CTFS (1B)

- Continuous Time Fourier Series

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Fourier Series

$$f(x) = a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

$$a_0 = \frac{1}{2\pi} \int_{-\pi}^{+\pi} f(x)\, dx$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{+\pi} f(x) \cos nx\, dx$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{+\pi} f(x) \sin nx\, dx$$

$$n = 1,\ 2,\ 3,\ \ldots$$

Trigonometric Identities

$$\cos\theta \cos\phi = \frac{1}{2}\left(\cos(\theta-\phi) + \cos(\theta+\phi)\right) \qquad \frac{1}{2}\left(1 + \cos(\theta+\phi)\right) \quad \textit{when} \quad \theta=\phi$$

$$\sin\theta \sin\phi = \frac{1}{2}\left(\cos(\theta-\phi) - \cos(\theta+\phi)\right) \qquad \frac{1}{2}\left(1 - \cos(\theta+\phi)\right) \quad \textit{when} \quad \theta=\phi$$

$$\sin\theta \cos\phi = \frac{1}{2}\left(\sin(\theta+\phi) + \sin(\theta-\phi)\right) \qquad \frac{1}{2}\left(\sin(\theta+\phi)\right) \quad \textit{when} \quad \theta=\phi$$

$$\cos\theta \sin\phi = \frac{1}{2}\left(\sin(\theta+\phi) - \sin(\theta-\phi)\right) \qquad \frac{1}{2}\left(\sin(\theta+\phi)\right) \quad \textit{when} \quad \theta=\phi$$

$$\int_{-\pi}^{+\pi} \cos \boldsymbol{n} x \cos \boldsymbol{m} x \, dx = 0 \quad (n \neq m)$$

$$\int_{-\pi}^{+\pi} \sin \boldsymbol{n} x \sin \boldsymbol{m} x \, dx = 0 \quad (n \neq m)$$

$$\int_{-\pi}^{+\pi} \sin \boldsymbol{n} x \cos \boldsymbol{m} x \, dx = 0$$

$$\int_{-\pi}^{+\pi} \cos \boldsymbol{n} x \sin \boldsymbol{m} x \, dx = 0$$

$$\int_{-\pi}^{+\pi} \cos \boldsymbol{n} x \cos \boldsymbol{m} x \, dx = \pi \quad (n = m)$$

$$\int_{-\pi}^{+\pi} \sin \boldsymbol{n} x \sin \boldsymbol{m} x \, dx = \pi \quad (n = m)$$

Trigonometric Orthogonality

$$f(x) = a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

$$a_0 = \frac{1}{2\pi} \int_{-\pi}^{+\pi} f(x)\, dx$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{+\pi} f(x) \cos nx\, dx$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{+\pi} f(x) \sin nx\, dx$$

$$n = 1, 2, 3, \ldots$$

$$\int_{-\pi}^{+\pi} \cos \boldsymbol{n} x \cos \boldsymbol{m} x\, dx = 0 \quad (n \neq m)$$

$$\int_{-\pi}^{+\pi} \sin \boldsymbol{n} x \sin \boldsymbol{m} x\, dx = 0 \quad (n \neq m)$$

$$\int_{-\pi}^{+\pi} \sin \boldsymbol{n} x \cos \boldsymbol{m} x\, dx = 0$$

$$\int_{-\pi}^{+\pi} \cos \boldsymbol{n} x \sin \boldsymbol{m} x\, dx = 0$$

$$\int_{-\pi}^{+\pi} \cos \boldsymbol{n} x \cos \boldsymbol{m} x\, dx = \pi \quad (n = m)$$

$$\int_{-\pi}^{+\pi} \sin \boldsymbol{n} x \sin \boldsymbol{m} x\, dx = \pi \quad (n = m)$$

$$a_{\boldsymbol{n}} \leftarrow f(x) \cdot \cos \boldsymbol{n} x = a_0 \cdot \cos \boldsymbol{n} x + \sum_{m=1}^{\infty} (a_m \cos \boldsymbol{m} x \cdot \cos \boldsymbol{n} x + b_m \sin \boldsymbol{m} x \cdot \cos \boldsymbol{n} x)$$

$$b_{\boldsymbol{n}} \leftarrow f(x) \cdot \sin \boldsymbol{n} x = a_0 \cdot \sin \boldsymbol{n} x + \sum_{m=1}^{\infty} (a_m \cos \boldsymbol{m} x \cdot \sin \boldsymbol{n} x + b_m \sin \boldsymbol{m} x \cdot \sin \boldsymbol{n} x)$$

