DFT Analysis (5B)

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Frequency View of a DFT Matrix

$row\ \ 0$	$f\ =\ 0$	$(0\,cycle)$	0
$row\ \ 1$	$f\ =\ -1 * f_o$	$(-1\,cycle)$	$\frac{-1}{N}$
$row\ \ 2$	$f\ =\ -2 * f_o$	$(-2\,cycles)$	$\frac{-2}{N}$
$row\ \ (\frac{N}{2}-1)$	$f\ =\ -(\frac{N}{2}-1) * f_o$	$(-(\frac{N}{2}-1)\,cycles)$	$\frac{-1}{2}+\frac{1}{N}$
$row\ \ (\frac{N}{2})$	$f\ =\ -(\frac{N}{2}) * f_o$	$(\frac{N}{2}\ \ cycles)$	$\frac{-1}{2}$
$row\ \ (\frac{N}{2}+1)$	$f\ =\ +(\frac{N}{2}-1) * f_o$	$(\frac{N}{2}-1\,cycles)$	$\frac{1}{2}-\frac{1}{N}$
$row\ \ N-2$	$f\ =\ +2 * f_o$	$(2\,cycles)$	$\frac{2}{N}$
$row\ \ N-1$	$f\ =\ +1 * f_o$	$(1\,cycle)$	$\frac{1}{N}$

