\frac{\partial S_r}{\partial \beta_1} = 2\sum_{i=1}^{n} (y_i - \beta_0 - \beta_1 x_{i1} - \beta_2 x_{i2} - \cdots - \beta_m x_{im})(-x_{i1}) = 0$$

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$$\frac{\partial S_r}{\partial \beta_m} = 2\sum_{i=1}^{n} (y_i - \beta_0 - \beta_1 x_{i1} - \beta_2 x_{i2} - \cdots - \beta_m x_{im})(-x_{im}) = 0$$

Multiple Linear Regression – General (2)

$$
\begin{pmatrix}
\left(\sum_{i=1}^{n} 1\right) & \left(\sum_{i=1}^{n} x_{i1}\right) & \left(\sum_{i=1}^{n} x_{i2}\right) & \cdots & \left(\sum_{i=1}^{n} x_{im}\right) \\
\left(\sum_{i=1}^{n} x_{i1}\right) & \left(\sum_{i=1}^{n} x_{i1}^{2}\right) & \left(\sum_{i=1}^{n} x_{i1} x_{i2}\right) & \cdots & \left(\sum_{i=1}^{n} x_{i1} x_{im}\right) \\
\left(\sum_{i=1}^{n} x_{i2}\right) & \left(\sum_{i=1}^{n} x_{i2} x_{i1}\right) & \left(\sum_{i=1}^{n} x_{i2}^{2}\right) & \cdots & \left(\sum_{i=1}^{n} x_{i1} x_{im}\right) \\
\vdots & \vdots & \vdots & & \vdots \\
\left(\sum_{i=1}^{n} x_{im}\right) & \left(\sum_{i=1}^{n} x_{im} x_{i1}\right) & \left(\sum_{i=1}^{n} x_{im} x_{i2}\right) & \cdots & \left(\sum_{i=1}^{n} x_{im}^{2}\right)
\end{pmatrix}
\begin{pmatrix}
\beta_0 \\ \beta_1 \\ \beta_2 \\ \vdots \\ \beta_m
\end{pmatrix}
=
\begin{pmatrix}
\left(\sum_{i=1}^{n} y_i\right) \\
\left(\sum_{i=1}^{n} x_{i1} y_i\right) \\
\left(\sum_{i=1}^{n} x_{i2} y_i\right) \\
\vdots \\
\left(\sum_{i=1}^{n} x_{im} y_i\right)
\end{pmatrix}
$$

Multiple Linear Regression – General (3)

$m = 1$ *measured data*

$$\begin{matrix} 1 & x_{11} & x_{12} & \cdots & x_{1m} \\ x_{11} & x_{11}^2 & x_{11}x_{12} & \cdots & x_{11}x_{1m} \\ x_{12} & x_{12}x_{11} & x_{12}^2 & \cdots & x_{11}x_{1m} \\ \vdots & \vdots & \vdots & & \vdots \\ x_{1m} & x_{1m}x_{11} & x_{1m}x_{12} & \cdots & x_{1m}^2 \end{matrix}$$

$m = 2$ *measured data*

$$\begin{matrix} 1 & x_{21} & x_{22} & \cdots & x_{2m} \\ x_{21} & x_{21}^2 & x_{21}x_{22} & \cdots & x_{21}x_{2m} \\ x_{22} & x_{22}x_{21} & x_{22}^2 & \cdots & x_{21}x_{2m} \\ \vdots & \vdots & \vdots & & \vdots \\ x_{2m} & x_{2m}x_{21} & x_{2m}x_{22} & \cdots & x_{2m}^2 \end{matrix}$$

$m = 3$ *measured data*

$$\begin{matrix} 1 & x_{31} & x_{32} & \cdots & x_{3m} \\ x_{31} & x_{31}^2 & x_{31}x_{32} & \cdots & x_{31}x_{3m} \\ x_{32} & x_{32}x_{31} & x_{32}^2 & \cdots & x_{31}x_{3m} \\ \vdots & \vdots & \vdots & & \vdots \\ x_{3m} & x_{3m}x_{31} & x_{3m}x_{32} & \cdots & x_{3m}^2 \end{matrix}$$

