Down-Sampling (4B)

Please send corrections (or suggestions) to youngwlim@hotmail.com.

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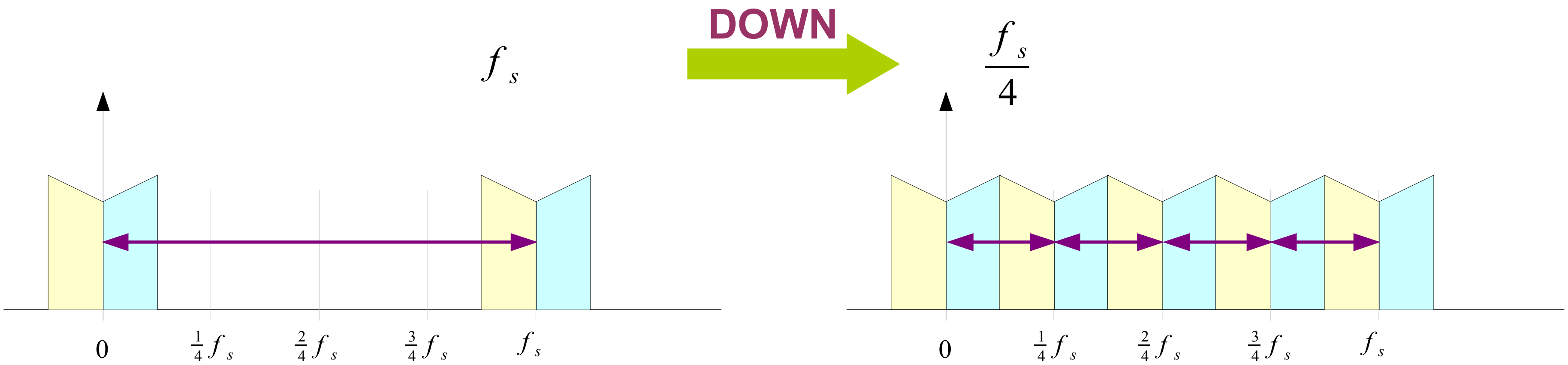
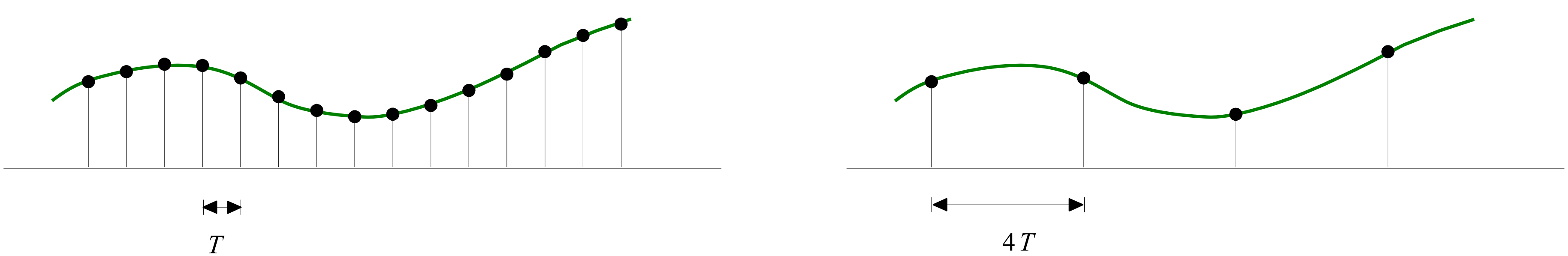
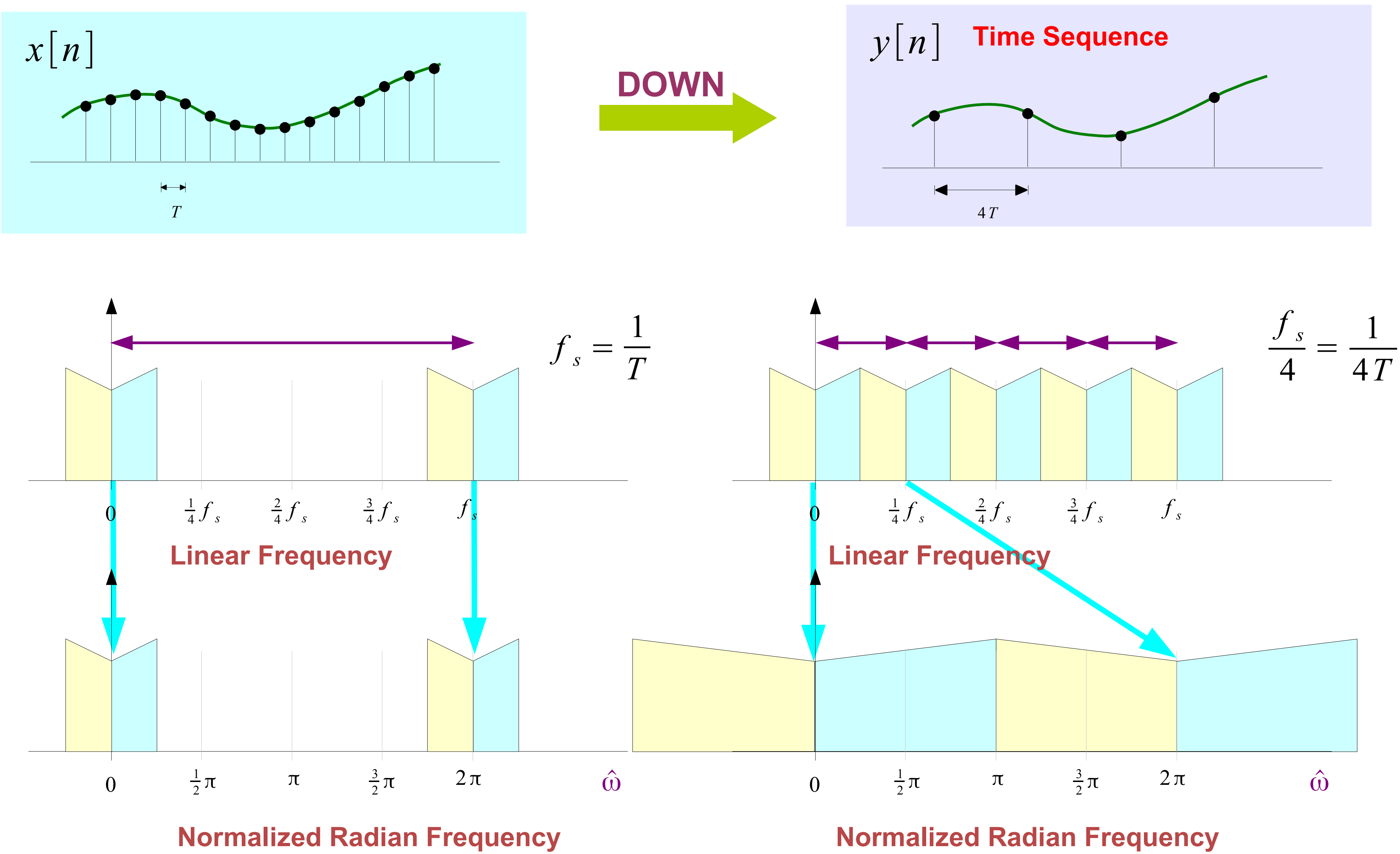
Decreasing Sampling Frequency



Sampling Frequency f_s	**Sampling Frequency** $f'_s = \frac{1}{4} f_s$
Sampling Time $T = \frac{1}{f_s}$	**Sampling Time** $T' = \frac{4}{f_s}$

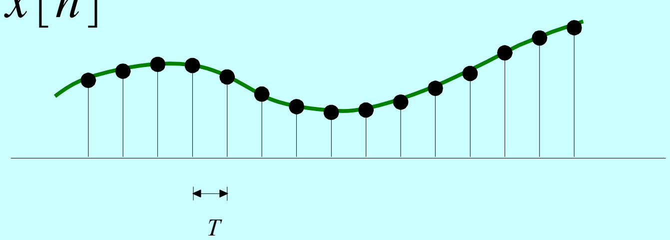


Coarse Sequence & Spectrum



Normalized Radian Frequency

$x[n]$



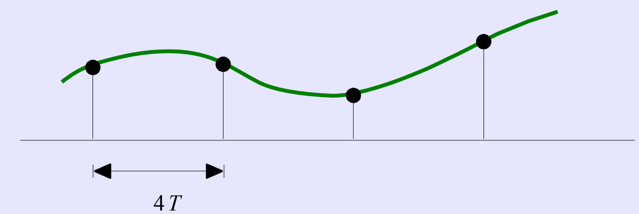
$$\hat{\omega} = \omega \cdot T_s = \frac{\omega}{1/T_s}$$

$$\hat{\omega} = \frac{\omega}{f_s} = 2\pi \frac{f}{f_s}$$

Normalized to f_s

Normalized Radian Frequency

$y[n]$ **Time Sequence** f_B



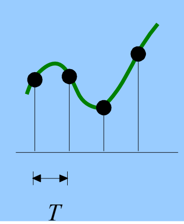
$$\frac{f}{f_s} = \frac{f_B}{1/4T} = f_B \cdot 4T$$

The Same

- **Time Sequence**
- **Normalized Radian Frequency**

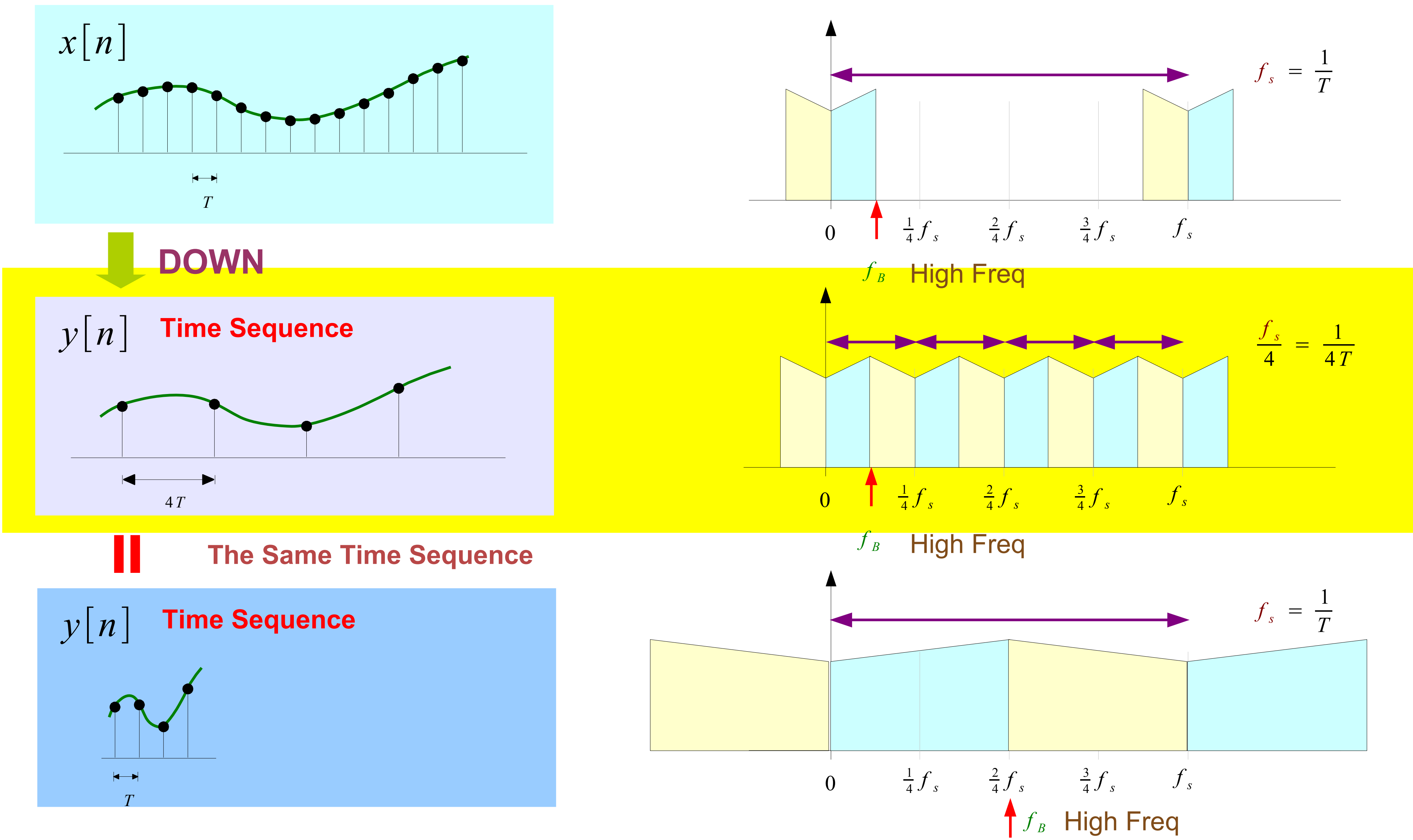
$$\frac{f}{f_s} = \frac{4 f_B}{1/T} = f_B \cdot 4T$$