Any Period $p = 2L$

$$g(v) = a_0 + \sum_{n=1}^{\infty} (a_n \cos nv + b_n \sin nv)$$

$$a_0 = \frac{1}{2\pi} \int_{-\pi}^{+\pi} g(v)\, dv$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{+\pi} g(v) \cos nv\, dv$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{+\pi} g(v) \sin nv\, dv$$

$$n = 1,\ 2,\ \ldots$$

$$v : [-\pi,\ +\pi]$$

$$f(x) = a_0 + \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi}{L} x + b_n \sin \frac{n\pi}{L} x\right)$$

$$a_0 = \frac{1}{2L} \int_{-L}^{+L} f(x)\, dx$$

$$a_n = \frac{1}{L} \int_{-L}^{+L} f(x) \cos \frac{n\pi x}{L}\, dx$$

$$b_n = \frac{1}{L} \int_{-L}^{+L} f(x) \sin \frac{n\pi x}{L}\, dx$$

$$n = 1,\ 2,\ 3,\ \ldots$$

$$x : [-L,\ +L]$$

$$v = \frac{\pi}{L} x$$

$$dv = \frac{\pi}{L} dx$$

Time and Frequency

$$f(x) = a_0 + \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi}{L} x + b_n \sin \frac{n\pi}{L} x \right)$$

$$a_0 = \frac{1}{2L} \int_{-L}^{+L} f(x)\, dx$$

$$a_n = \frac{1}{L} \int_{-L}^{+L} f(x) \cos \frac{n\pi x}{L}\, dx$$

$$b_n = \frac{1}{L} \int_{-L}^{+L} f(x) \sin \frac{n\pi x}{L}\, dx$$

$$n = 1,\ 2,\ 3,\ \ldots$$

$$x:\ [-L,\ +L]$$

$$x(t) = a_0 + \sum_{n=1}^{\infty} \left(a_n \cos \frac{2\pi n}{T} t + b_n \sin \frac{2\pi n}{T} t \right)$$

$$a_0 = \frac{1}{T} \int_{0}^{T} x(t)\, dt$$

$$a_n = \frac{2}{T} \int_{0}^{T} x(t) \cos \frac{2\pi n t}{T}\, dt$$

$$b_n = \frac{2}{T} \int_{0}^{T} x(t) \sin \frac{2\pi n t}{T}\, dt$$

$$n = 1,\ 2,\ \ldots$$

$$t:\ [0,\ T]$$

$$2L = T$$

Continuous Time Periodic Signal $x(t)$

Harmonic Frquency

$$x(t) = a_0 + \sum_{n=1}^{\infty}\left(a_n \cos\frac{2\pi n}{T}t + b_n \sin\frac{2\pi n}{T}t\right)$$

$$a_0 = \frac{1}{T}\int_0^T x(t)\,dt$$

$$a_n = \frac{2}{T}\int_0^T x(t)\cos\frac{2\pi n t}{T}\,dt$$

$$b_n = \frac{2}{T}\int_0^T x(t)\sin\frac{2\pi n t}{T}\,dt$$

$$n = 1,\ 2,\ \ldots$$

$$t: [0,\ T]$$

$$x(t) = a_0 + \sum_{n=1}^{\infty}\left(a_n \cos(n 2\pi f_0 t) + b_n \sin(n 2\pi f_0 t)\right)$$

$$a_0 = \frac{1}{T}\int_0^T x(t)\,dt$$

$$a_n = \frac{2}{T}\int_0^T x(t)\cos(n 2\pi f_0 t)\,dt \qquad n = 1,\ 2,\ \ldots$$

$$b_n = \frac{2}{T}\int_0^T x(t)\sin(n 2\pi f_0 t)\,dt \qquad n = 1,\ 2,\ \ldots$$

$$t: [0,\ T]$$

resolution frequency $\quad f_0 = \frac{1}{T}$

n-th harmonic frequency $\quad f_n = \boldsymbol{n} f_0 = \boldsymbol{n}\frac{1}{T}$

