CORDIC in VHDL (1A)

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Please send corrections (or suggestions) to youngwlim@hotmail.com.
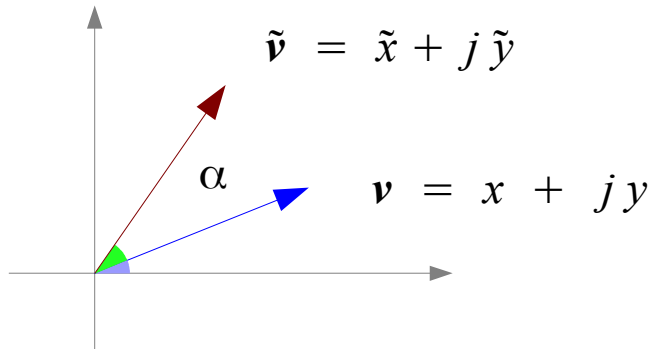
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CORDIC Background

1. G Hampson,
A VHDL Implementation of a CORDIC Arithmetic Processor Chip
Monash University, Technical Report 94-9, 1994

Angle Expansion

$$\tilde{v} = v\, e^{j\alpha}$$

$$\tilde{v} = \tilde{x} + j\,\tilde{y}$$

α $\qquad$ $v = x + jy$

elementary angle

$$\theta_0 = \tan^{-1}(2^0) =$$
$$\theta_1 = \tan^{-1}(2^{-1}) =$$
$$\theta_2 = \tan^{-1}(2^{-2}) =$$
$$\theta_3 = \tan^{-1}(2^{-3}) =$$

α $\quad$ can be expanded by

a set of elementary angles $\qquad \alpha_i$

pseudo-digits $\qquad\qquad\qquad q_i$

$$\alpha_i \begin{cases} \pi/2 & i = -1 \\ \\ \tan^{-1}(2^{-i}) & i = 0,\,1,\,2,\,\cdots,\,n-1 \end{cases}$$

$$q_i \begin{cases} -1 \\ \\ +1 \end{cases}$$

$$\alpha = \sum_{i=-1}^{n-1} q_i \cdot \alpha_i + z_n$$

angle expansion error $\qquad z_n$

$$|z_n| \le 2^{-(n-1)}$$

Rotating Vector

$$\tilde{\boldsymbol{v}} = \boldsymbol{v}\, e^{j\alpha}$$

$$= \boldsymbol{v}\, \exp\left(j\left(\sum_{i=-1}^{n-1} q_i \cdot \alpha_i + z_n \right) \right)$$

$$= \boldsymbol{v} \cdot \left(\prod_{i=-1}^{n-1} e^{j\, q_i\, \alpha_i} \right) \cdot e^{jz_n}$$

$$= \boldsymbol{v} \cdot (jq_{-1}) \left(\prod_{i=0}^{n-1} e^{j\, q_i\, \alpha_i} \right) \cdot e^{jz_n}$$

$$= \boldsymbol{v} \cdot (jq_{-1}) \left(\prod_{i=0}^{n-1} \cos(\alpha_i) \cdot (1 + j\, q_i\, 2^{-i}) \right) \cdot e^{jz_n}$$

$$= \boldsymbol{v} \cdot (jq_{-1}) \left(\prod_{i=0}^{n-1} \cos(\alpha_i) \right) \cdot \left(\prod_{i=0}^{n-1} (1 + j\, q_i\, 2^{-i}) \right) \cdot e^{jz_n}$$

⬅ $$\alpha = \sum_{i=-1}^{n-1} q_i \cdot \alpha_i + z_n$$

⬅ $$e^{j\theta} = \cos\theta + j\sin\theta$$

⬅ $$e^{j\, q_{-1}\, \alpha_0} = e^{j\, q_{-1}\frac{\pi}{2}} = jq_{-1} \quad \left(e^{\pm j\frac{\pi}{2}} = \pm j \right)$$

⬅ $$e^{j\, q_i\, \alpha_i} = \cos(q_i\, \alpha_i) + j\sin(q_i\, \alpha_i)$$

$$= \cos(q_i\, \alpha_i) \cdot (1 + j\tan(q_i\, \alpha_i))$$

$$= \cos(q_i\, \alpha_i) \cdot (1 + j\, q_i\, 2^{-i})$$

$$= \cos(\alpha_i) \cdot (1 + j\, q_i\, 2^{-i})$$

$$(\cos(\pm\alpha_i) = \cos(\alpha_i))$$

Rotating Vector

$$\tilde{\boldsymbol{v}} = \boldsymbol{v}\, e^{j\alpha} \quad \text{rotated by } \alpha$$