$y'[n]$ **Time Sequence** $4 f_B$

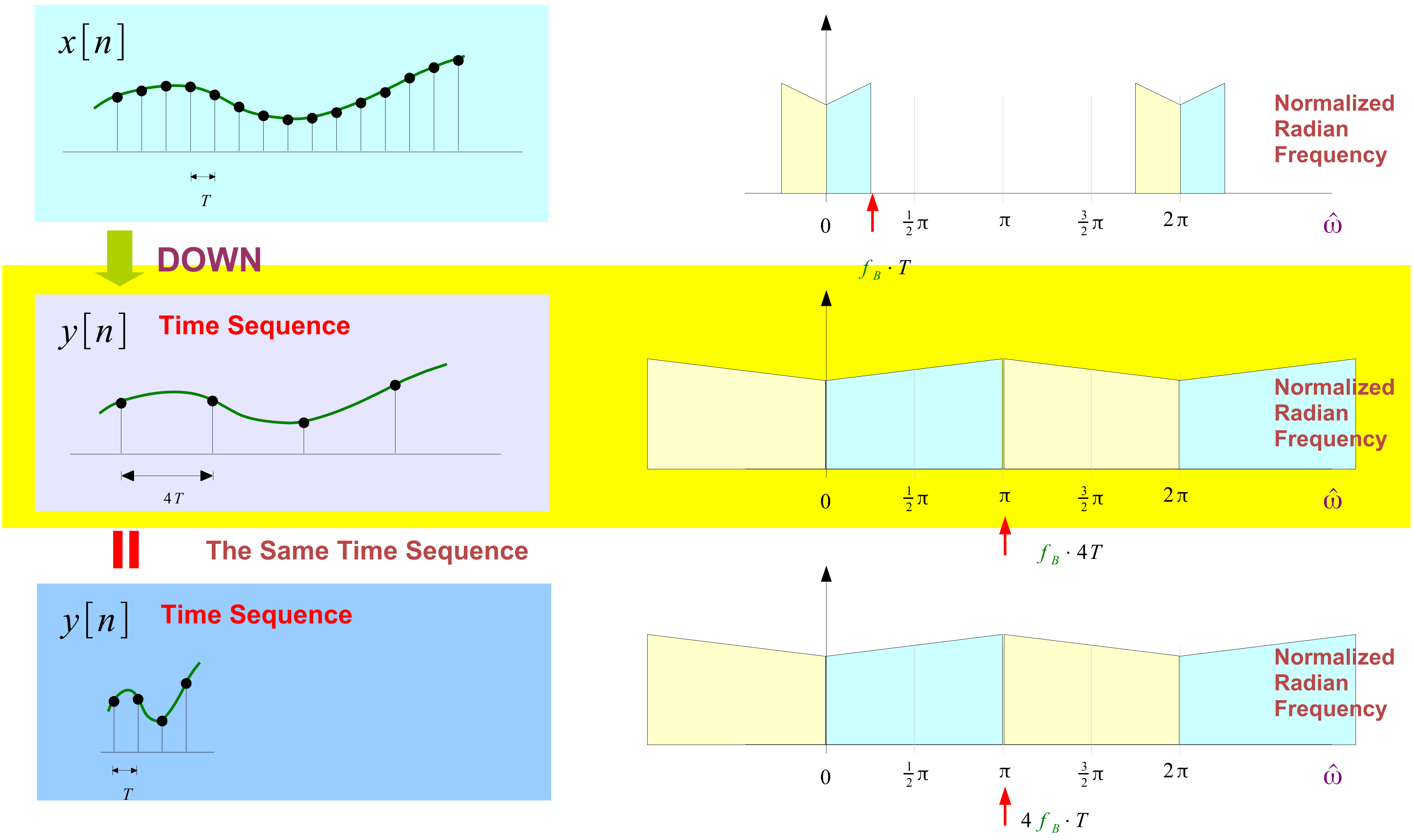


The Highest Frequency:

Coarse Sequence Spectrum – Linear Frequency



Coarse Sequence Spectrum – Normalized Frequency



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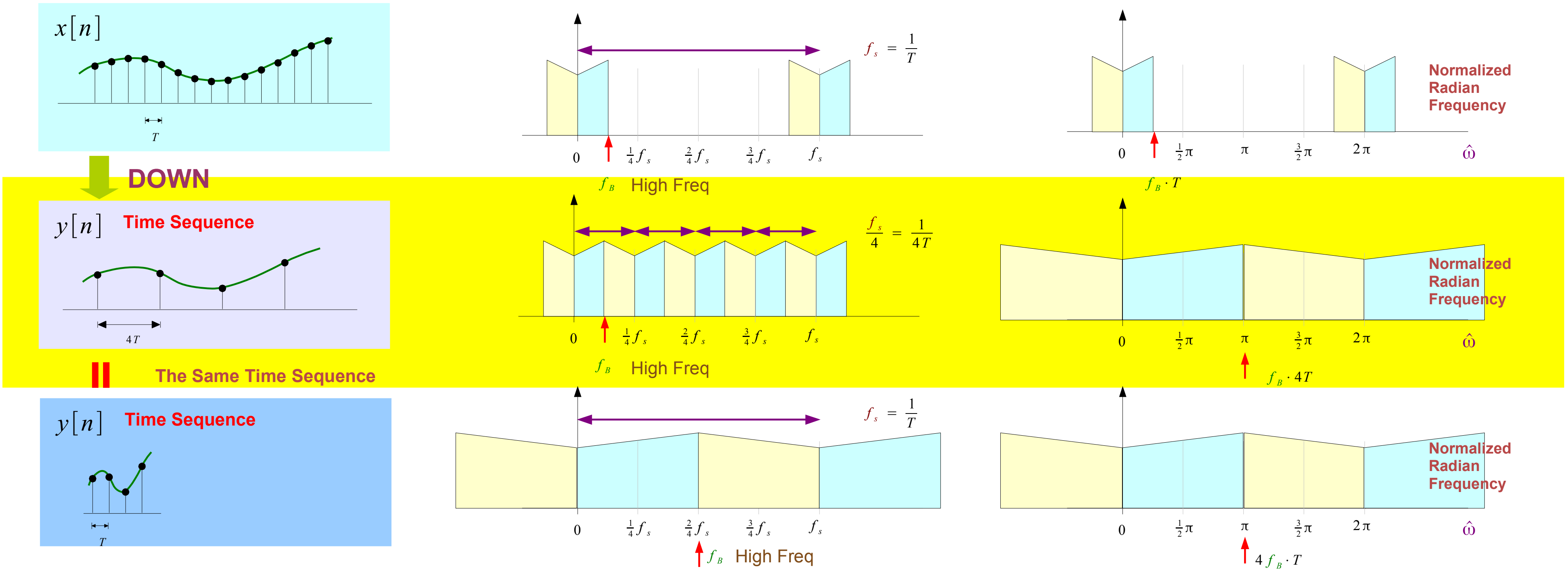
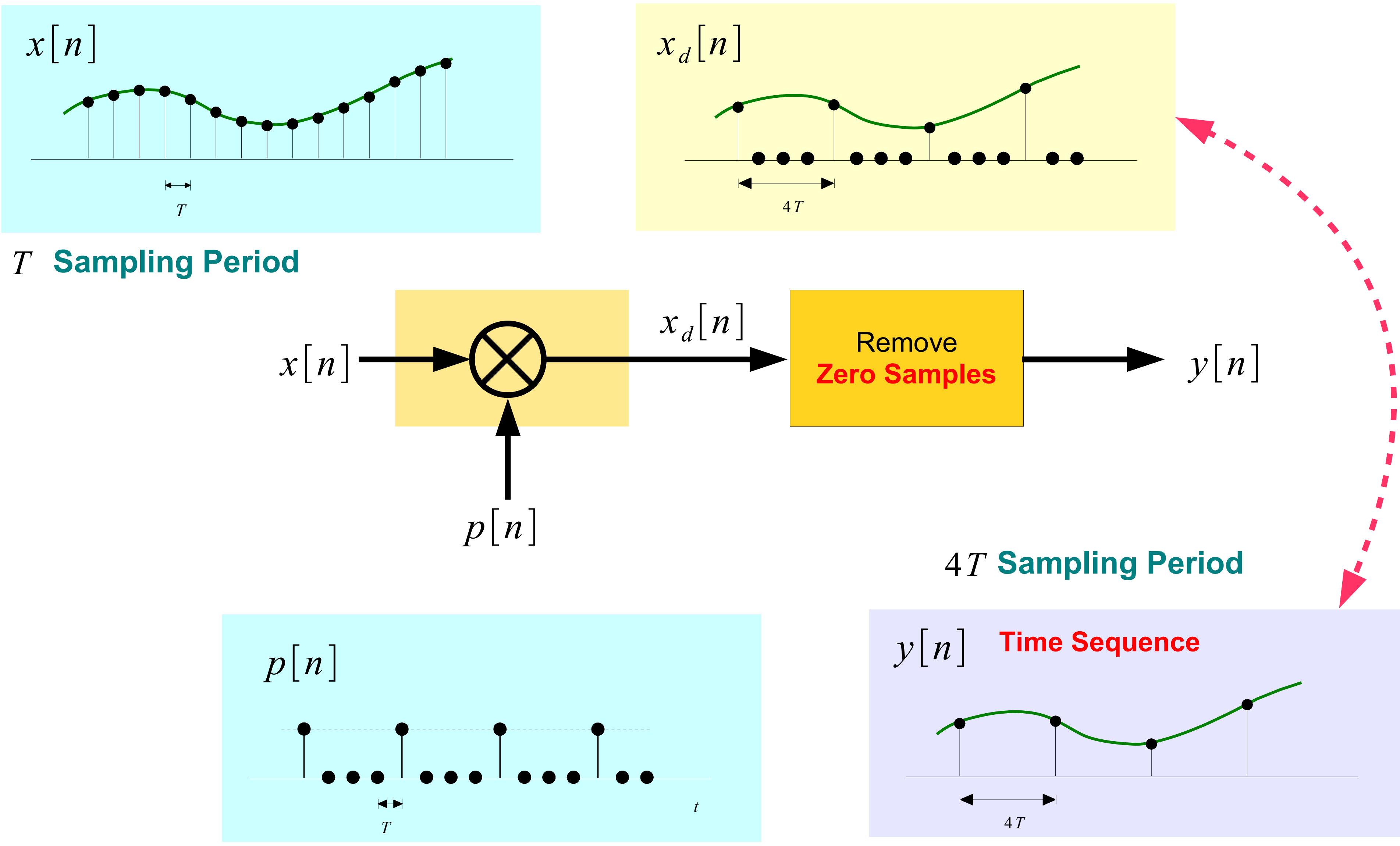
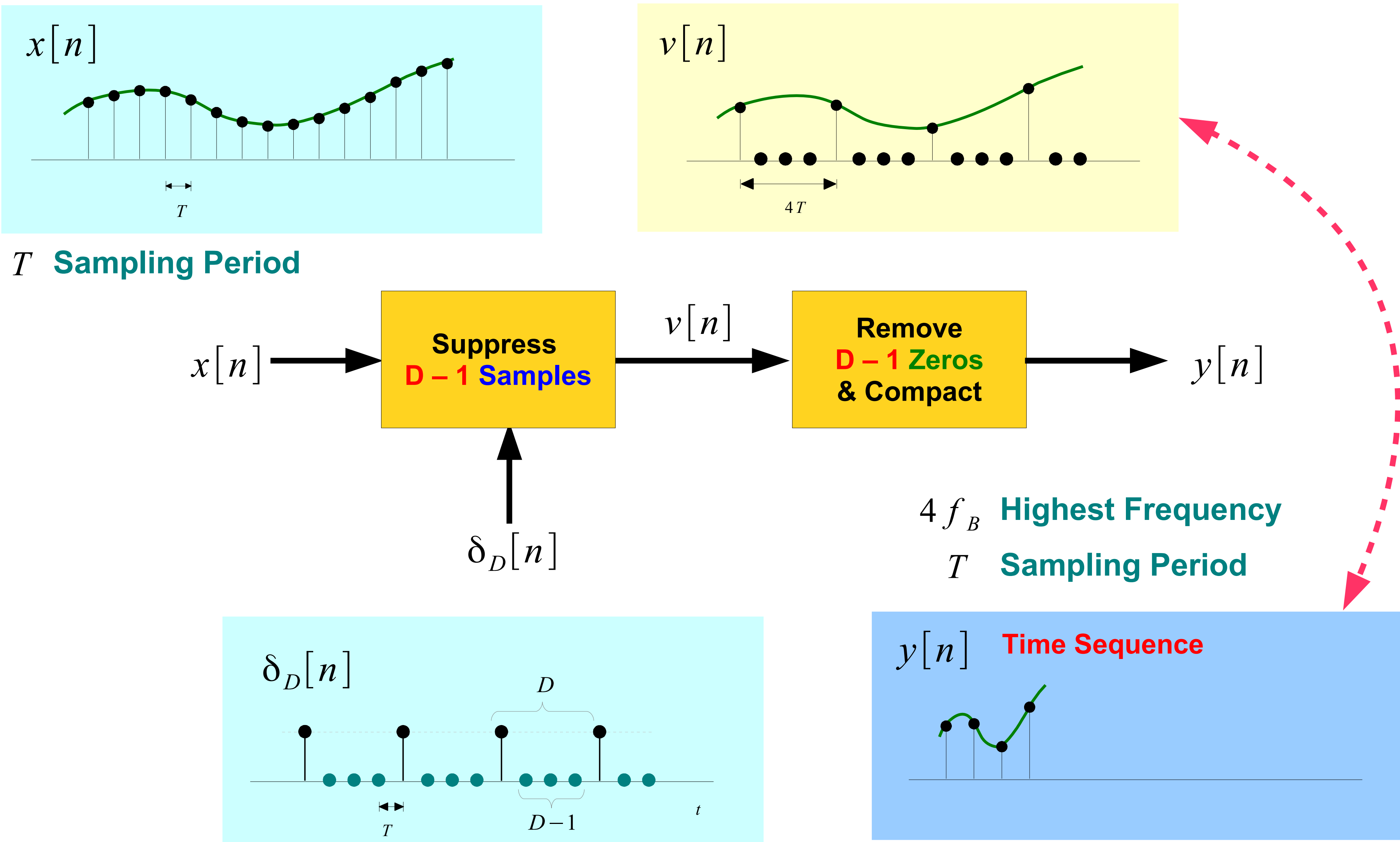
Coarse Sequence Spectrum