Rows 1 to $(\frac{N}{2}-1)$: $\frac{N}{2}-1$

Rows $(\frac{N}{2}+1)$ to $N-1$: $\frac{N}{2}-1$

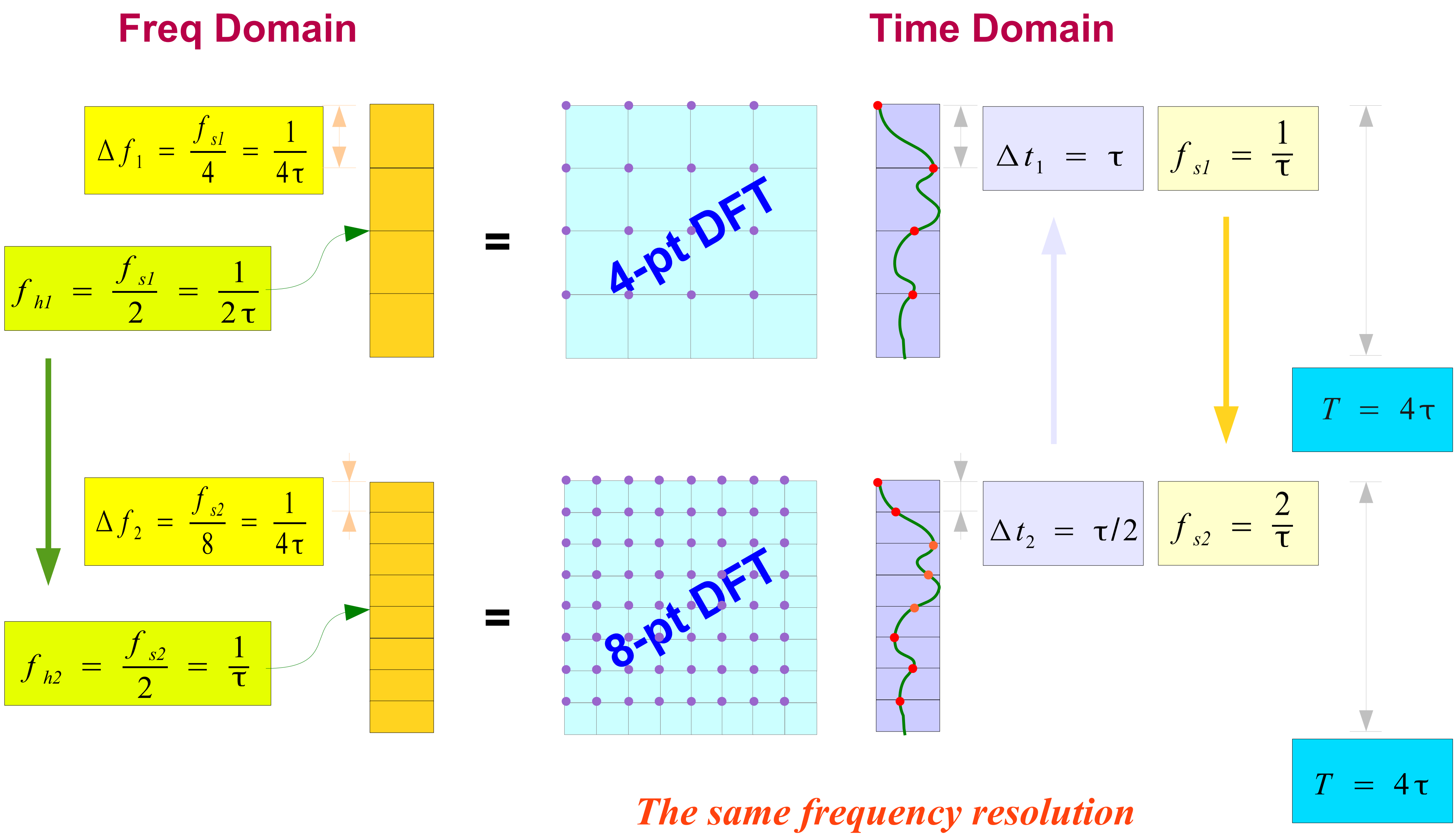
Normalized Frequency

$$f_o\ =\ \frac{f_s}{N}$$

Frequency View of a X[i] Vector

				Normalized Frequency	
	$X[0]$	$f = 0$	0	0	$(0\,cycle)$
$\frac{N}{2} - 1$	$X[1]$	$f = +1 * f_o$	$\frac{1}{N} f_s$	$\frac{1}{N}$	$(1\,cycle)$
	$X[2]$	$f = +2 * f_o$	$\frac{2}{N} f_s$	$\frac{2}{N}$	$(2\,cycles)$
	$X[\frac{N}{2}-1]$	$f = +(\frac{N}{2} - 1) * f_o$	$\frac{(\frac{N}{2}-1)}{N} f_s$	$\frac{1}{2} - \frac{1}{N}$	$(\frac{N}{2} - 1\,cycles)$
	$X[\frac{N}{2}]$	$f = +(\frac{N}{2}) * f_o$	$\frac{1}{2} f_s$	$\frac{1}{2}$	$(\frac{N}{2}\quad cycles)$
$\frac{N}{2} - 1$	$X[\frac{N}{2}+1]$	$f = -(\frac{N}{2} - 1) * f_o$	$-\frac{(\frac{N}{2}-1)}{N} f_s$	$\frac{-1}{2} + \frac{1}{N}$	$(-(\frac{N}{2} - 1)\,cycles)$
	$X[N-2]$	$f = -2 * f_o$	$-\frac{2}{N} f_s$	$\frac{-2}{N}$	$(-2\,cycles)$
	$X[N-1]$	$f = -1 * f_o$	$-\frac{1}{N} f_s$	$\frac{-1}{N}$	$(-1\,cycle)$

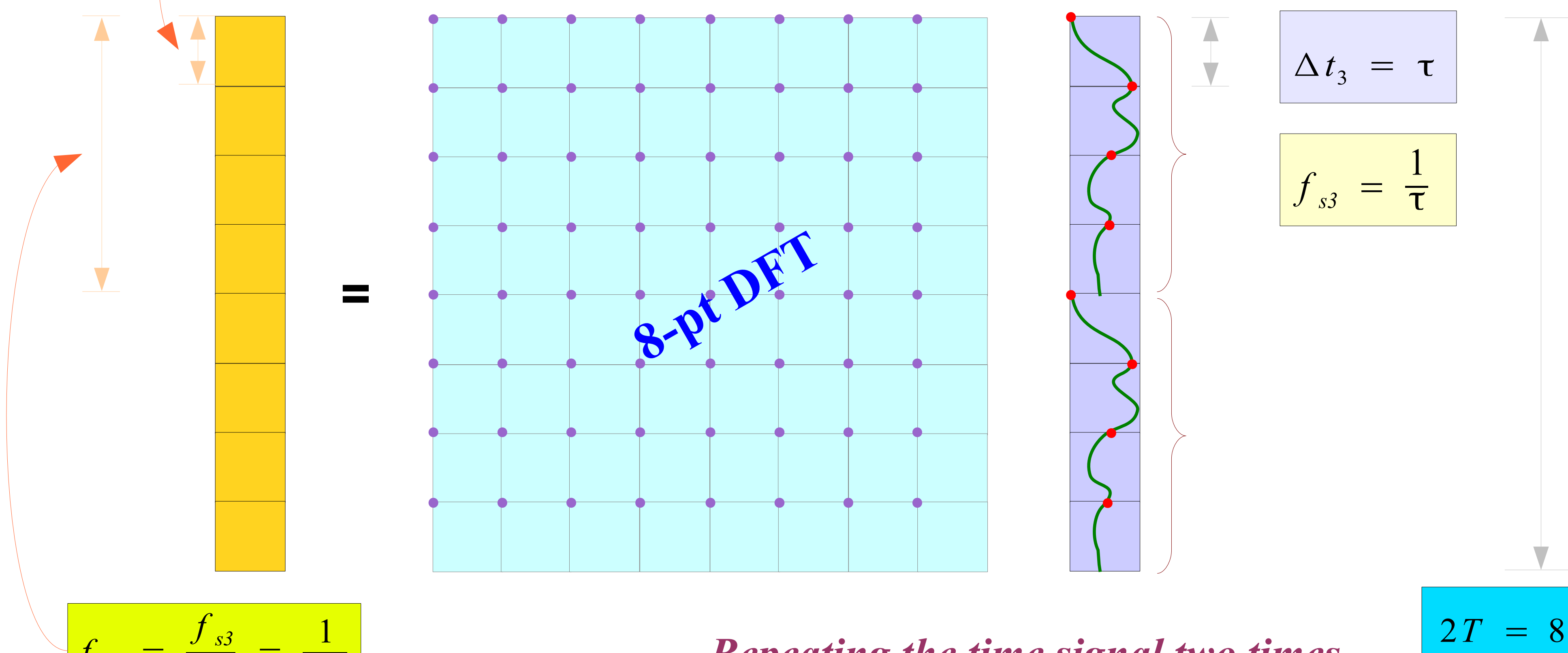
Frequency and Time Interval (1)



Frequency and Time Interval (2)