Radial Frequency

$$x(t) = a_0 + \sum_{n=1}^{\infty} \left(a_n \cos(n 2\pi f_0 t) + b_n \sin(n 2\pi f_0 t)\right)$$

$$a_0 = \frac{1}{T} \int_0^T x(t)\, dt$$

$$a_n = \frac{2}{T} \int_0^T x(t) \cos(n 2\pi f_0 t)\, dt \qquad n = 1, 2, \ldots$$

$$b_n = \frac{2}{T} \int_0^T x(t) \sin(n 2\pi f_0 t)\, dt \qquad n = 1, 2, \ldots$$

$$x(t) = a_0 + \sum_{n=1}^{\infty} \left(a_n \cos(n \omega_0 t) + b_n \sin(n \omega_0 t)\right)$$

$$a_0 = \frac{1}{T} \int_0^T x(t)\, dt$$

$$a_n = \frac{2}{T} \int_0^T x(t) \cos(n \omega_0 t)\, dt$$

$$b_n = \frac{2}{T} \int_0^T x(t) \sin(n \omega_0 t)\, dt$$

$$n = 1, 2, \ldots$$

$t: [0, T]$

$t: [0, T]$

linear frequency f

angular (radial) frequency $2\pi f$

Complex Fourier Series Coefficients

$$x(t) = a_0 + \sum_{n=1}^{\infty} \left(a_n \cos(n\omega_0 t) + b_n \sin(n\omega_0 t)\right)$$

$$a_0 = \frac{1}{T}\int_0^T x(t)\,dt$$

$$a_n = \frac{2}{T}\int_0^T x(t)\cos(n\omega_0 t)\,dt$$

$$b_n = \frac{2}{T}\int_0^T x(t)\sin(n\omega_0 t)\,dt$$

$$n = 1,\ 2,\ \ldots$$

$$t : [0,\ T]$$

$$x(t) = A_0 + \sum_{n=1}^{\infty} \left(A_n e^{jn\omega_0 t} + B_n e^{-jn\omega_0 t}\right)$$

$$A_0 = \frac{1}{T}\int_0^T x(t)\,dt$$

$$A_n = \frac{1}{T}\int_0^T x(t)\,e^{-jn\omega_0 t}\,dt$$

$$B_n = \frac{1}{T}\int_0^T x(t)\,e^{+jn\omega_0 t}\,dt$$

$$t : [0,\ T]$$

Real coefficients

$a_{0,}\ a_n,\ b_n,\quad n = 1,\ 2,\ \ldots$

→ **one-sided spectrum**
only positive frequencies

Complex coefficients

$A_{0,}\ A_n,\ B_n,\quad n = 1,\ 2,\ \ldots$

→ **two-sided spectrum**
Both pos and neg frequencies

Euler Equation (1)

$$x(t) = a_0 + \sum_{n=1}^{\infty} \left(a_n \cos(n\omega_0 t) + b_n \sin(n\omega_0 t)\right)$$

$$a_0 = \frac{1}{T}\int_0^T x(t)\, dt$$

$$a_n = \frac{2}{T}\int_0^T x(t) \cos(n\omega_0 t)\, dt$$

$$b_n = \frac{2}{T}\int_0^T x(t) \sin(n\omega_0 t)\, dt$$

$$n = 1, 2, \ldots$$

$$e^{+j\omega t} = \cos\omega t + j\sin\omega t$$

$$e^{-j\omega t} = \cos\omega t - j\sin\omega t$$

$$\cos\omega t = \frac{e^{j\omega t} + e^{-j\omega t}}{2}$$

$$\sin\omega t = \frac{e^{j\omega t} - e^{-j\omega t}}{2j}$$

$$a_n \cos(n\omega_0 t) + b_n \sin(n\omega_0 t)$$

$$= a_n \frac{1}{2}\left(e^{jn\omega_0 t} + e^{-jn\omega_0 t}\right) + b_n \frac{1}{2j}\left(e^{jn\omega_0 t} - e^{-jn\omega_0 t}\right)$$