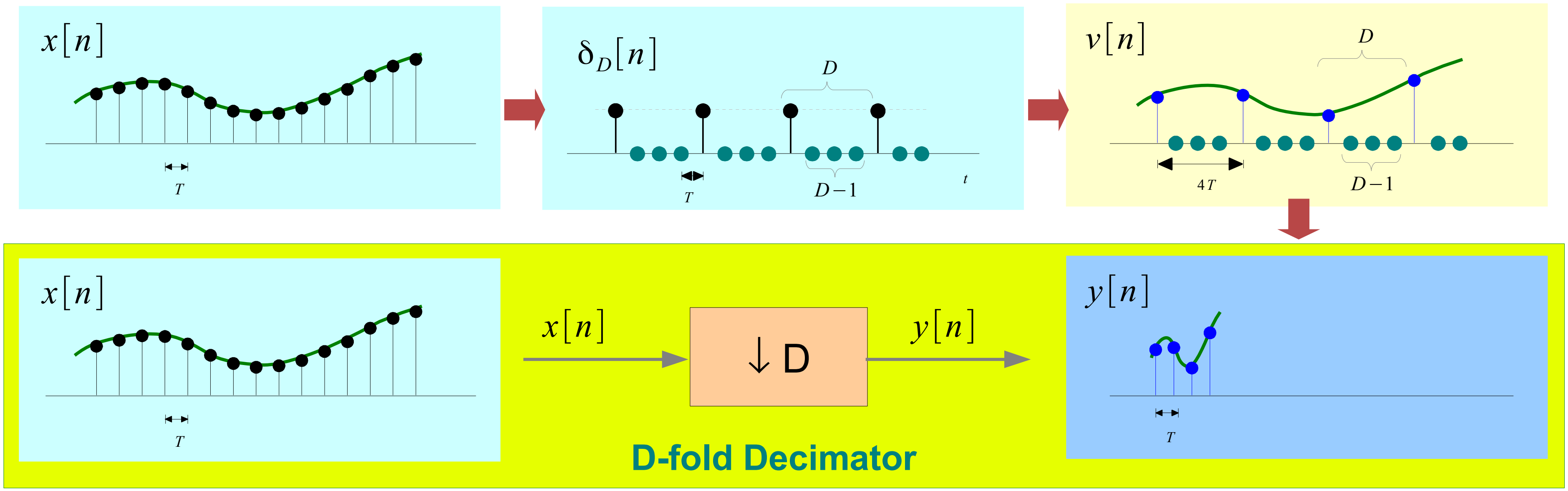
Coarse Sequence Generation



Down Sampling in Two Steps



Down-Sampling Operator



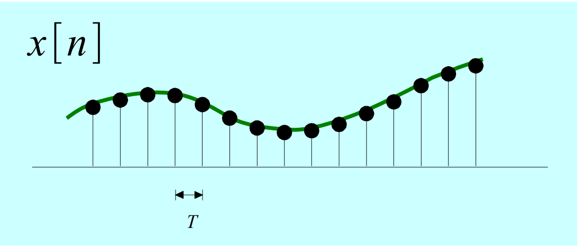
$$y[n] = S_{1/D}\, x[n] = x[nD]$$

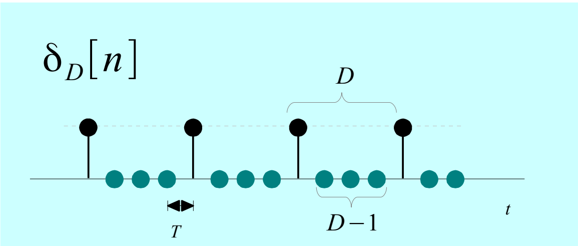
Decrease sampling frequency by a factor of 1/D

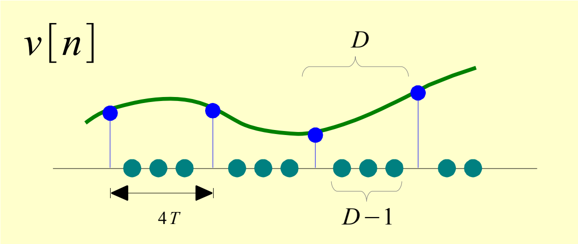
Increase sampling period by a factor of D

$$y[1] = x[1D]$$
$$y[2] = x[2D]$$
$$y[3] = x[3D]$$
$$\cdots$$

Suppressing D-1 Samples





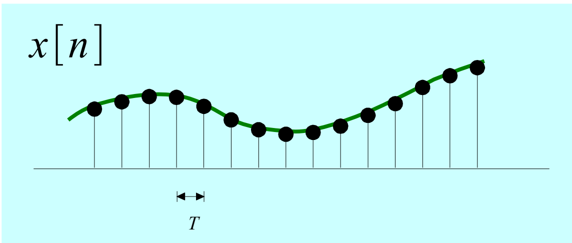


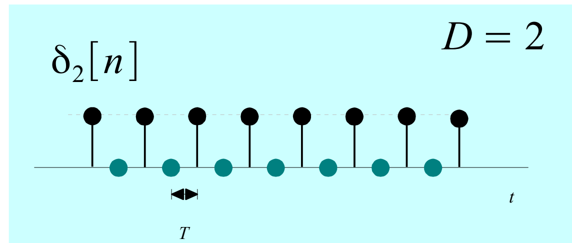
$$\delta_D[n] = \begin{cases} 1 & \textbf{if mod}(n, D) = 0 \\ 0 & \text{otherwise} \end{cases} \qquad v[n] = \delta_D[n]\,x[n]$$

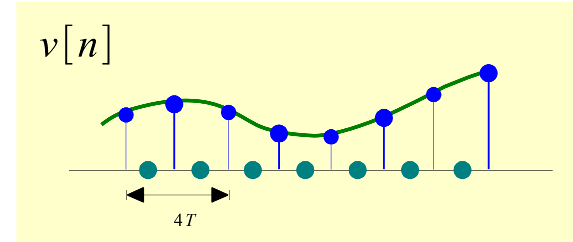
$$V[z] = \cdots + v[0]z^{0} + v[D]z^{-D} + v[2D]z^{-2D} + \cdots$$

$$V[z] = \sum_{n=-\infty}^{+\infty} v[n]z^{-n} = \sum_{m=-\infty}^{+\infty} v[mD]z^{-mD} = F(z^{D})$$