$$\Delta f_3 = \frac{f_{s3}}{8} = \frac{1}{8\tau}$$

The finer frequency resolution



$$\Delta t_3 = \tau$$

$$f_{s3} = \frac{1}{\tau}$$

$$f_{h3} = \frac{f_{s3}}{2} = \frac{1}{2\tau}$$

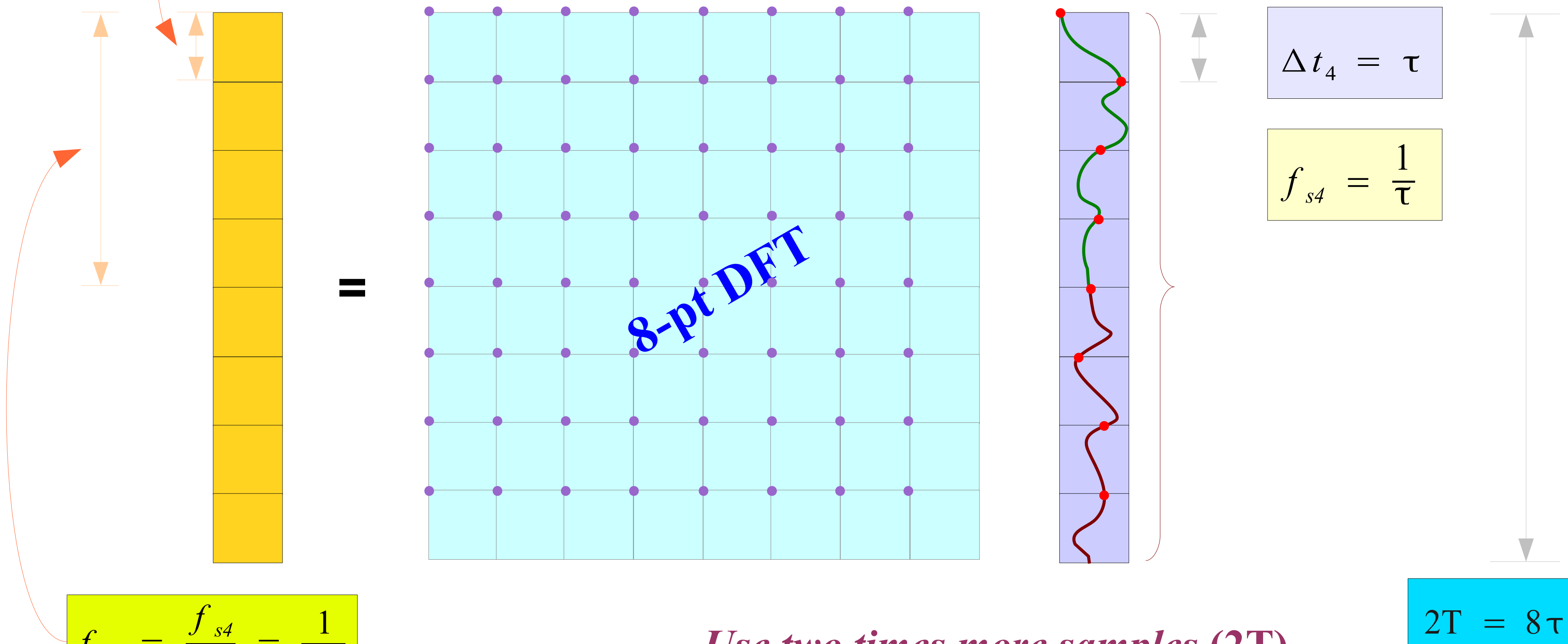
Repeating the time signal two times

$$2T = 8\tau$$

Frequency and Time Interval (3)

