

# Matrices and Matrix Operations (3A)

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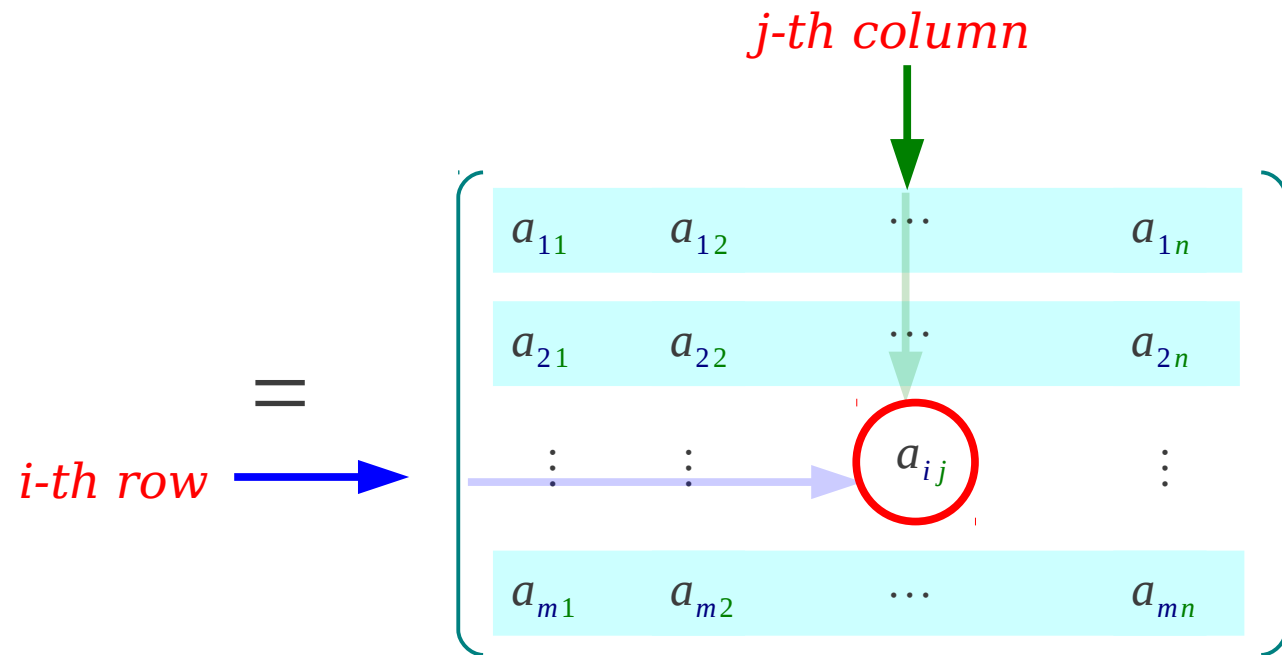
# A General $m \times n$ Matrix

$$A = [a_{ij}]_{m \times n} = [a_{ij}]$$

$$= \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}$$

# A Element of a Matrix

$$A = [a_{ij}]_{m \times n} = [a_{ij}]$$



$$(A)_{m \times n} = a_{ij}$$

# Matrix Multiplication (1)

1st row →

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix}$$

2nd row →

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix}$$

m-th row →

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix}$$

$$\mathbf{A} = [a_{ij}]_{m \times n}$$

$$\mathbf{x} = [x_{ij}]_{n \times 1}$$

$$\mathbf{b} = [b_{ij}]_{m \times 1}$$

$$\mathbf{A} \mathbf{x} = \mathbf{b}$$

$$(m) \times n \quad n \times (1) \quad (m \times 1)$$

# Matrix Multiplication (2)

1st row →

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \\ \vdots & \vdots \\ b_{n1} & b_{n2} \end{pmatrix} = \begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \\ \vdots & \vdots \\ c_{m1} & c_{m2} \end{pmatrix}$$

↓ 1st column

$$\mathbf{A} = [a_{ij}]_{m \times n}$$

$$\mathbf{B} = [b_{ij}]_{n \times 2}$$

$$\mathbf{C} = [c_{ij}]_{m \times 2}$$

2nd row →

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \\ \vdots & \vdots \\ b_{n1} & b_{n2} \end{pmatrix} = \begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \\ \vdots & \vdots \\ c_{m1} & c_{m2} \end{pmatrix}$$

↓ 1st column

$$\mathbf{A} \mathbf{B} = \mathbf{C}$$

$$(m \times n) (n \times 2) = (m \times 2)$$

# Matrix Multiplication (3)

1st row →

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \begin{pmatrix} b_{11} \\ b_{12} \\ b_{21} \\ b_{22} \\ \vdots \\ \vdots \\ b_{n1} \\ b_{n2} \end{pmatrix} = \begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \\ \vdots & \vdots \\ c_{m1} & c_{m2} \end{pmatrix}$$