$m = 4$ *measured data*

$$\begin{matrix} 1 & x_{41} & x_{42} & \cdots & x_{4m} \\ x_{41} & x_{41}^2 & x_{41}x_{42} & \cdots & x_{41}x_{4m} \\ x_{42} & x_{42}x_{41} & x_{42}^2 & \cdots & x_{41}x_{4m} \\ \vdots & \vdots & \vdots & & \vdots \\ x_{4m} & x_{4m}x_{41} & x_{4m}x_{42} & \cdots & x_{4m}^2 \end{matrix}$$

Multiple Linear Regression – General (4)

$E\{1\}$	$E\{x_1\}$	$E\{x_2\}$	$E\{x_m\}$
1	$\overline{x_1}$	$\overline{x_2}$	$\overline{x_m}$
$\frac{1}{n}\sum_{i=1}^{n} 1$	$\frac{1}{n}\sum_{i=1}^{n} x_{i1}$	$\frac{1}{n}\sum_{i=1}^{n} x_{i2}$	$\frac{1}{n}\sum_{i=1}^{n} x_{im}$

$$
\begin{matrix}
1 & \overline{x_1} & \overline{x_2} & \cdots & \overline{x_m} \\
\overline{x_1} & \overline{x_1^2} & \overline{x_1 x_2} & \cdots & \overline{x_1 x_m} \\
\overline{x_2} & \overline{x_2 x_1} & \overline{x_2^2} & \cdots & \overline{x_1 x_m} \\
\vdots & \vdots & \vdots & & \vdots \\
\overline{x_m} & \overline{x_m x_1} & \overline{x_m x_2} & \cdots & \overline{x_m^2}
\end{matrix}
$$

$$
\begin{matrix}
1 & x_{11} & x_{12} & \cdots & x_{1m} \\
x_{11} & x_{11}^2 & x_{11} x_{12} & \cdots & x_{11} x_{1m} \\
x_{12} & x_{12} x_{11} & x_{12}^2 & \cdots & x_{11} x_{1m} \\
\vdots & \vdots & \vdots & & \vdots \\
x_{1m} & x_{1m} x_{11} & x_{1m} x_{12} & \cdots & x_{1m}^2
\end{matrix}
$$

$m = 4$ *measured data*

$m = 3$ *measured data*

$m = 2$ *measured data*

$m = 1$ *measured data*

Least Square (1)

Sum of the square of the residuals

$$S_r = \sum_{i=1}^{n} \epsilon_i^2 = \sum_{i=1}^{n} \left(y_i - f(x_i, \boldsymbol{\beta}) \right)^2$$

$$\epsilon_i = \left(y_i - f(x_i, \boldsymbol{\beta}) \right)$$

$\beta_1,\ \beta_2, \cdots,\ \beta_m$ *unknowns*

$(x_{i1},\ x_{i2}, \cdots, x_{im},\ y_i)$ *measured data*

random

Minimum Condition

$$\frac{\partial S_r}{\partial \beta_j} = 2 \sum_{i=1}^{n} \epsilon_i \frac{\partial \epsilon_i}{\partial \beta_j} = 0 \qquad j = 1, \ldots, m$$

$$\frac{\partial S_r}{\partial \beta_j} = -2 \sum_{i=1}^{n} \epsilon_i \frac{\partial f(x_i, \boldsymbol{\beta})}{\partial \beta_j} = 0 \qquad j = 1, \ldots, m$$

$$\frac{\partial \epsilon_i}{\partial \beta_j} = -\frac{\partial f(x_i, \boldsymbol{\beta})}{\partial \beta_j}$$

$$\epsilon_i = \left(y_i - f(x_i, \boldsymbol{\beta}) \right)$$

Least Square (2)

Sum of the square of the residuals

$$S_r = \sum_{i=1}^{n} \epsilon_i^2 = \sum_{i=1}^{n} \left(y_i - f(x_i, \boldsymbol{\beta}) \right)^2$$