$$\Delta f_4 = \frac{f_{s4}}{8} = \frac{1}{8\tau}$$

The finer frequency resolution



$$\Delta t_4 = \tau$$

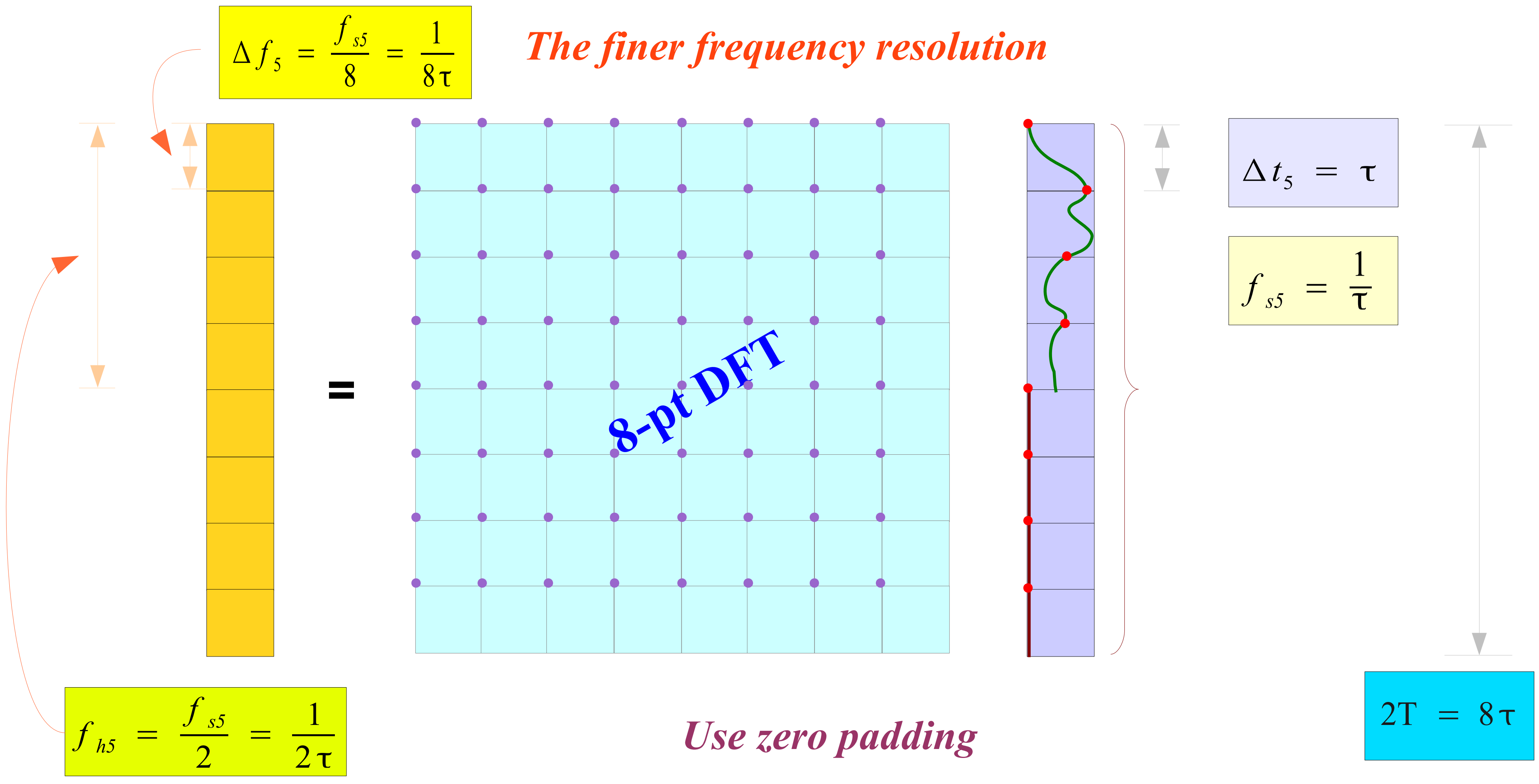
$$f_{s4} = \frac{1}{\tau}$$

$$f_{h4} = \frac{f_{s4}}{2} = \frac{1}{2\tau}$$

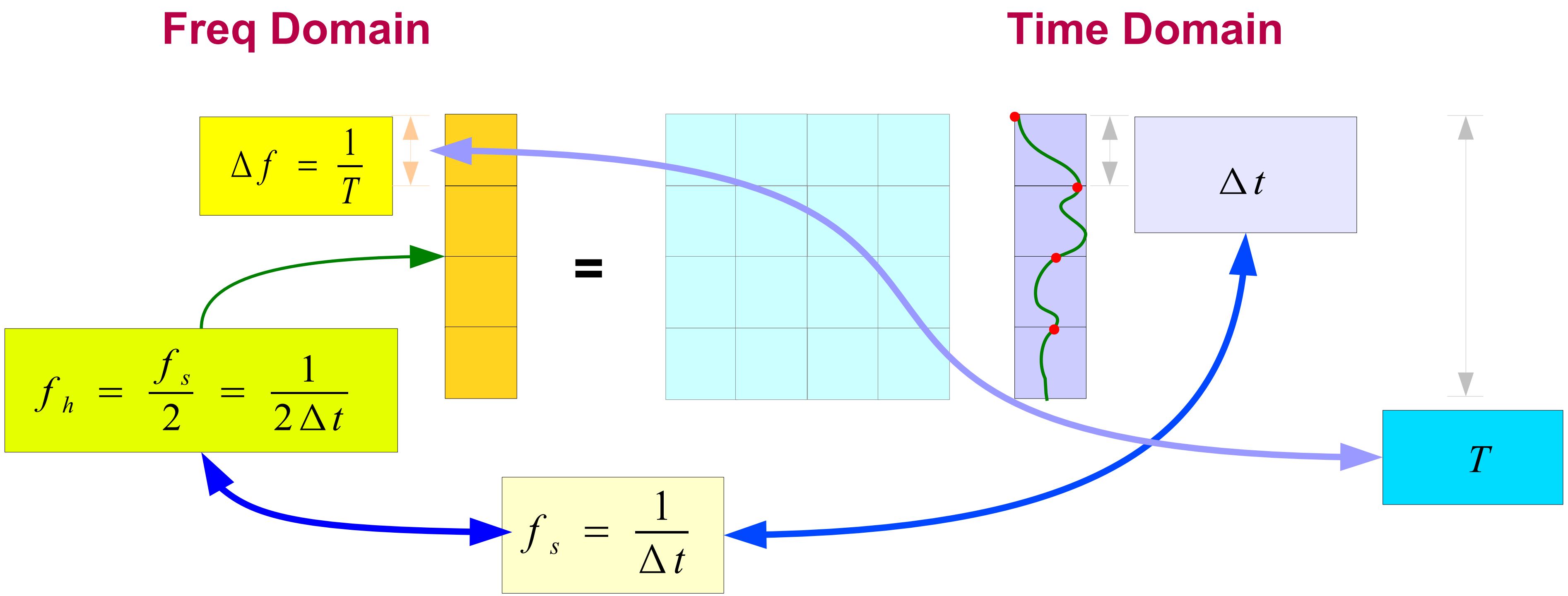
Use two times more samples (2T)

$$2T = 8\tau$$

Frequency and Time Interval (4)



Frequency and Time Interval (5)



Single-Sided Spectrum

$$x(t) = a_0 + \sum_{k=1}^{\infty} \left(a_k \cos(k\omega_0 t) + b_k \sin(k\omega_0 t) \right)$$

$$x(t) = g_0 + \sum_{k=1}^{\infty} g_k \cos(k\omega_0 t + \phi_k)$$

$$a_0 = \frac{1}{T}\int_0^T x(t)\, dt$$

$$a_k = \frac{2}{T}\int_0^T x(t) \cos(k\omega_0 t)\, dt$$

$$b_k = \frac{2}{T}\int_0^T x(t) \sin(k\omega_0 t)\, dt$$

$\boldsymbol{k} = +1, +2, \ldots$

$$g_0 = a_0$$

$$g_k = \sqrt{a_k^2 + b_k^2}$$

$$\phi_k = \tan^{-1}\left(-\frac{b_k}{a_k}\right)$$

$\boldsymbol{k} = +1, +2, \ldots$

$$\cos(\alpha+\beta) = \cos(\alpha)\cos(\beta) - \sin(\alpha)\sin(\beta)$$

$$g_k \cos(k\omega_0 t + \phi_k) = g_k \cos(\phi_k)\cos(k\omega_0 t) - g_k \sin(\phi_k)\sin(k\omega_0 t)$$

$$a_k \cos(k\omega_0 t) + b_k \sin(k\omega_0 t)$$

$$a_k = g_k \cos(\phi_k)$$

$$-b_k = g_k \sin(\phi_k)$$

$$a_k^2 + b_k^2 = g_k^2$$

$$-\frac{b_k}{a_k} = \tan(\phi_k)$$

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Two-Sided Spectrum

$$x(t) = \sum_{k=-\infty}^{+\infty} C_k \, e^{+jk\omega_0 t}$$

$$C_k = \frac{1}{T}\int_0^T x(t)\, e^{-jk\omega_0 t}\, dt$$

$$k = \dots, -2, -1, 0, +1, +2, \dots$$

$$C_k = \begin{cases} a_0 & (k = 0) \\ \frac{1}{2}(a_k - jb_k) & (k > 0) \\ \frac{1}{2}(a_k + jb_k) & (k < 0) \end{cases}$$

$$|C_k| = \begin{cases} a_0 & (k = 0) \\ \frac{1}{2}\sqrt{a_k^2 + b_k^2} & (k \neq 0) \end{cases}$$

$$Arg(C_k) = \begin{cases} \tan^{-1}(-b_k / a_k) & (k > 0) \\ \tan^{-1}(+b_k / a_k) & (k < 0) \end{cases}$$