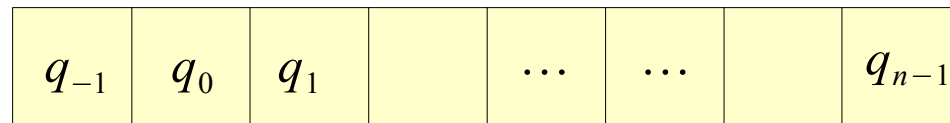$$\alpha = \sum_{i=-1}^{n-1} q_i \cdot \alpha_i + z_n$$

$$= \boldsymbol{v} \cdot \left(\prod_{i=0}^{n-1} \cos(\alpha_i) \right) \cdot (j q_{-1}) \cdot \left(\prod_{i=0}^{n-1} \left(1 + j\, q_i\, 2^{-i}\right) \right) \cdot e^{j z_n}$$

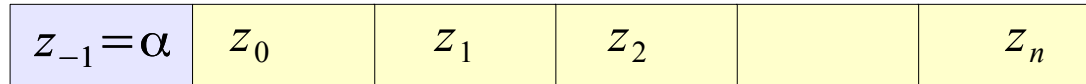$$K_n = \prod_{i=0}^{n-1} \frac{1}{\sqrt{1 + 2^{-2i}}}$$

series rotations of α_i

n iterations

q_{-1}	q_0	q_1		$\cdots$	$\cdots$		q_{n-1}

Rotating Vector

$$((((\alpha - q_{-1}\alpha_{-1}) - q_0\alpha_0) - q_1\alpha_1) \quad \cdots \quad - q_{n-1}\alpha_{n-1})$$

$z_{-1} = \alpha$	z_0	z_1	z_2		z_n

$\Leftarrow$ $z_{i+1} = z_i - q_i \alpha_i$

$if \ (z_0 \geq 0) \quad q_0 = +1$
$if \ (z_0 < 0) \quad q_0 = -1$

$if \ (z_1 \geq 0) \quad q_1 = +1$
$if \ (z_1 < 0) \quad q_1 = -1$

$if \ (z_2 \geq 0) \quad q_2 = +1$
$if \ (z_2 < 0) \quad q_2 = -1$

$if \ (z_i \geq 0) \quad q_i = +1$
$if \ (z_i < 0) \quad q_i = -1$

$if \ (z_{-1} \geq 0) \quad q_{-1} = +1$
$if \ (z_{-1} < 0) \quad q_{-1} = -1$

$if \ (z_{n-1} \geq 0) \quad q_{n-1} = +1$
$if \ (z_{n-1} < 0) \quad q_{n-1} = -1$

q_{-1}	q_0	q_1	q_2		q_{n-1}

Rotating Vector

$$\tilde{v} = v \, e^{j\alpha} \qquad \text{rotated by } \alpha$$

$$\alpha = \sum_{i=-1}^{n-1} q_i \cdot \alpha_i + z_n$$

Rotated vector after n iteration

$$v_n = v \cdot e^{j\alpha} \cdot e^{-jz_n}$$

$$z_n = \alpha - \sum_{i=-1}^{n-1} q_i \cdot \alpha_i$$

$$v_{i+1} = v_i \cdot \left(1 + j\,q_i \cdot 2^{-i}\right)$$

$$v_0 = v_{-1} \cdot \left(j\,q_{-1}\right)$$

$$x_{i+1} + jy_{i+1} = (x_i + jy_i) \cdot \left(1 + j\,q_i \cdot 2^{-i}\right)$$

$$x_0 + jy_0 = (x_{-1} + jy_{-1}) \cdot \left(j\,q_{-1}\right)$$

$$= \left(x_i - y_i \cdot q_i \cdot 2^{-i}\right) + j\left(y_i + x_i \cdot q_i \cdot 2^{-i}\right)$$

$$= \left(-q_{-1} \cdot y_{-1}\right) + j\left(q_{-1} \cdot x_{-1}\right)$$

$$\begin{cases} x_{i+1} = \left(x_i - y_i \cdot q_i \cdot 2^{-i}\right) \\ y_{i+1} = \left(y_i + x_i \cdot q_i \cdot 2^{-i}\right) \end{cases}$$

$$\begin{cases} x_0 = -q_{-1} \cdot y_{-1} \\ y_0 = q_{-1} \cdot x_{-1} \end{cases}$$