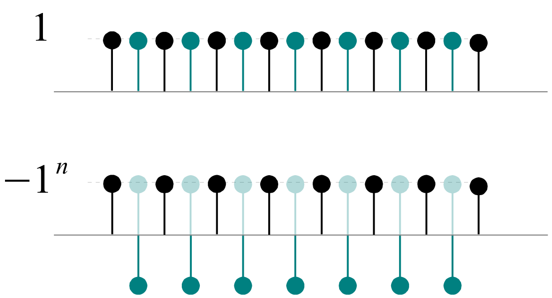
Example When D=2 (1)







$$x[n] = e^{j\omega n}$$



$$\delta_2[n] = \tfrac{1}{2}\left(1 + (-1)^n\right)$$
$$= \tfrac{1}{2}\left(1 + e^{-j\pi n}\right)$$
$$\left(e^{-j\pi} = -1\right)$$

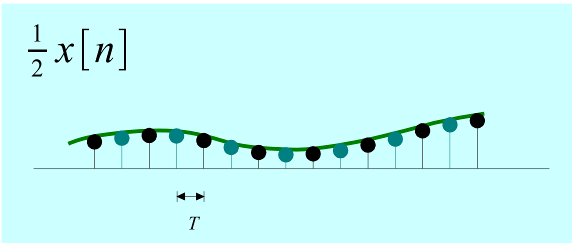
$$v[n] = \tfrac{1}{2}x[n] + \tfrac{1}{2}e^{-j\pi n}x[n]$$
$$= \tfrac{1}{2}e^{j\omega n} + \tfrac{1}{2}e^{-j\pi n}e^{j\omega n}$$
$$= \tfrac{1}{2}e^{j\omega n} + \tfrac{1}{2}e^{+j(\omega-\pi)n}$$

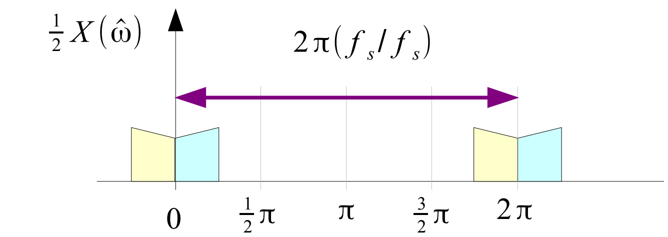
$$V(z) = \tfrac{1}{2}\sum_{n=-\infty}^{+\infty}\left(x[n]z^{-n} + x[n](-z)^{-n}\right)$$
$$= \tfrac{1}{2}X(z) + \tfrac{1}{2}X(-z)$$

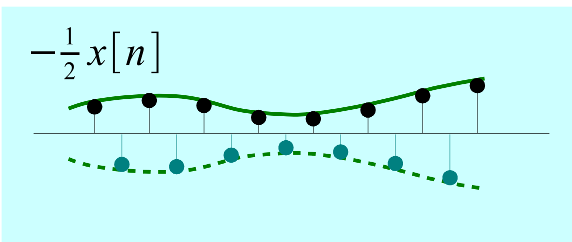
$$V(e^{j\hat{\omega}}) = \tfrac{1}{2}X(e^{j\hat{\omega}}) + \tfrac{1}{2}X(e^{-j\pi}e^{j\hat{\omega}})$$

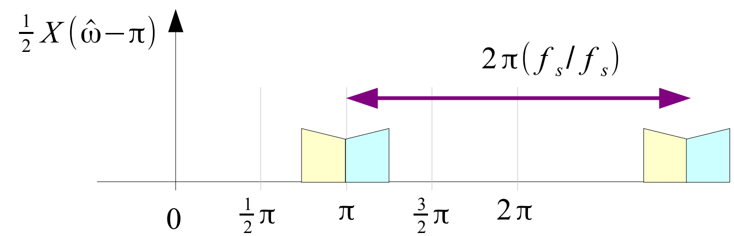
$$V(\hat{\omega}) = \tfrac{1}{2}X(\hat{\omega}) + \tfrac{1}{2}X(\hat{\omega} - \pi)$$

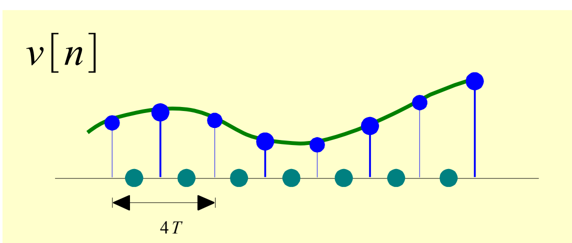
Example When D=2 (2)

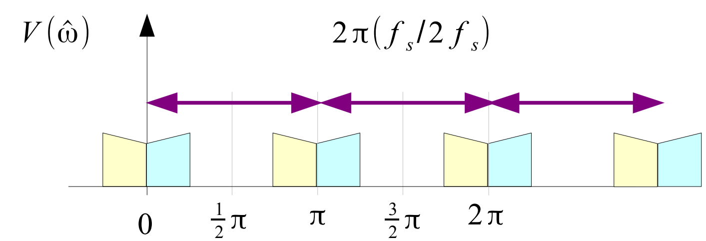












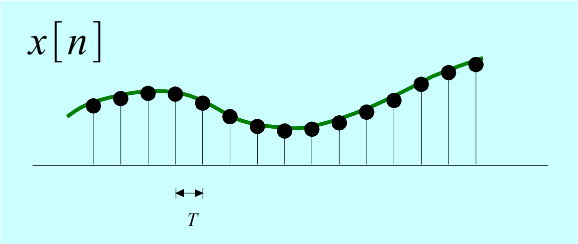
$$v[n] = \frac{1}{2}x[n] + \frac{1}{2}e^{-j\pi n}x[n]$$

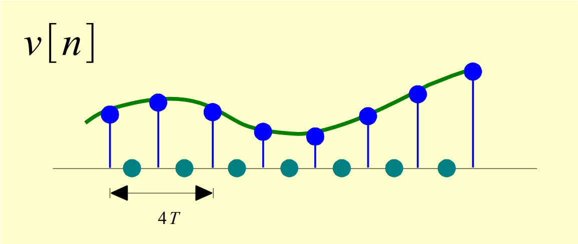
$$V(z) = \frac{1}{2}X(z) + \frac{1}{2}X(-z)$$

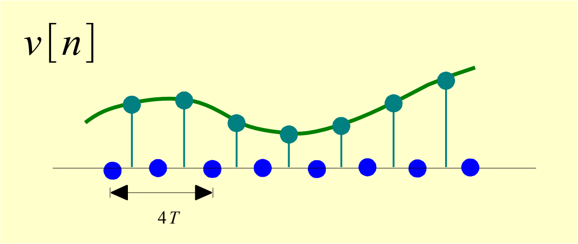
$$V(e^{j\hat{\omega}}) = \frac{1}{2}X(e^{j\hat{\omega}}) + \frac{1}{2}X(e^{-j\pi}e^{j\hat{\omega}})$$

$$V(\hat{\omega}) = \frac{1}{2}X(\hat{\omega}) + \frac{1}{2}X(\hat{\omega} - \pi)$$

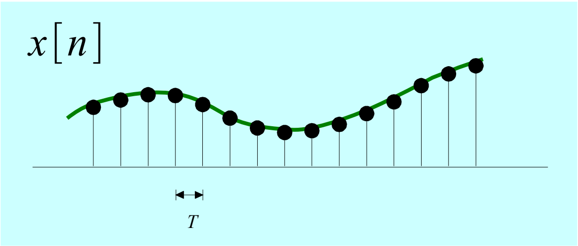
Example When D=2 (3)

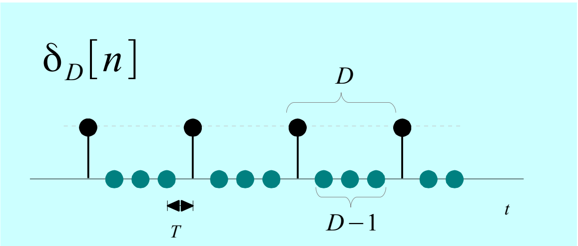


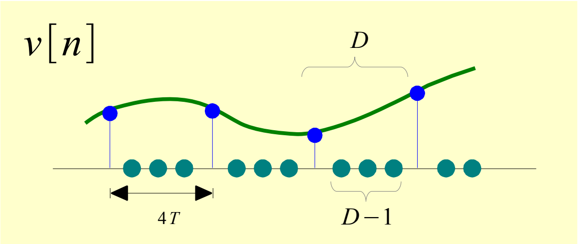




Z-Transform & Suppressing D-1 Samples (1)







$$\delta_D[n] = \frac{1}{D}\sum_{k=0}^{D-1} e^{-j2\pi k n/D} = \begin{cases} \frac{1}{D} D = 1 & \text{if } \mathbf{mod}(n, D) = 0 \\ \frac{1}{D}\frac{1-e^{-j2\pi n}}{1-e^{-j2\pi n/D}} = 0 & \text{otherwise} \end{cases}$$

$$v[n] = \delta_D[n]x[n]$$

$$v[n] = \delta_D[n]x[n] = \frac{1}{D}\sum_{k=0}^{D-1} x[n]e^{-j2\pi k n/D}$$

$$Z\{x[n]e^{-j\hat\omega_k n}\} = \sum_{n=-\infty}^{+\infty} x[n](e^{j\hat\omega_k}\cdot z)^{-n} = X(e^{j\hat\omega_k}\cdot z)$$

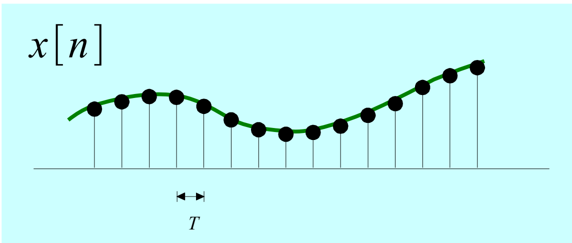
$$\hat\omega_k = 2\pi k/D$$

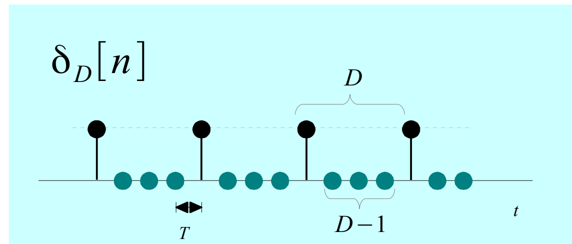
$$e^{-j2\pi k/D}\cdot z = e^{-j2\pi k/D}e^{j\hat\omega} = e^{j(\hat\omega - 2\pi k/D)}$$

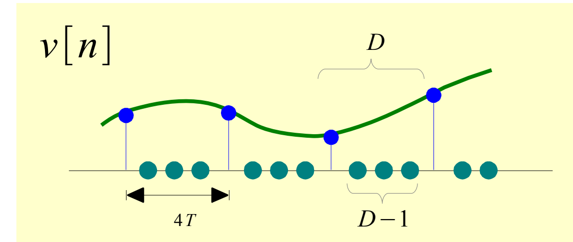
$$V(z) = \frac{1}{D}\sum_{k=0}^{D-1} X(e^{-j2\pi k/D}\cdot z)$$

$$V(\hat\omega) = \frac{1}{D}\sum_{k=0}^{D-1} X(\hat\omega - 2\pi k/D)$$

Z-Transform & Suppressing D-1 Samples (2)