Power Spectrum   *Two-Sided*

$$|C_k|^2 + |C_{-k}|^2 = \frac{1}{2} g_k^2 = \frac{1}{2}(a_k^2 + b_k^2)$$

$$x(t) = \sum_{k=-\infty}^{+\infty} C_k \, e^{+jk\omega_0 t}$$

$$C_k = \frac{1}{T}\int_0^T x(t)\, e^{-jk\omega_0 t}\, dt$$

$$k = \dots, -2, -1, 0, +1, +2, \dots$$

$$C_k = \begin{cases} a_0 & (k = 0) \\ \frac{1}{2} g_k \, e^{+j\phi_k} & (k > 0) \\ \frac{1}{2} g_k \, e^{-j\phi_k} & (k < 0) \end{cases}$$

$$|C_k| = \begin{cases} a_0 & (k = 0) \\ \frac{1}{2}|g_k| & (k \neq 0) \end{cases}$$

$$Arg(C_k) = \begin{cases} +\phi_k & (k > 0) \\ -\phi_k & (k < 0) \end{cases}$$

Periodogram   *One-Sided*

$$2 \cdot |C_k| = g_k = \sqrt{a_k^2 + b_k^2}$$

CTFS and DTFS (1)

$$x(t) = \sum_{k=-\infty}^{+\infty} C_k \, e^{+jk\omega_0 t}$$ **CTFS**

$$C_k = \frac{1}{T}\int_0^T x(t)\, e^{-jk\omega_0 t}\, dt$$

$$k = \ldots, -2, -1, 0, +1, +2, \ldots$$

$$x(t) \approx \sum_{k=-M}^{+M} C_k \, e^{+jk\omega_0 t} \qquad N = 2M+1$$

Truncate coefficeints

$$k = -M, \ldots, -1, 0, +1, \ldots, +M$$

N Time Samples: N equations

$$t \rightarrow nTs = n\left(\tfrac{T}{N}\right)$$

$$x[n] = \sum_{k=-M}^{+M} \gamma_k \, e^{+jk\omega_0 n\left(\frac{T}{N}\right)}$$

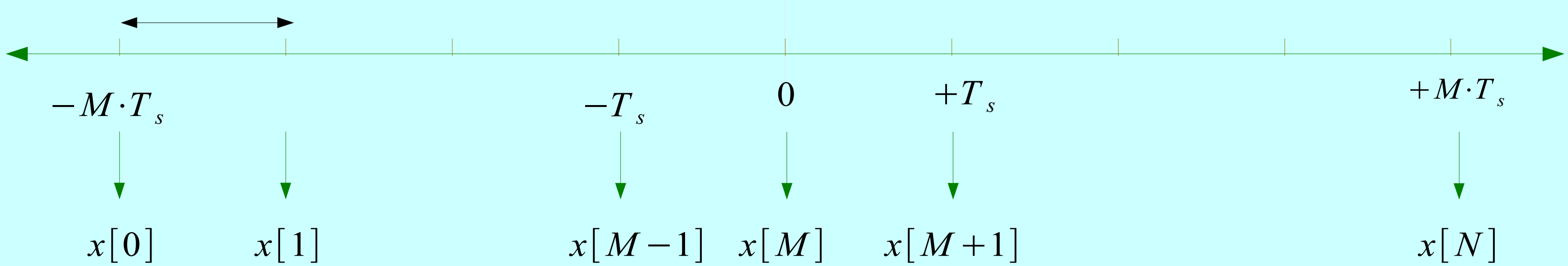
$$jk\omega_0 t \rightarrow k\left(\frac{2\pi}{T}\right) n\left(\frac{T}{N}\right) = \left(\frac{2\pi}{T}\right) nk$$

$$x[n] = \sum_{k=-M}^{+M} \gamma_k \, e^{+j\left(\frac{2\pi}{N}\right)nk}$$

$$n = 0, 1, 2, \ldots, N-1,$$

$$T_s = \frac{T}{N}$$

Time Samples



$0 \le n < L$ $\qquad T_s \rightarrow 0$

CTFS and DTFS (2)

$$x(t) = \sum_{k=-\infty}^{+\infty} C_k \, e^{+jk\omega_0 t}$$ **CTFS**

$$C_k = \frac{1}{T}\int_0^T x(t)\, e^{-jk\omega_0 t}\, dt$$

$$k = \ldots, -2, -1, 0, +1, +2, \ldots$$

$$x[n] = \sum_{k=-M}^{+M} \gamma_k \, e^{+j\left(\frac{2\pi}{N}\right)nk}$$

$$n = 0, 1, 2, \ldots, N-1,$$

$$\gamma_k = \frac{1}{N}\sum_{n=0}^{N-1} x[n]\, e^{-j\left(\frac{2\pi}{N}\right)nk}$$

$$k = -M, \ldots, 0, \ldots, +M$$

Approximate Continuous Signal
With the truncated coefficients

$$x_{FS}(t) = \sum_{k=-M}^{+M} \gamma_k \, e^{+jk\omega_0 t}$$ **DTFS**

$$C_k \approx \gamma_k = \frac{X[k]}{N}$$ **Approximated Fourier Coefficients**

$$X[k] = \sum_{n=0}^{N-1} x[n]\, e^{-j(2\pi/N)kn}$$ **DFT**

$$x[n] = \frac{1}{N}\sum_{k=0}^{N-1} X[k]\, e^{+j(2\pi/N)kn}$$ **IDFT**

$0 \le n < L$ $\qquad T_s \to 0$

CTFS and DTFS (3)