$$= \frac{(a_n - jb_n)}{2} e^{jn\omega_0 t} + \frac{(a_n + jb_n)}{2} e^{-jn\omega_0 t}$$

$$= A_n e^{jn\omega_0 t} + B_n e^{-jn\omega_0 t}$$

$$x(t) = A_0 + \sum_{n=1}^{\infty} \left(A_n e^{jn\omega_0 t} + B_n e^{-jn\omega_0 t}\right)$$

$$A_0 = a_0$$

$$A_n = \frac{1}{2}(a_n - jb_n)$$

$$B_n = \frac{1}{2}(a_n + jb_n)$$

Euler Equation (2)

$$x(t) = a_0 + \sum_{n=1}^{\infty} \left(a_n \cos(n\omega_0 t) + b_n \sin(n\omega_0 t) \right)$$

$$a_0 = \frac{1}{T}\int_0^T x(t)\, dt$$

$$a_n = \frac{2}{T}\int_0^T x(t) \cos(n\omega_0 t)\, dt$$

$$b_n = \frac{2}{T}\int_0^T x(t) \sin(n\omega_0 t)\, dt$$

$$n = 1,\ 2,\ \ldots$$

$$x(t) = A_0 + \sum_{n=1}^{\infty} \left(A_n e^{jn\omega_0 t} + B_n e^{-jn\omega_0 t} \right)$$

$$A_0 = a_0$$

$$A_n = \frac{1}{2}(a_n - j b_n)$$

$$B_n = \frac{1}{2}(a_n + j b_n)$$

$$A_n = \frac{1}{T}\int_0^T x(t) \left(\cos(n\omega_0 t) - j \sin(n\omega_0 t) \right) dt$$

$$B_n = \frac{1}{T}\int_0^T x(t) \left(\cos(n\omega_0 t) + j \sin(n\omega_0 t) \right) dt$$

$$A_n = \frac{1}{T}\int_0^T x(t)\, e^{-jn\omega_0 t}\, dt$$

$$B_n = \frac{1}{T}\int_0^T x(t)\, e^{+jn\omega_0 t}\, dt$$

$$x(t) = A_0 + \sum_{n=1}^{\infty} \left(A_n e^{+jn\omega_0 t} + B_n e^{-jn\omega_0 t} \right)$$

$$x(t) = \sum_{n=0}^{\infty} \left(A_n e^{+jn\omega_0 t} + B_n e^{-jn\omega_0 t} \right)$$

Complex Fourier Series

$$x(t) = a_0 + \sum_{n=1}^{\infty} \left(a_n \cos(n\omega_0 t) + b_n \sin(n\omega_0 t)\right)$$

$$a_0 = \frac{1}{T}\int_0^T x(t)\,dt$$

$$a_n = \frac{2}{T}\int_0^T x(t)\cos(n\omega_0 t)\,dt$$

$$b_n = \frac{2}{T}\int_0^T x(t)\sin(n\omega_0 t)\,dt$$

$$n = 1, 2, \ldots$$

$$x(t) = \sum_{n=0}^{\infty} \left(A_n e^{+jn\omega_0 t} + B_n e^{-jn\omega_0 t}\right)$$

$$A_n = \frac{1}{T}\int_0^T x(t)\, e^{-jn\omega_0 t}\,dt$$

$$n = 0, 1, 2, \ldots$$

$$B_n = \frac{1}{T}\int_0^T x(t)\, e^{+jn\omega_0 t}\,dt$$

$$n = 1, 2, \ldots$$

$$x(t) = A_0 + \sum_{n=1}^{\infty} \left(A_n e^{jn\omega_0 t} + B_n e^{-jn\omega_0 t}\right)$$

$$A_0 = a_0$$

$$A_n = \frac{1}{2}(a_n - jb_n)$$

$$B_n = \frac{1}{2}(a_n + jb_n)$$

$$n = 1, 2, \ldots$$

$$x(t) = \sum_{n=-\infty}^{+\infty} C_n e^{+jn\omega_0 t}$$

$$C_n = \frac{1}{T}\int_0^T x(t)\, e^{-jn\omega_0 t}\,dt$$

$$n = -2, -1, 0, +1, +2, \ldots$$

$$C_n = \begin{cases} A_0 & (n = 0) \\ A_n & (n > 0) \\ B_n & (n < 0) \end{cases}$$