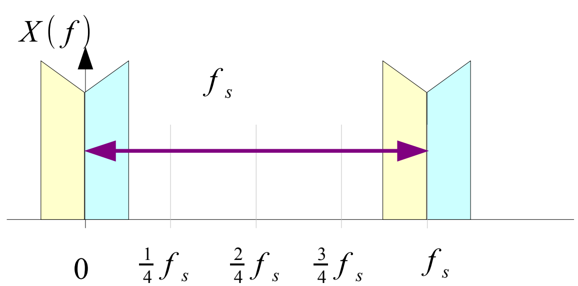


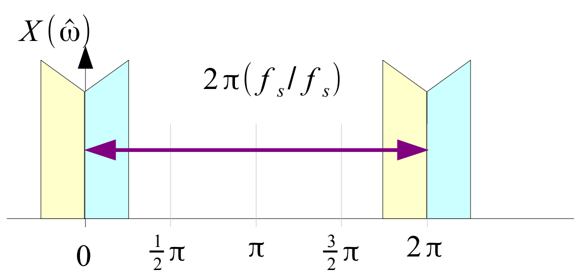


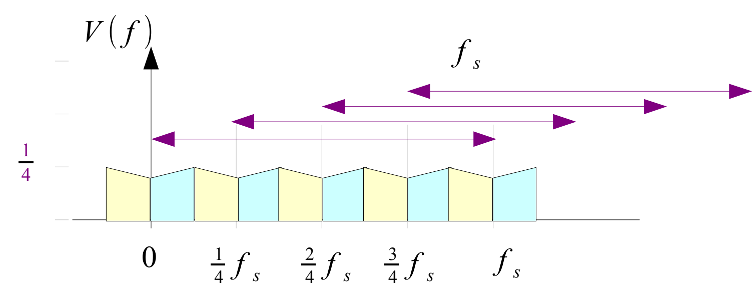


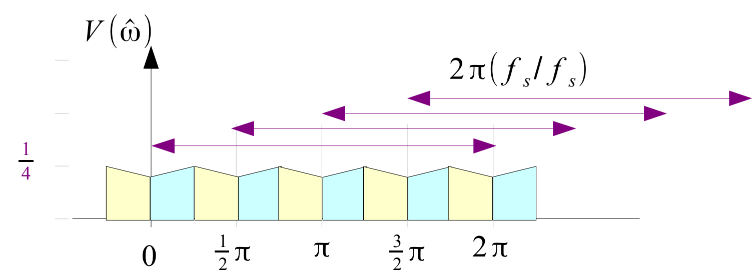
$$V(z) = \frac{1}{D}\sum_{k=0}^{D-1} X\left(e^{-j2\pi k/D}\cdot z\right)$$

$$V(\hat{\omega}) = \frac{1}{D}\sum_{k=0}^{D-1} X\left(\hat{\omega} - 2\pi k/D\right)$$

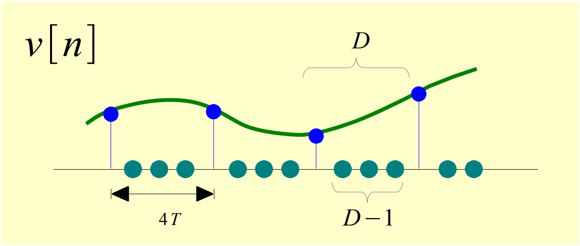


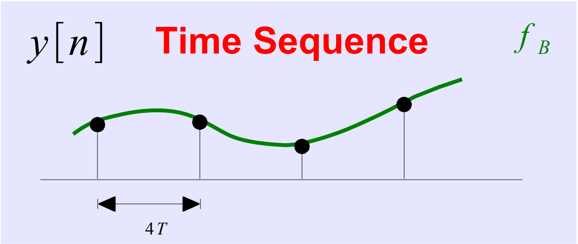


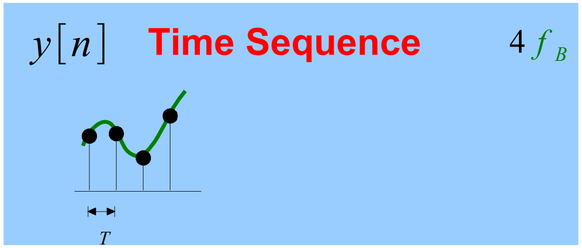




Z-Transform & Removing D-1 Zeros (1)







$$Y(z) = \cdots + y[0] + y[1]z^{-1} + y[2]z^{-2} + \cdots + y[n]z^{-n} + \cdots$$
$$= \cdots + v[0] + v[D]z^{-1} + v[2D]z^{-2} + \cdots + v[nD]z^{-n} + \cdots$$

$$Y(z) = V(z^{1/D})$$

$$V(z) = \frac{1}{D}\sum_{k=0}^{D-1} X(e^{-j2\pi k/D} \cdot z)$$

$$Y(z) = \frac{1}{D}\sum_{k=0}^{D-1} X(e^{-j2\pi k/D} \cdot z^{1/D})$$

$$V(\hat{\omega}) = \frac{1}{D}\sum_{k=0}^{D-1} X(\hat{\omega} - 2\pi k/D)$$

$$V(\hat{\omega}) = \frac{1}{D}\sum_{k=0}^{D-1} X(\hat{\omega}/D - 2\pi k/D)$$

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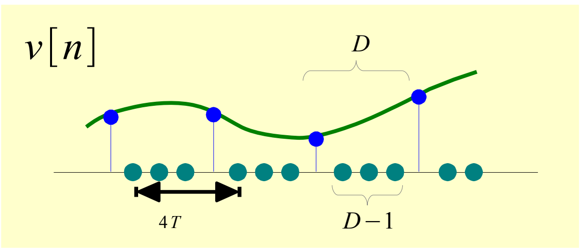
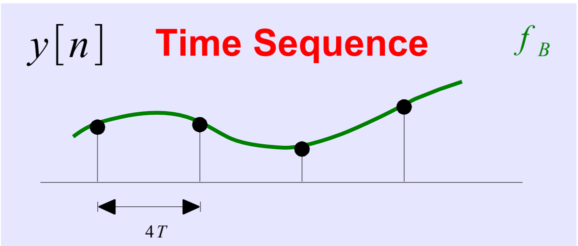
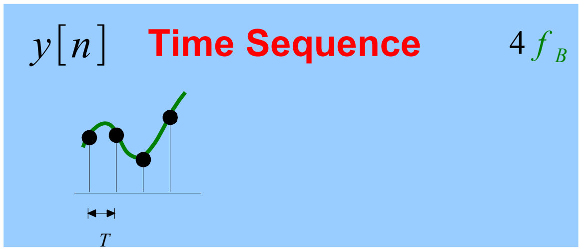
Z-Transform & Removing D-1 Zeros (2)





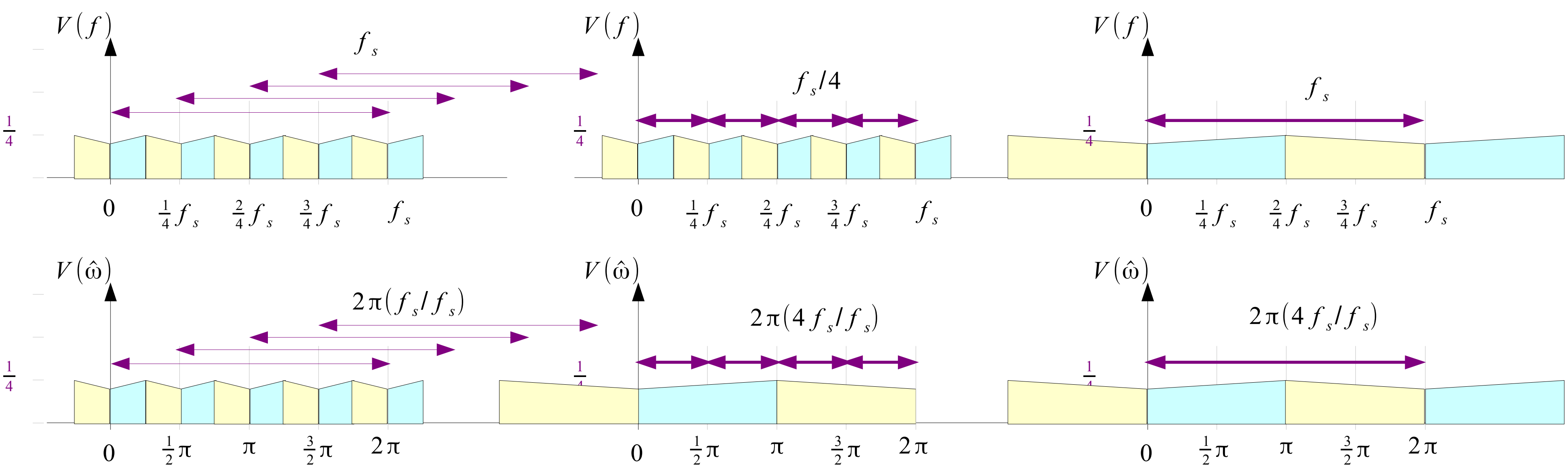


$$V(z) = \frac{1}{D}\sum_{k=0}^{D-1} X\left(e^{-j2\pi k/D}\cdot z\right)$$

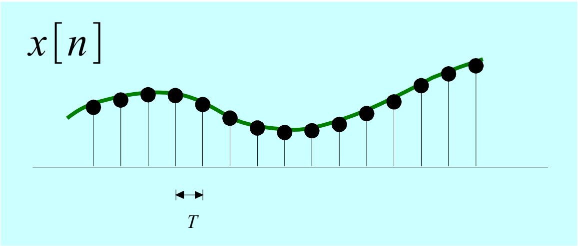
$$Y(z) = \frac{1}{D}\sum_{k=0}^{D-1} X\left(e^{-j2\pi k/D}\cdot z^{1/D}\right)$$

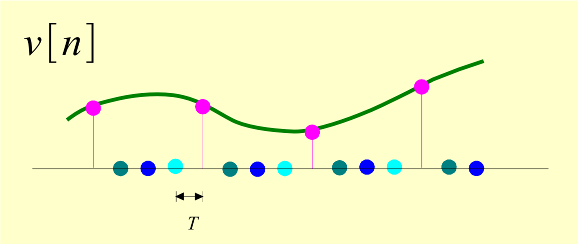
$$V(\hat{\omega}) = \frac{1}{D}\sum_{k=0}^{D-1} X\left(\hat{\omega} - 2\pi k/D\right)$$

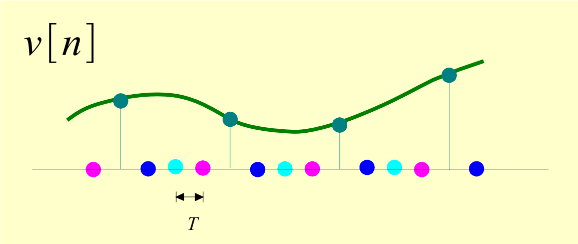
$$V(\hat{\omega}) = \frac{1}{D}\sum_{k=0}^{D-1} X\left(\hat{\omega}/D - 2\pi k/D\right)$$