Discrete Fourier Transform

$$X[k] = \sum_{n=0}^{N-1} x[n]\, e^{-j(2\pi/N)kn}$$

$$x[n] = \frac{1}{N}\sum_{k=0}^{N-1} X[k]\, e^{+j(2\pi/N)kn}$$

$$x[\boldsymbol{n}] = \sum_{\boldsymbol{k}=-M}^{+M} \gamma_{\boldsymbol{k}}\, e^{+j\left(\frac{2\pi}{N}\right)\boldsymbol{nk}}$$

$$\boldsymbol{n} = 0,\ 1,\ 2,\ \ldots\ ,\ N-1,$$

$$\gamma_{\boldsymbol{k}} = \frac{1}{N}\sum_{\boldsymbol{n}=0}^{N-1} x[\boldsymbol{n}]\, e^{-j\left(\frac{2\pi}{N}\right)\boldsymbol{nk}}$$

$$\boldsymbol{k} = -M,\ \ldots\ ,\ 0,\ \ldots\ ,\ +M$$

$$x(t) = \sum_{\boldsymbol{k}=-\infty}^{+\infty} C_{\boldsymbol{k}}\, e^{+j\boldsymbol{k}\omega_0 t}$$

CTFS

$$C_k \approx \gamma_{\boldsymbol{k}} = \frac{X[k]}{N}$$

Approximated Fourier Coefficients

Truncate Fourier Coefficients

$$x_{FS}(t) = \sum_{\boldsymbol{k}=-M}^{+M} \gamma_{\boldsymbol{k}}\, e^{+j\boldsymbol{k}\omega_0 t}$$

DTFS

Power Spectrum using FFT

Discrete Fourier Transform

$$X[k] = \sum_{n=0}^{N-1} x[n]\, e^{-j(2\pi/N)kn}$$

$$x[n] = \frac{1}{N} \sum_{k=0}^{N-1} X[k]\, e^{+j(2\pi/N)kn}$$

```
X = fft(x)
```

```
x = ifft(X)
```

$$x(t) = \sum_{k=-\infty}^{+\infty} C_k\, e^{+jk\omega_0 t}$$ CTFS

Approximated
Fourier Series Coefficients

$$x_{FS}(t) = \sum_{k=-M}^{+M} \gamma_k\, e^{+jk\omega_0 t}$$ DTFS

```
fc = fft(x)/N     = X/N
```

$$C_k \approx \gamma_k = \frac{X[k]}{N}$$

Approximated
Fourier Coefficients

```
x = ifft(fc)*N
```

$$|C_k|^2 \approx \frac{|X[k]|^2}{N^2}$$

Approximated
Power Spectrum

Periodogram using FFT

$$C_k \approx \gamma_k = \frac{X[k]}{N}$$ **Approximated Fourier Coefficients**

$$|C_k|^2 \approx \frac{|X[k]|^2}{N^2}$$ **Approximated Power Spectrum**

$$\frac{1}{N}\sum_{n=0}^{N-1} x^2[n] = \frac{1}{N^2}\sum_{k=0}^{N-1} |X[k]|^2$$ **Average Power**

$$\Rightarrow \left(\sqrt{\frac{\sum_{k=0}^{N-1} \frac{|X[k]|^2}{N}}{N}} \right)^2$$ **RMS of sq root Periodogram**

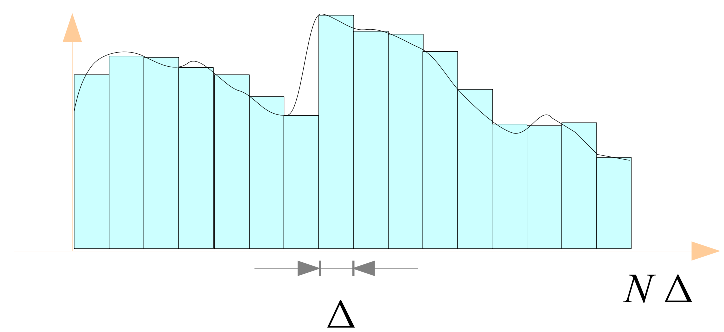
$$\frac{|X[k]|^2}{N} \quad k = 0, 1, \dots, N-1$$ **Approximated Periodogram**

$$\frac{|X[k]|}{\sqrt{N}} \quad k = 0, 1, \dots, N-1$$ **Square root Periodogram**

RMS in continuous time

$$\frac{1}{T}\int_0^T g^2(t)\, dt$$

RMS in discrete time



$$\frac{1}{N\Delta}\sum_{k=0}^{N-1} |g[k]|^2 \Delta = \frac{1}{N}\sum_{k=0}^{N-1} |g[k]|^2$$

From CTFS to CTFT

Continuous Time Fourier Series

$$C_k = \frac{1}{T}\int_0^T x(t)\, e^{-jk\omega_0 t}\, dt \quad \Longleftrightarrow \quad x(t) = \sum_{n=0}^{\infty} C_k\, e^{+jk\omega_0 t}$$

$$C_k = \frac{1}{T_0}\int_{-T_0/2}^{+T_0/2} x_{T_0}(t)\, e^{-jk\omega_0 t}\, dt \qquad x_{T_0}(t) = \sum_{n=0}^{\infty} C_k\, e^{+jk\omega_0 t} \cdot \frac{2\pi}{2\pi} \cdot \frac{T_0}{T_0}$$

$$C_k T_0 = \int_{-T_0/2}^{+T_0/2} x_{T_0}(t)\, e^{-jk\omega_0 t}\, dt \qquad x_{T_0}(t) = \frac{1}{2\pi}\sum_{k=0}^{\infty} C_k T_0\, e^{+jk\omega_0 t} \cdot \frac{2\pi}{T_0}$$

$$T_0 \to \infty \qquad \omega_0 = \frac{2\pi}{T_0} \to d\omega \qquad C_k T_0 \to X(j\omega) \qquad x_{T_0} \to x(t)$$

Continuous Time Fourier Transform

$$X(j\omega) = \int_{-\infty}^{+\infty} x(t)\, e^{-j\omega t}\, dt \quad \Longleftrightarrow \quad x(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} X(j\omega)\, e^{+j\omega t}\, d\omega$$