Single-Sided Spectrum

$$x(t) = a_0 + \sum_{n=1}^{\infty} \left(a_n \cos(n\omega_0 t) + b_n \sin(n\omega_0 t)\right)$$

$$a_0 = \frac{1}{T} \int_0^T x(t)\, dt$$

$$a_n = \frac{2}{T} \int_0^T x(t) \cos(n\omega_0 t)\, dt$$

$$b_n = \frac{2}{T} \int_0^T x(t) \sin(n\omega_0 t)\, dt$$

$$n = 1, 2, \ldots$$

$$x(t) = g_0 + \sum_{n=1}^{\infty} g_n \cos(n\omega_0 t + \phi_n)$$

$$g_0 = a_0$$

$$g_n = \sqrt{a_n^2 + b_n^2}$$

$$\phi_n = \tan^{-1}\left(-\frac{b_n}{a_n}\right)$$

$$n = 1, 2, \ldots$$

$$\cos(\alpha \pm \beta) = \cos(\alpha)\cos(\beta) \mp \sin(\alpha)\sin(\beta)$$

$$g_n \cos(n\omega_0 t + \phi_n) = g_n \cos(\phi_n)\cos(n\omega_0 t) - g_n \sin(\phi_n)\sin(n\omega_0 t)$$

Two-Sided Spectrum

$$x(t) = a_0 + \sum_{n=1}^{\infty} \left(a_n \cos(n\omega_0 t) + b_n \sin(n\omega_0 t)\right)$$

$$a_0 = \frac{1}{T}\int_0^T x(t)\,dt$$

$$a_n = \frac{2}{T}\int_0^T x(t)\cos(n\omega_0 t)\,dt$$

$$b_n = \frac{2}{T}\int_0^T x(t)\sin(n\omega_0 t)\,dt$$

$$n = 1,\ 2,\ \ldots$$

$$x(t) = A_0 + \sum_{n=1}^{\infty} \left(A_n e^{jn\omega_0 t} + B_n e^{-jn\omega_0 t}\right)$$

$$A_0 = a_0$$

$$A_n = \frac{1}{2}(a_n - j b_n)$$

$$B_n = \frac{1}{2}(a_n + j b_n)$$

$$x(t) = \sum_{n=-\infty}^{+\infty} C_n e^{+jn\omega_0 t}$$

$$C_n = \frac{1}{T}\int_0^T x(t)\, e^{-jn\omega_0 t}\,dt$$

$$n = -2,\ -1,\ 0,\ +1,\ +2,\ \ldots$$

$$C_n = \begin{cases} A_0 & (n = 0) \\ A_n & (n > 0) \\ B_n & (n < 0) \end{cases}$$

$$|C_n| = \frac{A_n}{2} \quad (n \neq 0)$$

$$Arg(C_n) = \begin{cases} +\phi_n = \tan^{-1}\left(\dfrac{-b_n}{a_n}\right) & (n > 0) \\ -\phi_n = \tan^{-1}\left(\dfrac{-b_n}{a_n}\right) & (n < 0) \end{cases}$$