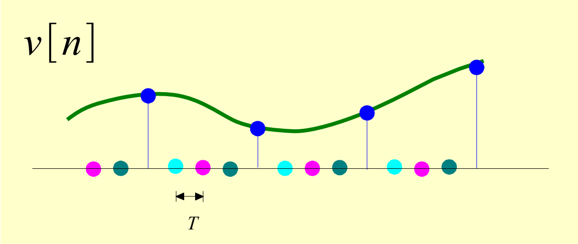


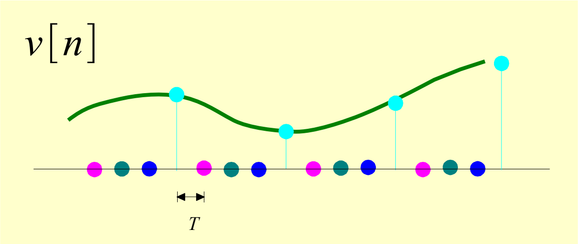
Example When D=2 (3)



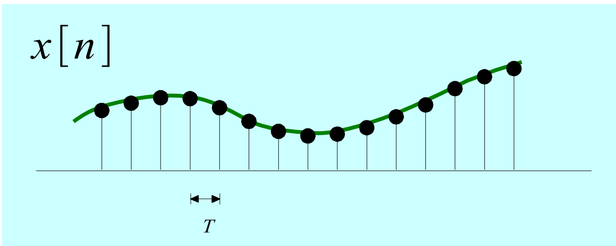


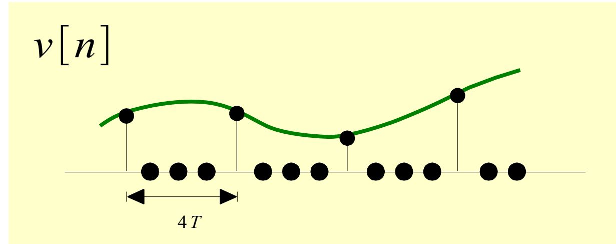


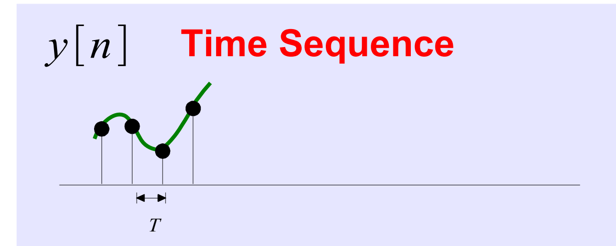




Down Sampling

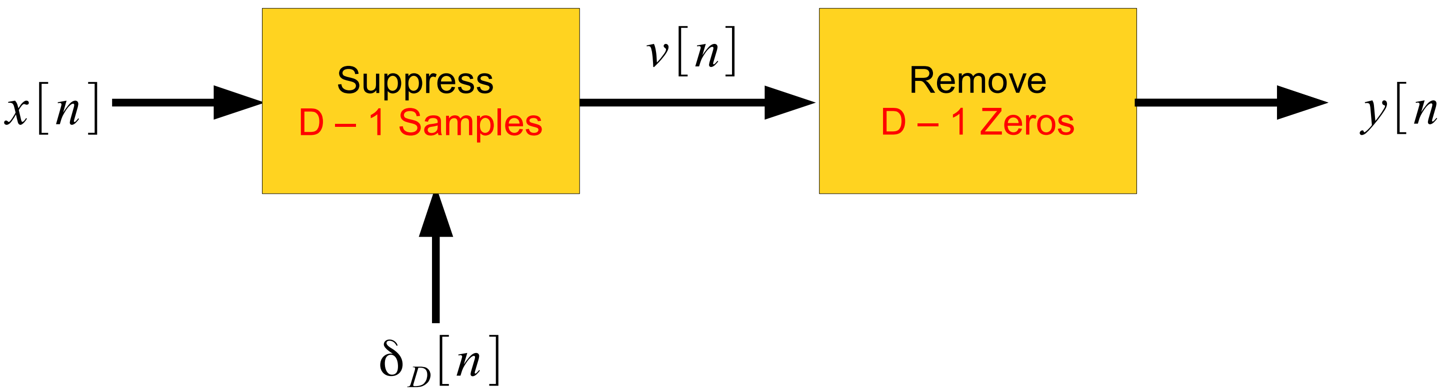






T **Sampling Period**

Ideal Sampling



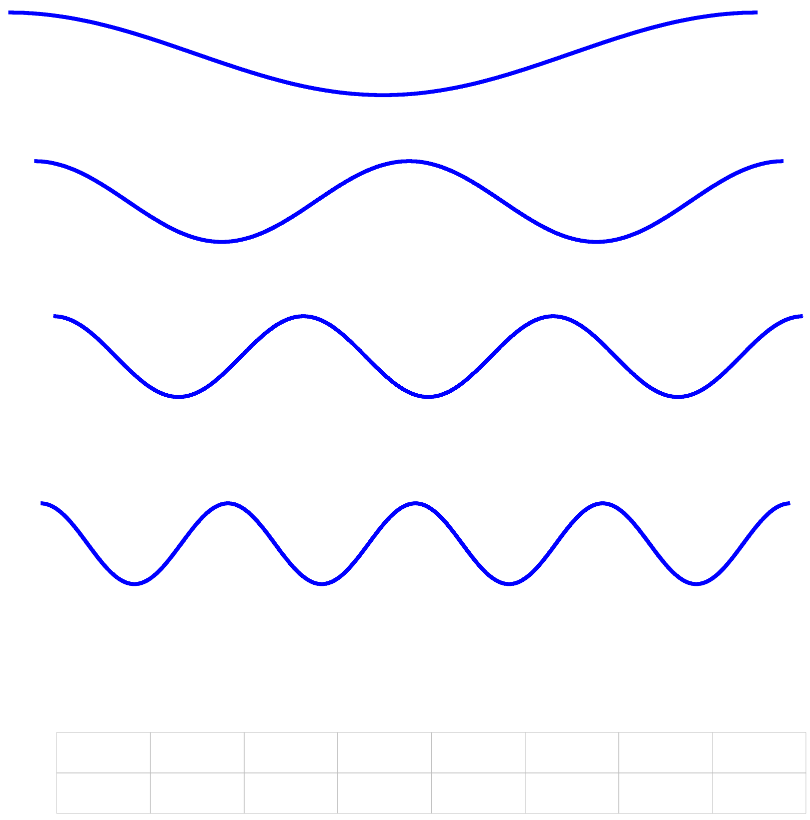
Amplitude Axis Scaling
1 / D

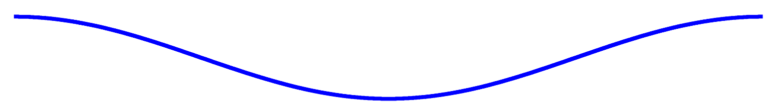
Time Axis Scaling
1 / D

Spectrum Replication
f_s / D

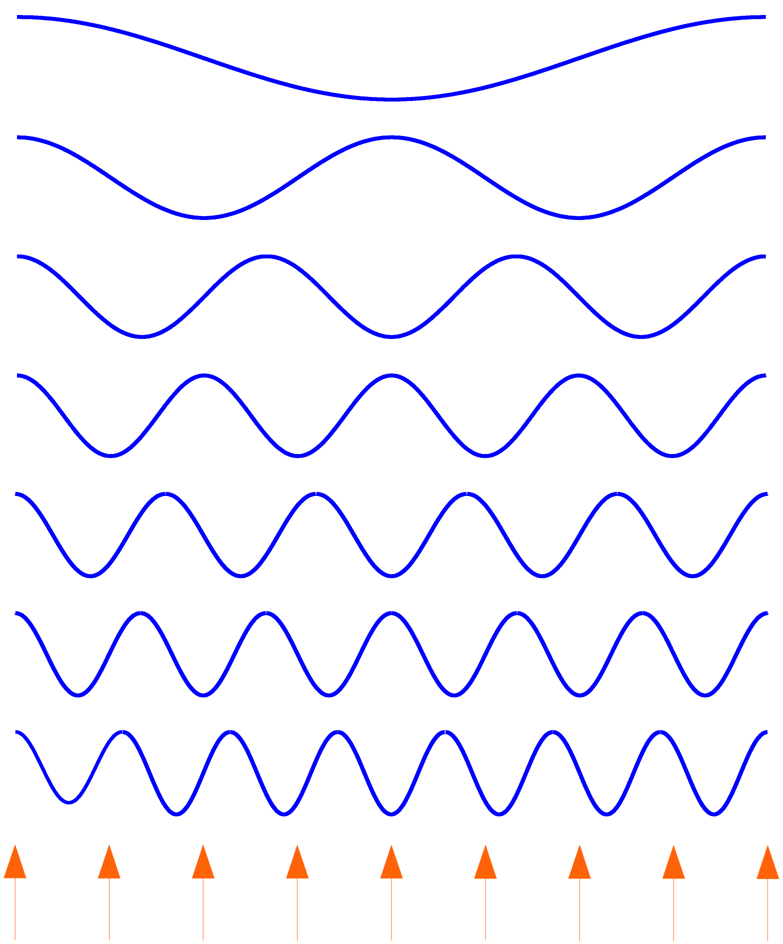
Stretching Spectrum
ω / D

Measuring Rotation Rate





Signals with Harmonic Frequencies (1)