CTFS and CTFT

Discrete Fourier Transform

$$X[k] = \sum_{n=0}^{N-1} x[n]\, e^{-j(2\pi/N)kn}$$

$$x[n] = \frac{1}{N}\sum_{k=0}^{N-1} X[k]\, e^{+j(2\pi/N)kn}$$

$$x[\boldsymbol{n}] = \sum_{\boldsymbol{k}=-M}^{+M} \gamma_{\boldsymbol{k}}\, e^{+j\left(\frac{2\pi}{N}\right)\boldsymbol{nk}}$$

$$\boldsymbol{n} = 0,\ 1,\ 2,\ \ldots,\ N-1,$$

$$\gamma_{\boldsymbol{k}} = \frac{1}{N}\sum_{\boldsymbol{n}=0}^{N-1} x[\boldsymbol{n}]\, e^{-j\left(\frac{2\pi}{N}\right)\boldsymbol{nk}}$$

$$\boldsymbol{k} = -M,\ \ldots,\ 0,\ \ldots,\ +M$$

$$C_k \approx \gamma_{\boldsymbol{k}} = \frac{X[k]}{N}$$

Approximated Fourier Coefficients

Continuous Time Fourier Series

$$C_k T_0 = \int_{-T_0/2}^{+T_0/2} x_{T_0}(t)\, e^{-jk\omega_0 t}\, dt$$

$$x_{T_0}(t) = \frac{1}{2\pi}\sum_{k=0}^{\infty} C_k T_0\, e^{+jk\omega_0 t} \cdot \frac{2\pi}{T_0}$$

$$T_0 \to \infty,\quad \omega_0 \to 0\quad (\omega_0 \to d\omega)$$

Continuous Time Fourier Transform

$$X(j\omega) = \int_{-\infty}^{+\infty} x(t)\, e^{-j\omega t}\, dt$$

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} X(j\omega)\, e^{+j\omega t}\, d\omega$$

FS Coefficients of Periodic and Aperiodic Signals

	Periodic Signals	Aperiodic Signals
Frequency Spacing	$\Delta f = \frac{1}{N \Delta t}$	$\Delta f = \frac{1}{N \Delta t}$
Two Sided F.S. Coefficient	$\frac{1}{N} X(k)$	$\frac{\Delta t}{N} X(k)$
One Sided F.S. Coefficient	$\frac{1}{N} X(k) \quad k=0, \frac{N}{2}$ $\frac{2}{N} X(k) \quad k=1,\cdots,\frac{N}{2}-1$	$\frac{\Delta t}{N} X(k) \quad k=0, \frac{N}{2}$ $\frac{2\Delta t}{N} X(k) \quad k=1,\cdots,\frac{N}{2}-1$
Frequency Bin	$k \Delta f$	$k \Delta f$

Spectrum of Periodic Signals

Two-Sided Amplitude Spectrum

$$A_k = \frac{1}{N}|X[k]|$$
$$= \frac{1}{N}\sqrt{\Re^2\{X[k]\}+\Im^2\{X[k]\}}$$
$$k = 0,1,2,\cdots,N-1$$

One-Sided Amplitude Spectrum

$$\bar{A}_k = \frac{1}{N}|X[0]| \qquad k=0$$
$$\bar{A}_k = \frac{2}{N}|X[k]| \qquad k=1,2,\cdots,N/2$$

Frequency Bin

$$f = \frac{k}{N} f_s$$

Phase Spectrum

$$\phi_k = \tan^{-1}\left(\frac{\Im\{X[k]\}}{\Re\{X[k]\}}\right) \qquad k=0,1,2,\cdots,N-1$$

Two-Sided Power Spectrum

$$P_k = \frac{1}{N^2}|X[k]|^2$$
$$= \frac{1}{N^2}\left(\Re^2\{X[k]\}+\Im^2\{X[k]\}\right)$$
$$k = 0,1,2,\cdots,N-1$$

One-Sided Power Spectrum

$$\bar{P}_k = \frac{1}{N^2}|X[0]|^2 \qquad k=0$$
$$\bar{P}_k = \frac{2}{N^2}|X[k]|^2 \qquad k=1,2,\cdots,N/2$$

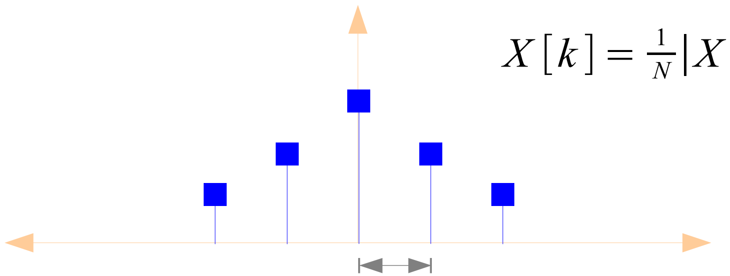
Frequency Bin

$$f = \frac{k}{N} f_s$$

Power Spectrum and Power Spectral Density

Power Spectrum

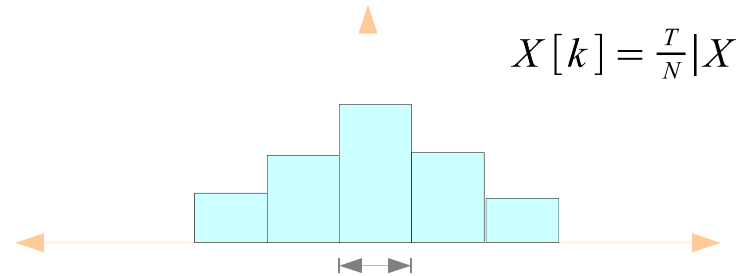
$X[k] = \frac{1}{N} |X[k]|^2$



$\Delta f = \frac{1}{N} f_s = \frac{1}{NT}$

Power Spectral Density — Hz

$X[k] = \frac{T}{N} |X[k]|^2$



$\Delta f = \frac{1}{N} f_s = \frac{1}{NT}$

$$\frac{1}{N} \sum_{n=0}^{N-1} x^2[n] = \frac{1}{N^2} \sum_{k=0}^{N-1} |X[k]|^2$$