Euler Equation (1)

$$e^{j\omega t} = \cos\omega t + j\sin\omega t$$
$$e^{-j\omega t} = \cos\omega t - j\sin\omega t$$

$$\cos\omega t = \frac{e^{j\omega t} + e^{-j\omega t}}{2}$$

$$\sin\omega t = \frac{e^{j\omega t} - e^{-j\omega t}}{2j}$$

$$\begin{aligned}
& a_n \cos(n\omega_0 t) + b_n \sin(n\omega_0 t) \\
&= a_n \frac{1}{2}(e^{jn\omega_0 t} + e^{-jn\omega_0 t}) + b_n \frac{1}{2j}(e^{jn\omega_0 t} - e^{-jn\omega_0 t}) \\
&= \frac{1}{2}(a_n - jb_n)\, e^{jn\omega_0 t} + \frac{1}{2}(a_n + jb_n)\, e^{-jn\omega_0 t} \\
&= \frac{1}{2}(a_n - jb_n)\, e^{jn\omega_0 t} + \frac{1}{2}(a_n + jb_n)\, e^{-jn\omega_0 t} \\
&= A_n\, e^{jn\omega_0 t} + B_n\, e^{-jn\omega_0 t}
\end{aligned}$$

$$x(t) = A_0 + \sum_{n=1}^{\infty}\left(A_n\, e^{jn\omega_0 t} + B_n\, e^{-jn\omega_0 t}\right)$$

Complex Fourier Series

$$x(t) = A_0 + \sum_{n=1}^{\infty} \left(A_n \, e^{+jn\omega_0 t} + B_n \, e^{-jn\omega_0 t} \right)$$

$$= \sum_{n=0}^{\infty} \left(A_n \, e^{+jn\omega_0 t} + B_n \, e^{-jn\omega_0 t} \right)$$

$$A_n = \frac{1}{T} \int_0^T x(t) \, e^{-jn\omega_0 t} \, dt$$

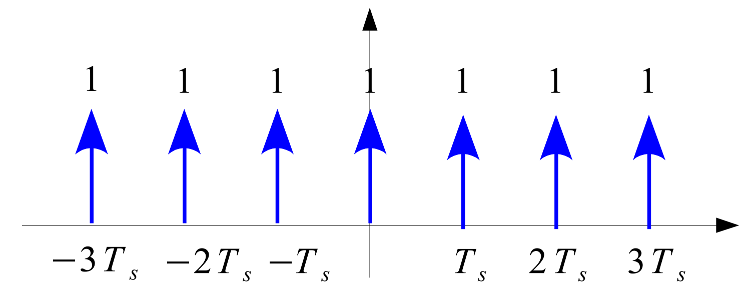
$$B_n = \frac{1}{T} \int_0^T x(t) \, e^{+jn\omega_0 t} \, dt$$

$$x(t) = \sum_{n=0}^{\infty} C_n \, e^{+jn\omega_0 t}$$

$$C_n = \frac{1}{T} \int_0^T x(t) \, e^{-jn\omega_0 t} \, dt$$

CTFS of Impulse Train (1)

$$p(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT_s)$$



Fourier Series Expansion of Impulse Train

$$p(t) = \sum_{k=-\infty}^{+\infty} a_k \, e^{+jk\omega_s t}$$

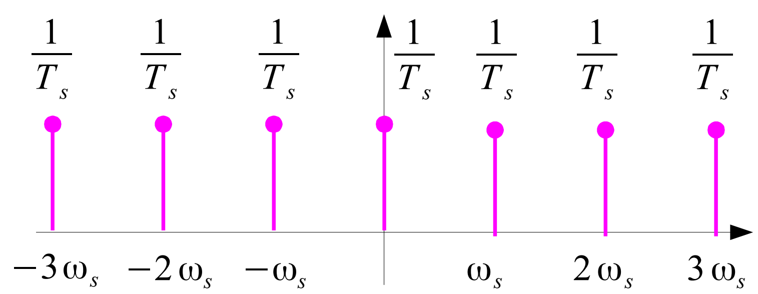
$$\omega_s = \frac{2\pi}{T_s}$$

Fourier Series Coefficients

$$a_k = \frac{1}{T_s} \int_{-T_s/2}^{+T_s/2} \delta(t)\, e^{-jk\omega_s t}\, dt$$