1 Hz *1 cycle / sec*

$$\cos(1\cdot 2\pi t) = \frac{e^{+j(1\cdot 2\pi)t} + e^{-j(1\cdot 2\pi)t}}{2}$$

2 Hz *2 cycles / sec*

$$\cos(2\cdot 2\pi t) = \frac{e^{+j(2\cdot 2\pi)t} + e^{-j(2\cdot 2\pi)t}}{2}$$

3 Hz *3 cycles / sec*

$$\cos(3\cdot 2\pi t) = \frac{e^{+j(3\cdot 2\pi)t} + e^{-j(3\cdot 2\pi)t}}{2}$$

4 Hz *4 cycles / sec*

$$\cos(4\cdot 2\pi t) = \frac{e^{+j(4\cdot 2\pi)t} + e^{-j(4\cdot 2\pi)t}}{2}$$

5 Hz *5 cycles / sec*

$$\cos(5\cdot 2\pi t) = \frac{e^{+j(5\cdot 2\pi)t} + e^{-j(5\cdot 2\pi)t}}{2}$$

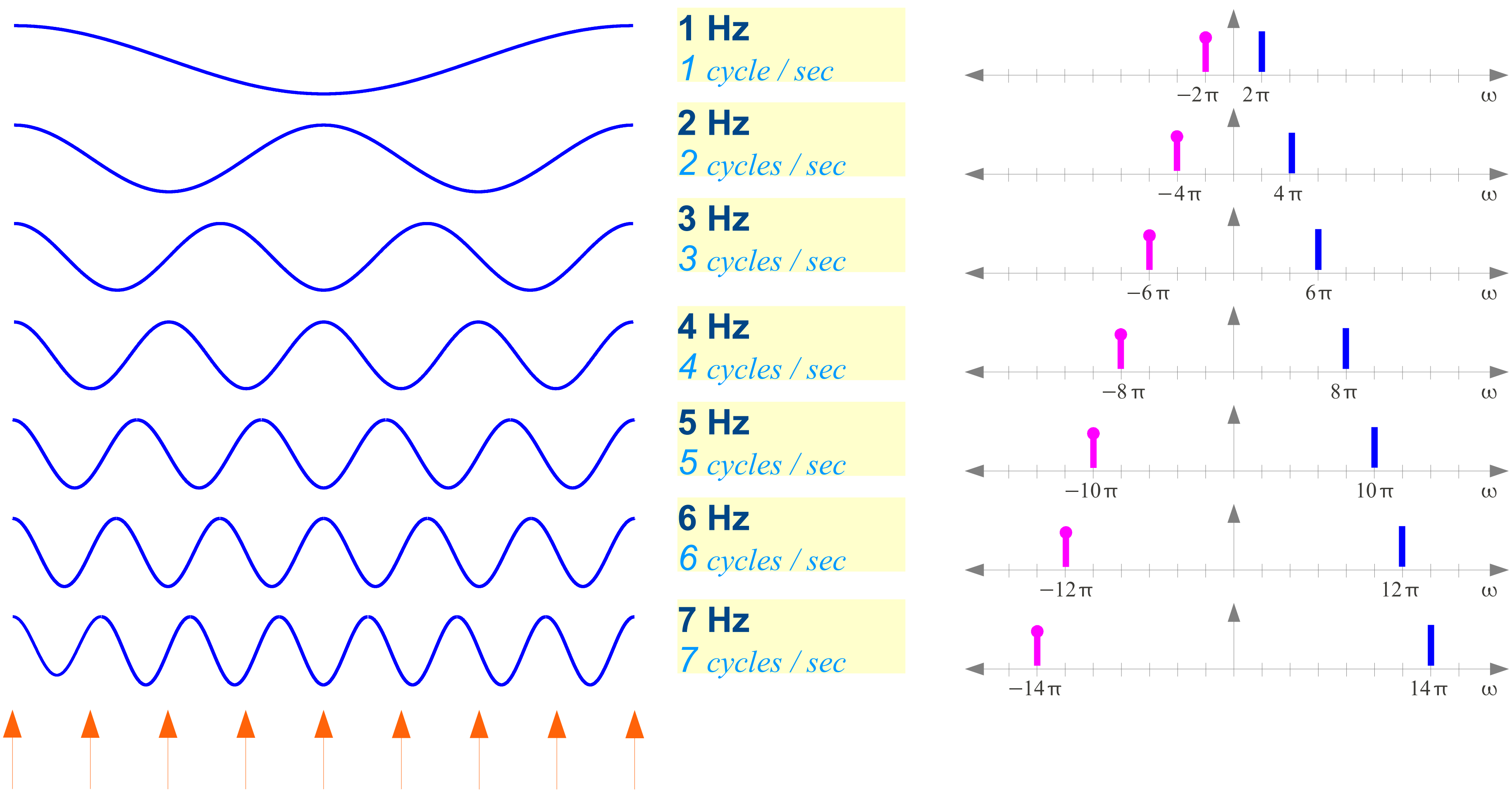
6 Hz *6 cycles / sec*

$$\cos(6\cdot 2\pi t) = \frac{e^{+j(6\cdot 2\pi)t} + e^{-j(6\cdot 2\pi)t}}{2}$$

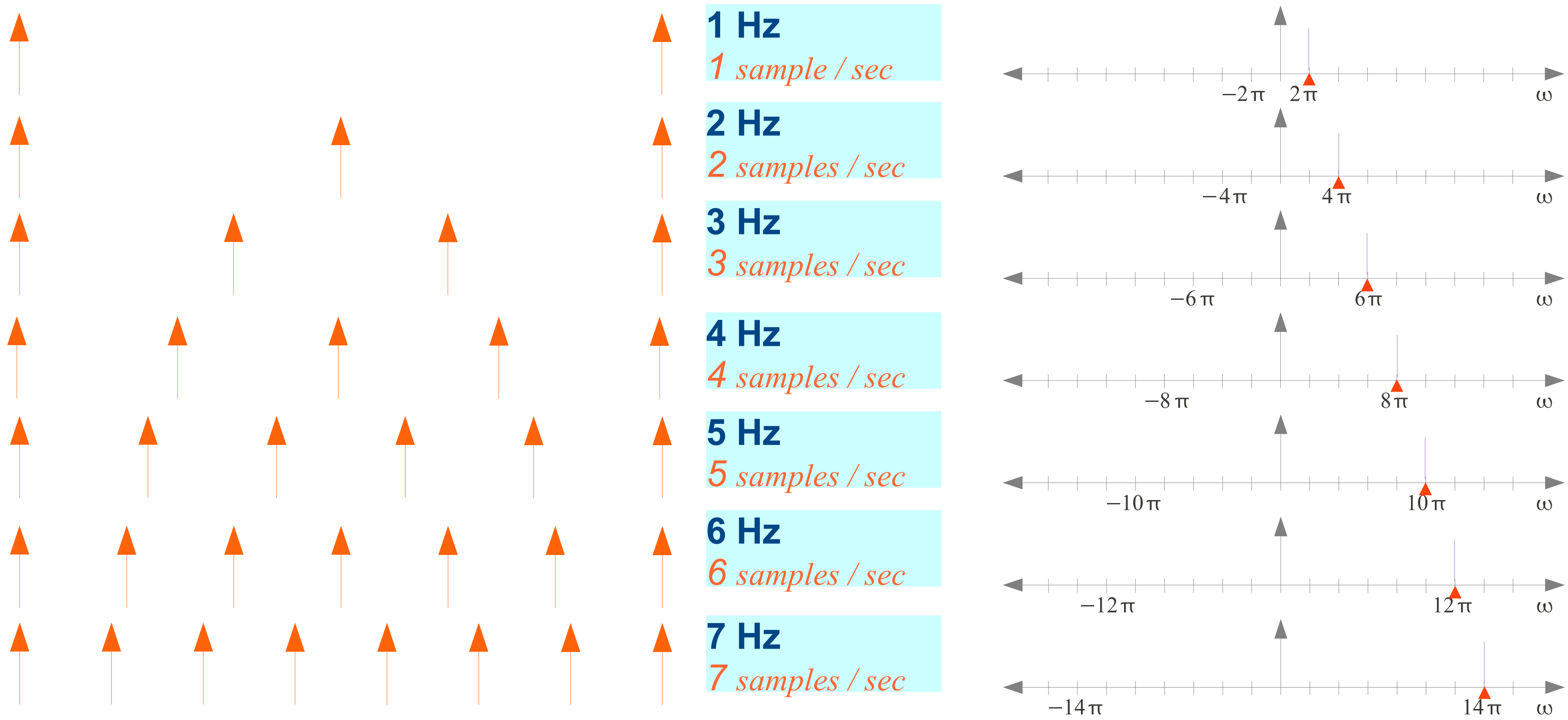
7 Hz *7 cycles / sec*

$$\cos(7\cdot 2\pi t) = \frac{e^{+j(7\cdot 2\pi)t} + e^{-j(7\cdot 2\pi)t}}{2}$$

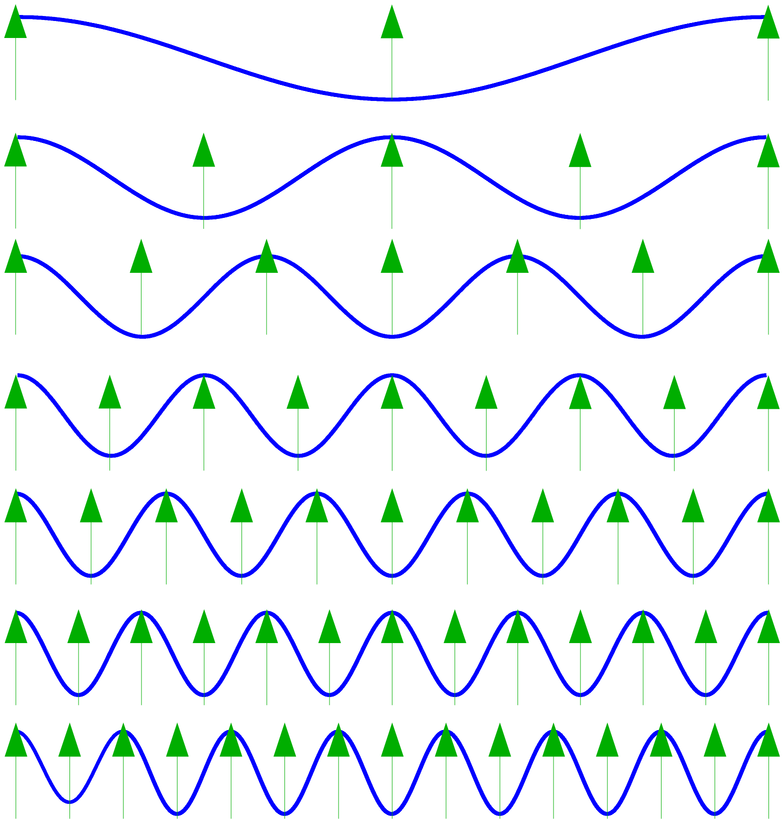
Signals with Harmonic Frequencies (2)