$\beta_1, \beta_2, \cdots, \beta_m$ *unknowns*

$(x_{i1}, x_{i2}, \cdots, x_{im}, y_i)$ *measured data*

random

Minimum Condition

$$\frac{\partial S_r}{\partial \beta_j} = -2 \sum_{i=1}^{n} \epsilon_i \frac{\partial f(x_i, \boldsymbol{\beta})}{\partial \beta_j} = 0 \qquad j = 1, \ldots, m$$

Linear Least Square

$$y = \sum_{j=1}^{m} x_j \beta_j = x_1\beta_1 + x_2\beta_2 + \cdots + x_m\beta_m$$

$$f(x_i, \boldsymbol{\beta}) = \sum_{j=1}^{m} x_{ij} \beta_j = x_{i1}\beta_{i1} + x_{i2}\beta_{i2} + \cdots + x_{im}\beta_{im}$$

i: *measuring index*

Linear Least Square (1)

Sum of the square of the residuals

$$S_r = \sum_{i=1}^{n} \epsilon_i^2 = \sum_{i=1}^{n} \left(y_i - \sum_{j=1}^{m} x_{ij}\beta_j \right)^2$$

$\beta_1, \beta_2, \cdots, \beta_m$ *unknowns*

$(x_{i1}, x_{i2}, \cdots, x_{im}, y_i)$ *measured data*

random

i: *measuring index*

$$y = \sum_{j=1}^{m} x_j \beta_j = x_1\beta_1 + x_2\beta_2 + \cdots + x_m\beta_m$$

Minimum Condition

$$\frac{\partial S_r}{\partial \beta_1} = 2\sum_{i=1}^{n} (y_i - \beta_1 x_{i1} - \beta_2 x_{i2} - \cdots - \beta_m x_{im})(-x_{i1}) = 0$$

$$\frac{\partial S_r}{\partial \beta_2} = 2\sum_{i=1}^{n} (y_i - \beta_1 x_{i1} - \beta_2 x_{i2} - \cdots - \beta_m x_{im})(-x_{i2}) = 0$$

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$$\frac{\partial S_r}{\partial \beta_m} = 2\sum_{i=1}^{n} (y_i - \beta_1 x_{i1} - \beta_2 x_{i2} - \cdots - \beta_m x_{im})(-x_{im}) = 0$$

Linear Least Square (2)

$$\begin{pmatrix} \left(\sum_{i=1}^{n} x_{i1}^2\right) & \left(\sum_{i=1}^{n} x_{i1}x_{i2}\right) & \left(\sum_{i=1}^{n} x_{i1}x_{i3}\right) & \cdots & \left(\sum_{i=1}^{n} x_{i1}x_{im}\right) \\ \left(\sum_{i=1}^{n} x_{i2}x_{i1}\right) & \left(\sum_{i=1}^{n} x_{i2}^2\right) & \left(\sum_{i=1}^{n} x_{i2}x_{i3}\right) & \cdots & \left(\sum_{i=1}^{n} x_{i2}x_{im}\right) \\ \left(\sum_{i=1}^{n} x_{i3}x_{i1}\right) & \left(\sum_{i=1}^{n} x_{i3}x_{i2}\right) & \left(\sum_{i=1}^{n} x_{i3}^2\right) & \cdots & \left(\sum_{i=1}^{n} x_{i3}x_{im}\right) \\ \vdots & \vdots & \vdots & & \vdots \\ \left(\sum_{i=1}^{n} x_{im}x_{i1}\right) & \left(\sum_{i=1}^{n} x_{im}x_{i2}\right) & \left(\sum_{i=1}^{n} x_{im}x_{i3}\right) & \cdots & \left(\sum_{i=1}^{n} x_{im}^2\right) \end{pmatrix} \begin{pmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \\ \vdots \\ \beta_m \end{pmatrix} = \begin{pmatrix} \left(\sum_{i=1}^{n} x_{i1}y_i\right) \\ \left(\sum_{i=1}^{n} x_{i2}y_i\right) \\ \left(\sum_{i=1}^{n} x_{i3}y_i\right) \\ \vdots \\ \left(\sum_{i=1}^{n} x_{im}y_i\right) \end{pmatrix}$$

i: *measuring index*

Linear Least Square (3)