$$= \frac{1}{T_s} \int_{-T_s/2}^{+T_s/2} \delta(t)\, e^{-jk\omega_s 0}\, dt$$

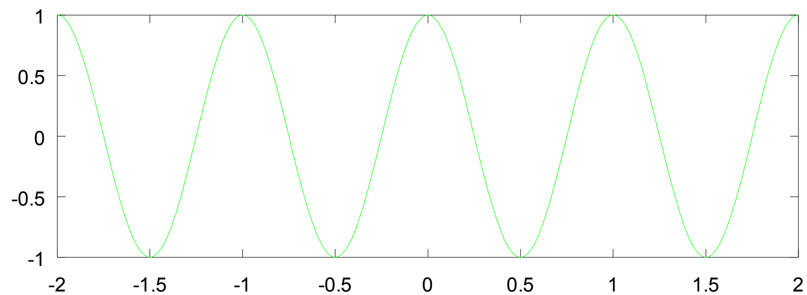
$$= \frac{1}{T_s} \int_{-T_s/2}^{+T_s/2} \delta(t)\, dt = \frac{1}{T_s}$$

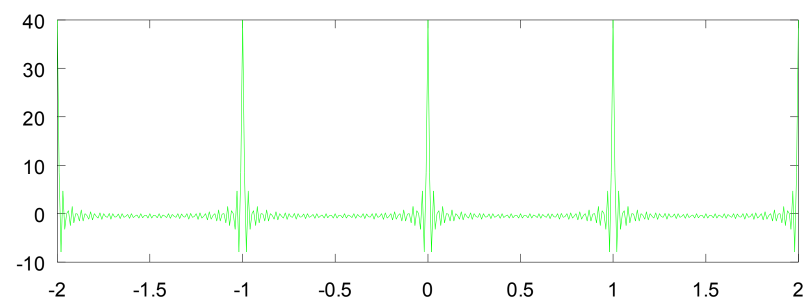


CTFS of Impulse Train (2)

$$p(t) = \frac{1}{T_s} \sum_{k=-\infty}^{+\infty} a_k \, e^{+jk\omega_s t} = \frac{1}{T_s} \sum_{k=-\infty}^{+\infty} \left(\cos k\omega_s t - j \sin k\omega_s t \right)$$