2nd column

$$A = [a_{ij}]_{m \times n}$$

$$B = [b_{ij}]_{n \times 2}$$

$$C = [c_{ij}]_{m \times 2}$$

2nd row →

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \begin{pmatrix} b_{11} \\ b_{12} \\ b_{21} \\ b_{22} \\ \vdots \\ \vdots \\ b_{n1} \\ b_{n2} \end{pmatrix} = \begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \\ \vdots & \vdots \\ c_{m1} & c_{m2} \end{pmatrix}$$

2nd column

$$A B = C$$

$$(m \times n) (n \times 2) = (m \times 2)$$

# Matrix Multiplication (4)

*m-th row* →

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \\ \vdots & \vdots \\ b_{n1} & b_{n2} \end{pmatrix} = \begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \\ \vdots & \vdots \\ c_{m1} & c_{m2} \end{pmatrix}$$

*1st column*

$$A = [a_{ij}]_{m \times n}$$

$$B = [b_{ij}]_{n \times 2}$$

$$C = [c_{ij}]_{m \times 2}$$

*m-th row* →

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \\ \vdots & \vdots \\ b_{n1} & b_{n2} \end{pmatrix} = \begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \\ \vdots & \vdots \\ c_{m1} & c_{m2} \end{pmatrix}$$

*2nd column*

$$A \quad B = C$$

$$(m \times n) \quad (n \times 2) \quad (m \times 2)$$



# Matrix Multiplication (4)

*m-th row* →

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \\ \vdots & \vdots \\ b_{n1} & b_{n2} \end{pmatrix} = \begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \\ \vdots & \vdots \\ c_{m1} & c_{m2} \end{pmatrix}$$

*1st column*

$$A = [a_{ij}]_{m \times n}$$

$$B = [b_{ij}]_{n \times 2}$$

$$C = [c_{ij}]_{m \times 2}$$

*m-th row* →

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \\ \vdots & \vdots \\ b_{n1} & b_{n2} \end{pmatrix} = \begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \\ \vdots & \vdots \\ c_{m1} & c_{m2} \end{pmatrix}$$

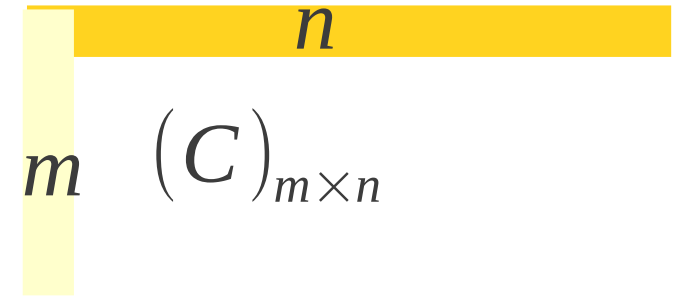
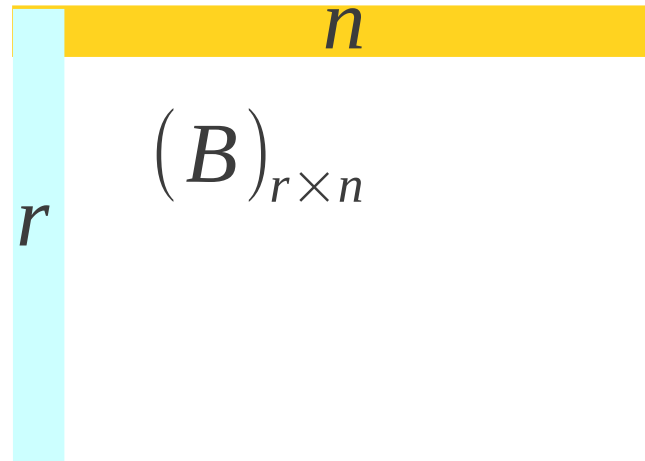
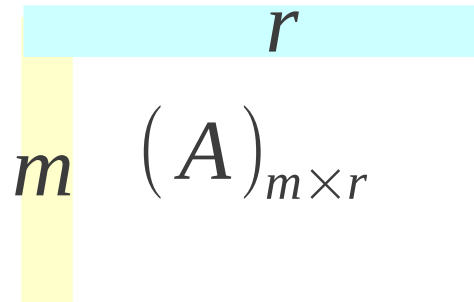
*2nd column*

$$A B = C$$

$$(m \times n) (n \times 2) = (m \times 2)$$

# Multiplication of Matrices (1)

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1r} \\ a_{21} & a_{22} & \cdots & a_{2r} \\ \vdots & \vdots & \cdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mr} \end{pmatrix} \begin{pmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2n} \\ \vdots & \vdots & \cdots & \vdots \\ b_{r1} & b_{r2} & \cdots & b_{rn} \end{pmatrix} = \begin{pmatrix} c_{11} & c_{12} & \cdots & c_{1n} \\ c_{21} & c_{22} & \cdots & c_{2n} \\ \vdots & \vdots & \boxed{c_{ii}} & \vdots \\ c_{r1} & c_{r2} & \cdots & c_{rn} \end{pmatrix}$$