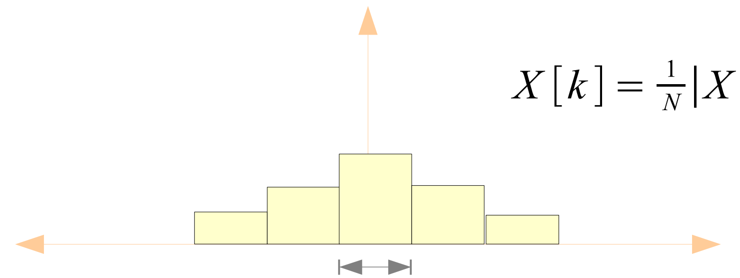
$$\Rightarrow \sum_{k=0}^{N-1} S[k] \Delta f$$

$$= \frac{1}{NT} \sum_{k=0}^{N-1} S[k] = \frac{1}{N^2} \sum_{k=0}^{N-1} |X[k]|^2$$

$$S[k] = \frac{T}{N} |X[k]|^2$$

Power Spectral Density — Hz ▪ sec

$X[k] = \frac{1}{N} |X[k]|^2$



$\frac{\Delta f}{f_s} = \frac{1}{N}$ *Normalized Frequency*

Periodogram → Power Spectral Density

FS Coefficients of Random Signals

Random Signals

Frequency Spacing	$\Delta f = \frac{1}{N \Delta t}$
Two Sided Power Spectral Density	$P = \sum_{k=0}^{N-1} S(k) \Delta f$
One Sided Power Spectral Density	$P = \sum_{k=0}^{N/2} S_1(k) \Delta f$ $S_1(k) = 2S(k) \quad k = 0, \frac{N}{2}$ $S_1(k) = S(k) \quad k = 1, \cdots, \frac{N}{2} - 1$
Frequency Bin	$k \Delta f$

$$\frac{1}{N \Delta t} \sum x^2 \Delta t$$

$$\sum S \Delta f = \frac{1}{N \Delta t} \sum S$$

$$S(k) = \frac{\Delta t}{N} |X(k)|^2$$

Signals without discontinuity
Signals with discontinuity

Sampling frequency is not an integer multiple of the FFT length

Leakage

$\left[0, \quad \frac{f_s}{2}\right]$

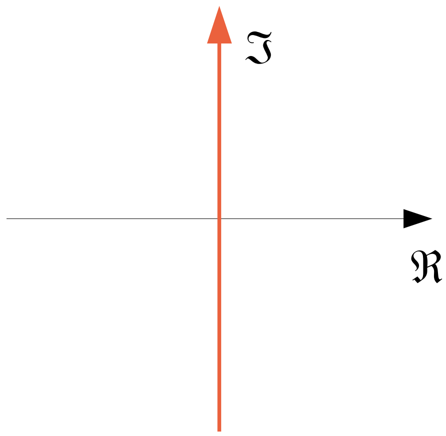
Fourier Transform

$f(t)$ A continuous sum of weighted exponential functions :

$f(t)\, e^{-j\omega t}$

$-\infty < \omega < +\infty$

Not so useful in transient analysis

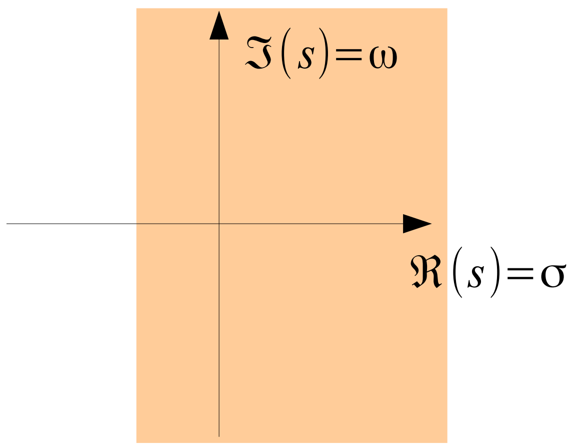


Laplace Transform

$$f(t)\, e^{-st} = f(t) e^{-(\sigma + j\omega)t}$$

Linear Time Domain Analysis

Initial Condition



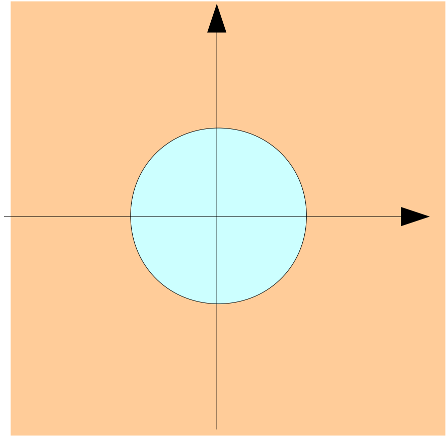
z Transform

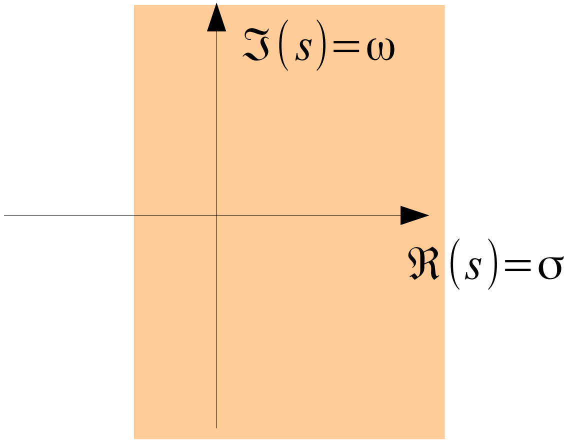
$$f[n]\, z^{-n}$$

Discrete Time System

Difference Equation

$$z = e^{sT} = e^{\sigma T}\, e^{j\omega T}$$





References

[1] http://en.wikipedia.org/
[2] J.H. McClellan, et al., Signal Processing First, Pearson Prentice Hall, 2003
[3] A “graphical interpretation” of the DFT and FFT, by Steve Mann