$m = 1$ *measured data*

$$\begin{matrix} x_{11}^2 & x_{11}x_{12} & x_{11}x_{13} & \cdots & x_{11}x_{1m} \\ x_{12}x_{11} & x_{12}^2 & x_{12}x_{13} & \cdots & x_{12}x_{1m} \\ x_{13}x_{11} & x_{13}x_{12} & x_{13}^2 & \cdots & x_{13}x_{1m} \\ \vdots & \vdots & \vdots & & \vdots \\ x_{1m}x_{11} & x_{1m}x_{12} & x_{1m}x_{13} & \cdots & x_{1m}^2 \end{matrix}$$

$m = 2$ *measured data*

$$\begin{matrix} x_{21}^2 & x_{21}x_{22} & x_{21}x_{23} & \cdots & x_{21}x_{2m} \\ x_{22}x_{21} & x_{22}^2 & x_{22}x_{23} & \cdots & x_{22}x_{2m} \\ x_{23}x_{21} & x_{23}x_{22} & x_{23}^2 & \cdots & x_{23}x_{2m} \\ \vdots & \vdots & \vdots & & \vdots \\ x_{2m}x_{21} & x_{2m}x_{22} & x_{2m}x_{23} & \cdots & x_{2m}^2 \end{matrix}$$

$m = 3$ *measured data*

$$\begin{matrix} x_{31}^2 & x_{31}x_{32} & x_{31}x_{33} & \cdots & x_{31}x_{3m} \\ x_{32}x_{31} & x_{32}^2 & x_{32}x_{33} & \cdots & x_{32}x_{3m} \\ x_{33}x_{31} & x_{33}x_{32} & x_{33}^2 & \cdots & x_{33}x_{3m} \\ \vdots & \vdots & \vdots & & \vdots \\ x_{3m}x_{31} & x_{3m}x_{32} & x_{3m}x_{33} & \cdots & x_{3m}^2 \end{matrix}$$

$m = 4$ *measured data*

$$\begin{matrix} x_{41}^2 & x_{41}x_{42} & x_{41}x_{43} & \cdots & x_{41}x_{4m} \\ x_{42}x_{41} & x_{42}^2 & x_{42}x_{43} & \cdots & x_{42}x_{4m} \\ x_{43}x_{41} & x_{43}x_{42} & x_{43}^2 & \cdots & x_{43}x_{4m} \\ \vdots & \vdots & \vdots & & \vdots \\ x_{4m}x_{41} & x_{4m}x_{42} & x_{4m}x_{43} & \cdots & x_{4m}^2 \end{matrix}$$

Linear Least Square (4)

$$E\{x_1\} \qquad E\{x_1 x_2\} \qquad E\{x_1 x_3\} \qquad E\{x_1 x_m\}$$

$$\overline{x_1^2} \qquad \overline{x_1 x_2} \qquad \overline{x_1 x_3} \qquad \overline{x_1 x_m}$$

$$\frac{1}{n}\sum_{i=1}^{n} x_{i1}^2 \qquad \frac{1}{n}\sum_{i=1}^{n} x_{i1} x_{i2} \qquad \frac{1}{n}\sum_{i=1}^{n} x_{i1} x_{i3} \qquad \frac{1}{n}\sum_{i=1}^{n} x_{i1} x_{im}$$

$$\begin{matrix} \overline{x_1^2} & \overline{x_1 x_2} & \overline{x_1 x_3} & \cdots & \overline{x_1 x_m} \\ \overline{x_2 x_1} & \overline{x_2^2} & \overline{x_2 x_3} & \cdots & \overline{x_2 x_m} \\ \overline{x_3 x_1} & \overline{x_3 x_2} & \overline{x_3^2} & \cdots & \overline{x_3 x_m} \\ \vdots & \vdots & \vdots & & \vdots \\ \overline{x_m x_1} & \overline{x_m x_2} & \overline{x_m x_3} & \cdots & \overline{x_m^2} \end{matrix}$$