# Multiplication of Matrices (2)

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1r} \\ a_{21} & a_{22} & \cdots & a_{2r} \\ \vdots & \vdots & \ddots & \vdots \\ a_{i1} & a_{i2} & \cdots & a_{ir} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mr} \end{pmatrix} \begin{pmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ b_{r1} & b_{r2} & \cdots & b_{rn} \end{pmatrix} = \begin{pmatrix} c_{11} & c_{12} & \cdots & c_{1n} \\ c_{21} & c_{22} & \cdots & c_{2n} \\ \vdots & \vdots & \boxed{c_{ij}} & \vdots \\ c_{r1} & c_{r2} & \cdots & c_{rn} \end{pmatrix}$$

$$(AB)_{ij} = c_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + a_{i3}b_{3j} \cdots + a_{ir}b_{rj}$$

$$= \sum_{k=1}^r a_{ik}b_{kj}$$

# Partitioned Matrix

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \\ \vdots & \vdots \\ b_{n1} & b_{n2} \end{pmatrix} = \begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \\ \vdots & \vdots \\ c_{m1} & c_{m2} \end{pmatrix}$$

$$A = [a_{ij}]_{m \times n}$$

$$B = [b_{ij}]_{n \times 2}$$

$$C = [c_{ij}]_{m \times 2}$$

$$\begin{matrix} B \\ n \times 2 \end{matrix} = \begin{bmatrix} \mathbf{b}_1 & \mathbf{b}_2 \end{bmatrix} \begin{matrix} n \times 1 & n \times 1 \end{matrix}$$

$$\begin{matrix} AB \\ m \times 2 \end{matrix} = \begin{bmatrix} A\mathbf{b}_1 & A\mathbf{b}_2 \end{bmatrix} \begin{matrix} m \times 1 & m \times 1 \end{matrix}$$

$$\begin{matrix} m \times n & n \times 2 & m \times n & n \times 1 & m \times n & n \times 1 \end{matrix}$$

# Partitioned Matrix

$$\begin{array}{l}
 \mathbf{a}_1 \rightarrow \\
 \mathbf{a}_2 \rightarrow \\
 \mathbf{a}_m \rightarrow
 \end{array}
 \begin{pmatrix}
 a_{11} & a_{12} & \cdots & a_{1n} \\
 a_{21} & a_{22} & \cdots & a_{2n} \\
 \vdots & \vdots & & \vdots \\
 a_{m1} & a_{m2} & \cdots & a_{mn}
 \end{pmatrix}
 \begin{pmatrix}
 b_{11} & b_{12} \\
 b_{21} & b_{22} \\
 \vdots & \vdots \\
 b_{n1} & b_{n2}
 \end{pmatrix}
 =
 \begin{pmatrix}
 c_{11} & c_{12} \\
 c_{21} & c_{22} \\
 \vdots & \vdots \\
 c_{m1} & c_{m2}
 \end{pmatrix}$$

$$\mathbf{A} = [a_{ij}]_{m \times n}$$

$$\mathbf{B} = [b_{ij}]_{n \times 2}$$

$$\mathbf{C} = [c_{ij}]_{m \times 2}$$

$$\begin{array}{l}
 \mathbf{A} \\
 m \times n
 \end{array}
 =
 \begin{bmatrix}
 \mathbf{a}_1 \\
 \mathbf{a}_2 \\
 \vdots \\
 \mathbf{a}_m
 \end{bmatrix}
 \begin{array}{l}
 1 \times n \\
 1 \times n \\
 \\
 1 \times n
 \end{array}
 \quad
 \begin{array}{l}
 \mathbf{A} \mathbf{B} \\
 m \times 2 \\
 m \times n \quad n \times 2
 \end{array}
 =
 \begin{bmatrix}
 \mathbf{a}_1 \\
 \mathbf{a}_2 \\
 \vdots \\
 \mathbf{a}_m
 \end{bmatrix}
 \begin{array}{l}
 1 \times 2 \quad 1 \times n \quad n \times 2 \\
 1 \times 2 \quad 1 \times n \quad n \times 2 \\
 \\
 1 \times 2 \quad 1 \times n \quad n \times 2
 \end{array}$$





## References

- [1] <http://en.wikipedia.org/>
- [2] Anton & Busby, "Contemporary Linear Algebra"
- [3] Anton & Rorres, "Elementary Linear Algebra"