Sampling Frequency

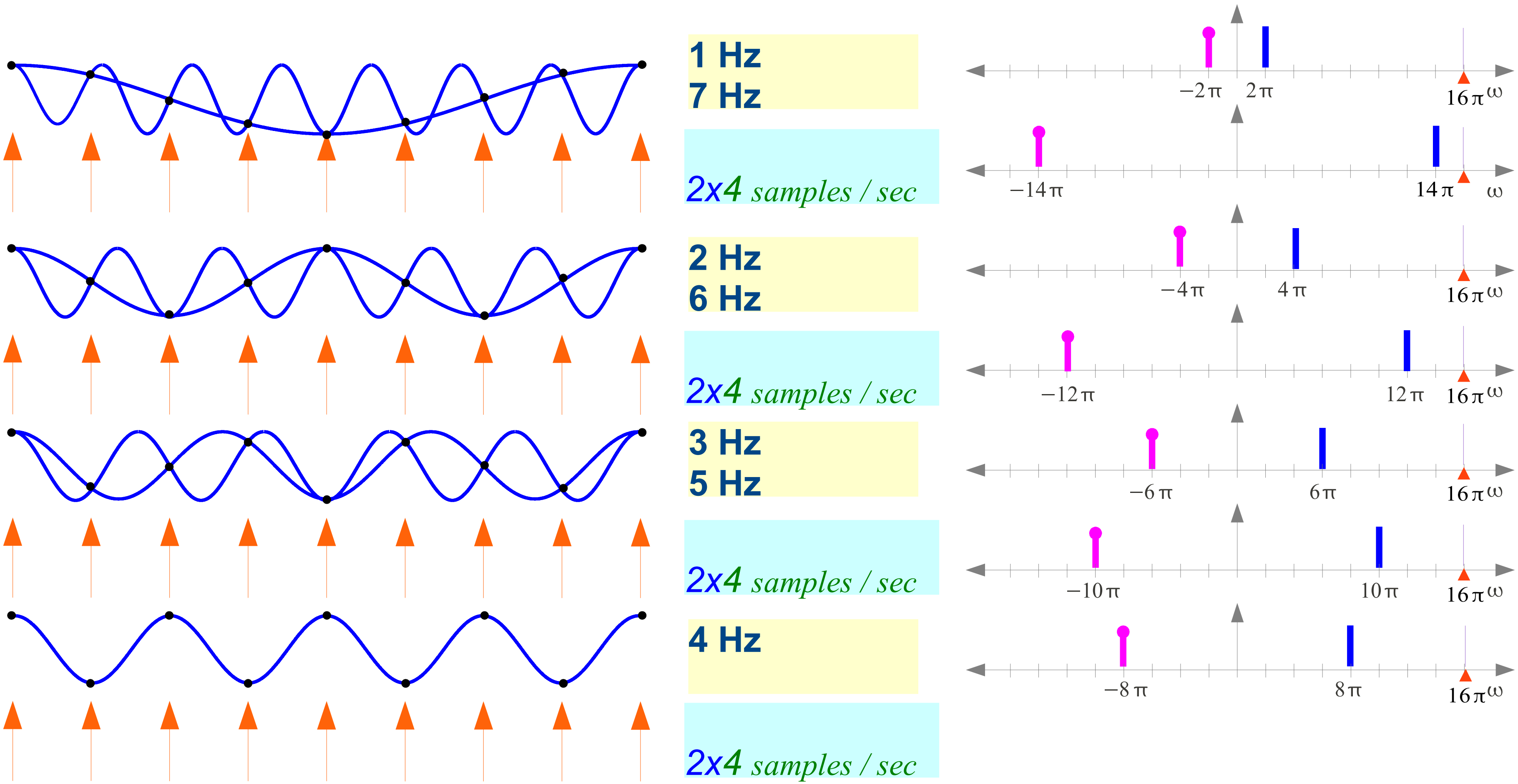


Nyquist Frequency



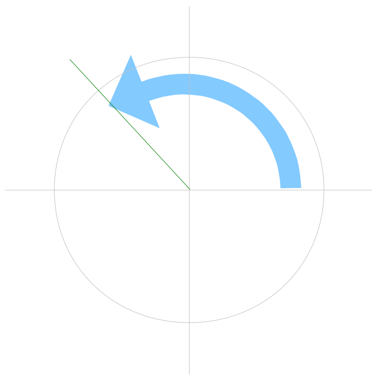
1 Hz *1 cycle / sec*	*2x1 sample / sec*
2 Hz *2 cycles / sec*	*2x2 samples / sec*
3 Hz *3 cycles / sec*	*2x3 samples / sec*
4 Hz *4 cycles / sec*	*2x4 samples / sec*
5 Hz *5 cycles / sec*	*2x5 samples / sec*
6 Hz *6 cycles / sec*	*2x6 samples / sec*
7 Hz *7 cycles / sec*	*2x7 samples / sec*

Aliasing


1 Hz
7 Hz
2x4 samples / sec
2 Hz
6 Hz
2x4 samples / sec
3 Hz
5 Hz
2x4 samples / sec
4 Hz
2x4 samples / sec
−2π 2π 16π ω
−14π 14π ω
−4π 4π 16π ω
−12π 12π 16π ω
−6π 6π 16π ω
−10π 10π 16π ω
−8π 8π 16π ω

Sampling

$$\omega_s = 2\pi f_s \ (rad/sec)$$



$$\omega_1 = 2\pi f_1$$

$$\omega_1 = \frac{\omega_s}{2} \ (rad/sec)$$

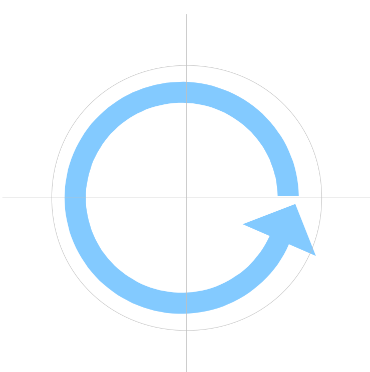
$$f_1 = \frac{f_s}{2} \ (rad/sec)$$

$$\omega_2 = 2\pi f_2$$

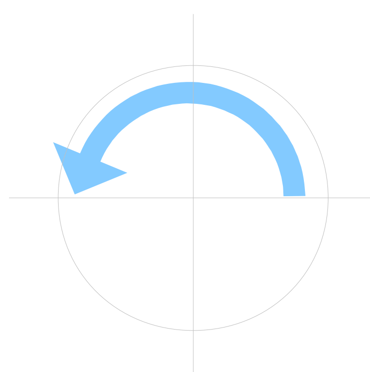
$$\omega_2 = -\frac{\omega_s}{2} \ (rad/sec)$$

$$f_2 = -\frac{f_s}{2} \ (rad/sec)$$

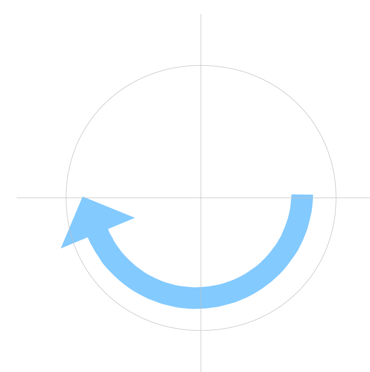
$$2\pi \ (rad) \ / \ T_s \ (sec)$$



$$\pi \ (rad) \ / \ T_s \ (sec)$$

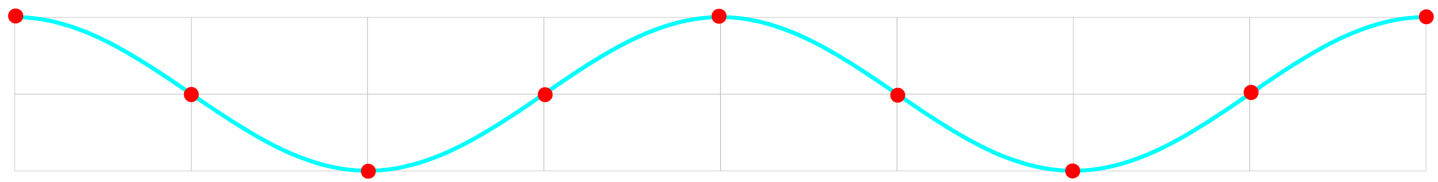


$$-\pi \ (rad) \ / \ T_s \ (sec)$$

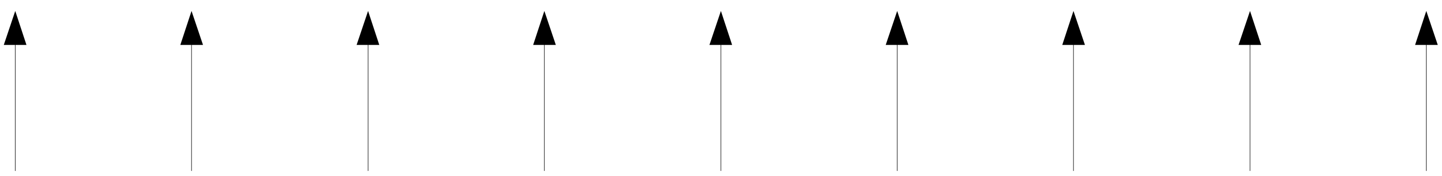


Sampling

$\omega_1 = 2\pi\, f_1 \;(rad/sec)$

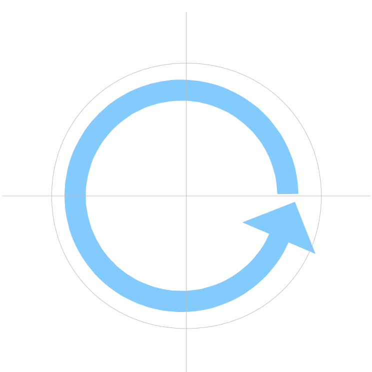


$\omega_s = 2\pi\, f_s \;(rad/sec)$

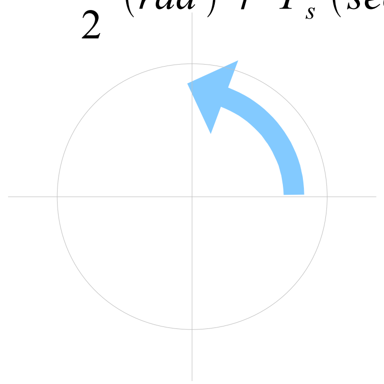


$T_s \;(= \tau)$

$2\pi\;(rad)\;/\;T_s\;(sec)$



$\frac{\pi}{2}\;(rad)\;/\;T_s\;(sec)$



For the period of T_s

Angular displacement $\frac{\pi}{2}\;(rad)$

$$\hat{\omega} = \omega \cdot T_s \;(rad)$$

$$= 2\pi f_1 \cdot T_s \;(rad)$$

$$= 2\pi \frac{f_s}{4} \cdot T_s \;(rad)$$

$$= \frac{\pi}{2} \;(rad)$$

Angular Frequencies in Sampling

continuous-time signals

Signal Frequency

$$f_0 = \frac{1}{T_0}$$

Signal Angular Frequency

$$\omega_0 = 2\pi f_0 \quad (rad/sec)$$

sampling sequence

Sampling Frequency

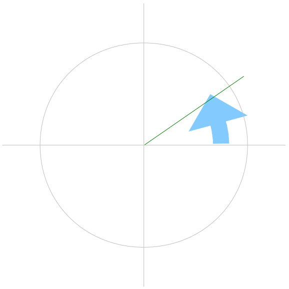
$$f_s = \frac{1}{T_s}$$

Sampling Angular Frequency

$$\omega_s = 2\pi f_s \quad (rad/sec)$$

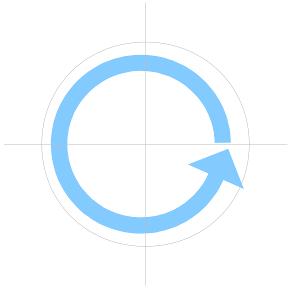
For 1 second

$\omega_0 = 2\pi f_0 \quad (rad/sec)$



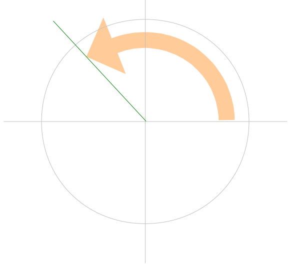
For 1 revolution

$2\pi \, (rad) \, / \, T_0 \, (sec)$



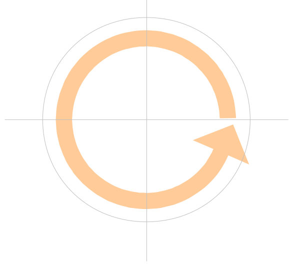
For 1 second

$\omega_s = 2\pi f_s \quad (rad/sec)$



For 1 revolution

$2\pi \, (rad) \, / \, T_s \, (sec)$



Young Won Lim
10/26/12

References

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[2] J.H. McClellan, et al., Signal Processing First, Pearson Prentice Hall, 2003
[3] A "graphical interpretation" of the DFT and FFT, by Steve Mann
[4] R. Cristi, "Modern Digital Signal Processing"