$$\begin{matrix} x_{11}^2 & x_{11} x_{12} & x_{11} x_{13} & \cdots & x_{11} x_{1m} \\ x_{12} x_{11} & x_{12}^2 & x_{12} x_{13} & \cdots & x_{12} x_{1m} \\ x_{13} x_{11} & x_{13} x_{12} & x_{13}^2 & \cdots & x_{13} x_{1m} \\ \vdots & \vdots & \vdots & & \vdots \\ x_{1m} x_{11} & x_{1m} x_{12} & x_{1m} x_{13} & \cdots & x_{1m}^2 \end{matrix}$$

$m = 4$ *measured data*

$m = 3$ *measured data*

$m = 2$ *measured data*

$m = 1$ *measured data*

Linear Least Square (5)

Sum of the square of the residuals

$$S_r = \sum_{i=1}^{n} \epsilon_i^2 = \sum_{i=1}^{n} \left(y_i - \sum_{j=1}^{m} x_{ij} \beta_j \right)^2$$

$\beta_1, \cdots, \beta_m$ *unknowns*

$(x_{i1}, x_{i2}, \cdots, x_{im}, y_i)$ *measured data*

random

$$y = \sum_{j=1}^{m} x_j \beta_j \quad \Rightarrow \quad y_i = \sum_{j=1}^{m} x_{ij} \beta_j \qquad \boldsymbol{y} = \boldsymbol{X} \boldsymbol{\beta}$$

i: *measuring index* i: *measuring index*

$$\boldsymbol{y} = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ \vdots \\ y_n \end{pmatrix} \qquad \boldsymbol{X} = \begin{pmatrix} x_{11} & x_{12} & x_{13} & \cdots & x_{1m} \\ x_{21} & x_{22} & x_{23} & \cdots & x_{2m} \\ x_{31} & x_{32} & x_{33} & \cdots & x_{3m} \\ \vdots & \vdots & \vdots & & \vdots \\ x_{n1} & x_{n2} & x_{n3} & \cdots & x_{nm} \end{pmatrix} \qquad \boldsymbol{\beta} = \begin{pmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \\ \vdots \\ \beta_m \end{pmatrix}$$

Linear Least Square (6)

Normal Equations

$$y_i = \sum_{j=1}^{m} x_{ij}\beta_j$$

$$X\beta = y$$

$\beta_1, \cdots, \beta_m$ *unknowns*

$(x_{i1}, x_{i2}, \cdots, x_{im}, y_i)$ *measured data*

random

$$\frac{\partial S_r}{\partial \beta_j} = 2\sum_{i=1}^{n} \epsilon_i \frac{\partial \epsilon_i}{\partial \beta_j} \quad (j = 1, 2, \ldots, m) \qquad \frac{\partial \epsilon_i}{\partial \beta_j} = -x_{ij}$$

$$\frac{\partial S_r}{\partial \beta_j} = 2\sum_{i=1}^{n} \left(y_i - \sum_{j=1}^{m} x_{ij}\beta_j \right)(-x_{ij}) = 0 \quad (j = 1, 2, \ldots, m)$$

$$\sum_{i=1}^{n} \left(x_{ij} y_i - \sum_{k=1}^{m} x_{ij} x_{ik} \hat{\beta}_k \right) = 0 \quad (j = 1, 2, \ldots, m)$$

$$\sum_{i=1}^{n} \sum_{k=1}^{n} x_{ij} x_{ik} \hat{\beta}_k = \sum_{i=1}^{n} x_{ij} y_i$$

$$X^t X \hat{\beta} = X^t y$$

References

[1] http://en.wikipedia.org/
[2] http://numericalmethods.eng.usf.edu/
[3] S.C. Chapra, Applied Numerical Methods W/ml Engineering And Science