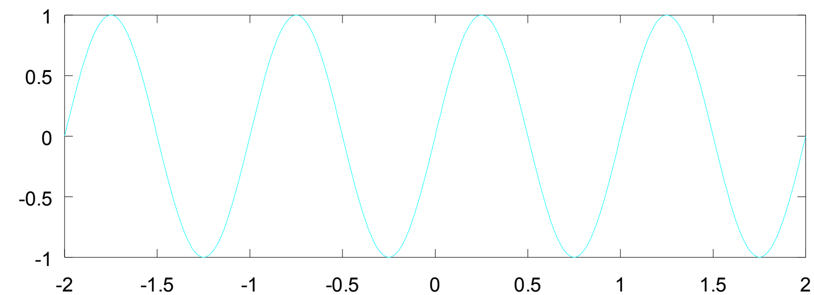
$\cos 2\pi \cdot 1 \cdot t$

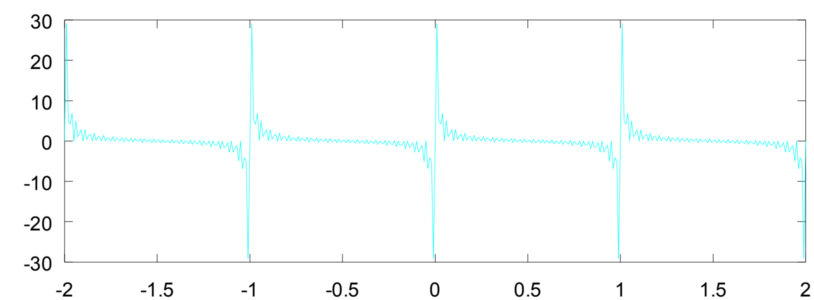




$$\sum_{k=1}^{40} \cos 2\pi \cdot k \cdot t$$

$\sin 2\pi \cdot 1 \cdot t$



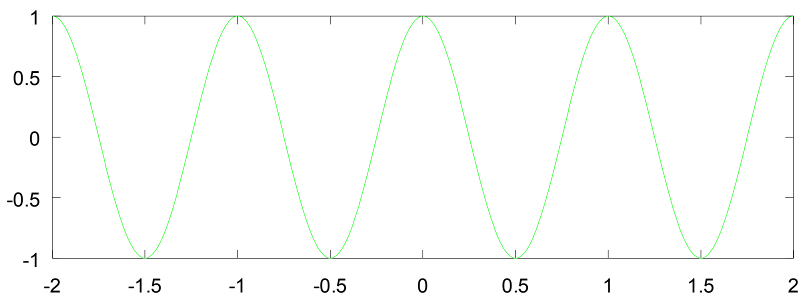


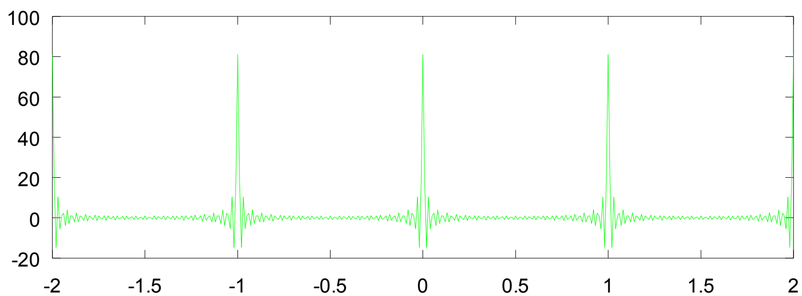
$$\sum_{k=1}^{40} \sin 2\pi \cdot k \cdot t$$

CTFS of Impulse Train (3)

$$p(t) = \frac{1}{T_s} \sum_{k=-\infty}^{+\infty} a_k \, e^{+jk\omega_s t} = \frac{1}{T_s} \sum_{k=-\infty}^{+\infty} \left(\cos k\omega_s t - j \sin k\omega_s t \right)$$

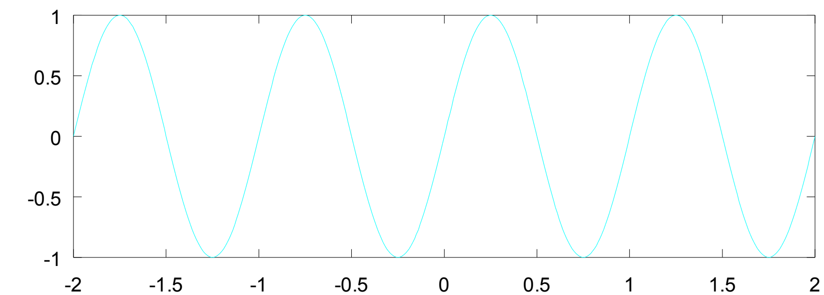
$\cos 2\pi \cdot 1 \cdot t$

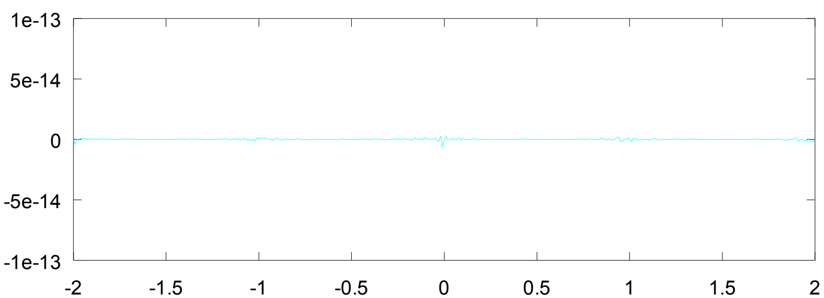




$$\sum_{k=-40}^{40} \cos 2\pi \cdot k \cdot t$$

$\sin 2\pi \cdot 1 \cdot t$





$$\sum_{k=-40}^{40} \sin 2\pi \cdot k \cdot t$$

References

[1] http://en.wikipedia.org/
[2] J.H. McClellan, et al., Signal Processing First, Pearson Prentice Hall, 2003
[3] G. Beale, http://teal.gmu.edu/~gbeale/ece_220/fourier_series_02.html