Rotating Vector
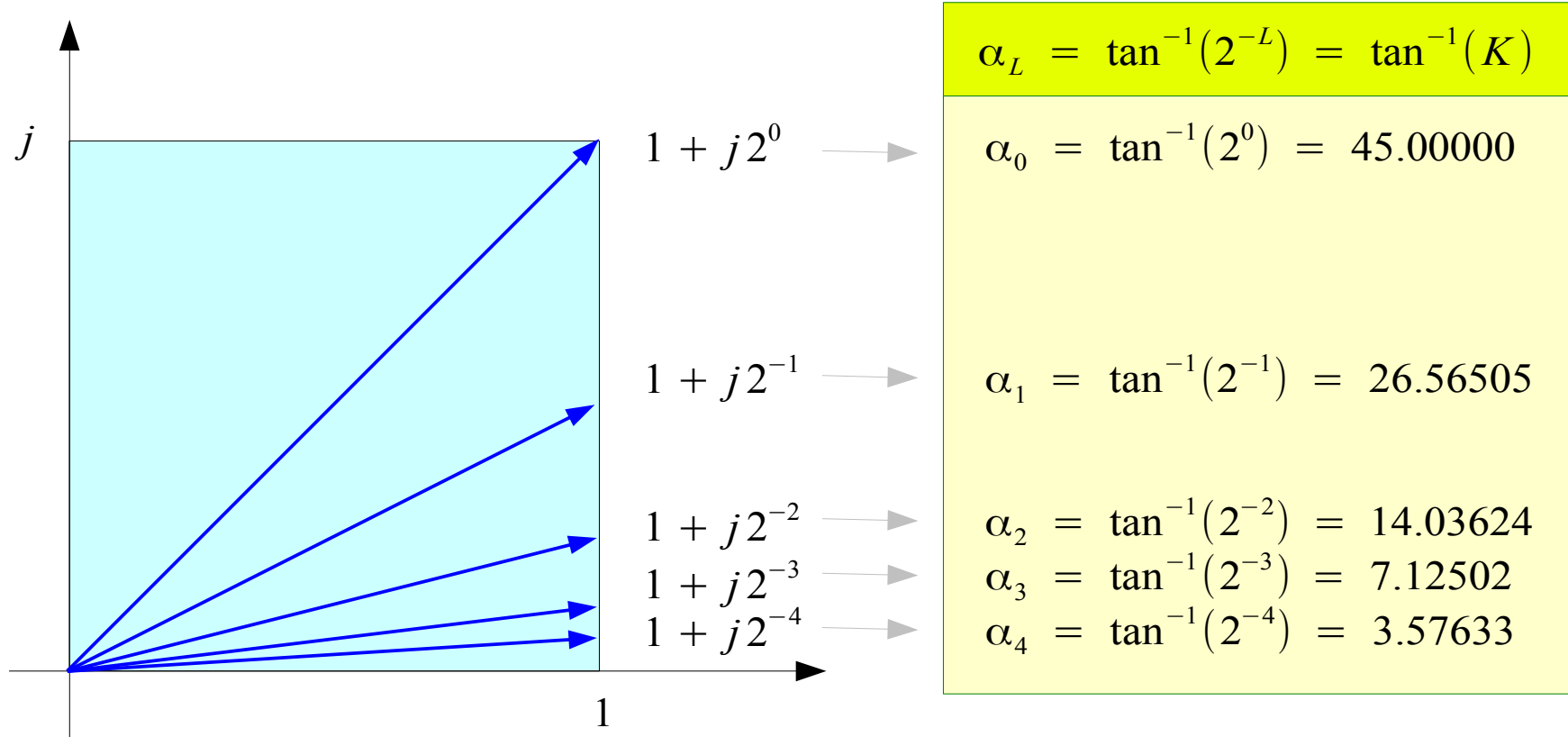
$$\tilde{v} = v \, e^{j\alpha} \quad \text{rotated by } \alpha$$

$$\alpha = \sum_{i=-1}^{n-1} q_i \cdot \alpha_i + z_n$$

$$= v \cdot \left(\prod_{i=0}^{n-1} \cos(\alpha_i) \right) \cdot (jq_{-1}) \cdot \left(\prod_{i=0}^{n-1} \left(1 + j \, q_i \, 2^{-i} \right) \right) \cdot e^{jz_n}$$

q_{-1}	q_0	q_1		...	...		q_{n-1}

Elementary Angle: tan⁻¹(K)

$$\alpha_L = \tan^{-1}(2^{-L}) = \tan^{-1}(K)$$

$$1 + j2^0 \longrightarrow \quad \alpha_0 = \tan^{-1}(2^0) = 45.00000$$

$$1 + j2^{-1} \longrightarrow \quad \alpha_1 = \tan^{-1}(2^{-1}) = 26.56505$$

$$1 + j2^{-2} \longrightarrow \quad \alpha_2 = \tan^{-1}(2^{-2}) = 14.03624$$
$$1 + j2^{-3} \longrightarrow \quad \alpha_3 = \tan^{-1}(2^{-3}) = 7.12502$$
$$1 + j2^{-4} \longrightarrow \quad \alpha_4 = \tan^{-1}(2^{-4}) = 3.57633$$

Represent arbitrary angle α

in terms of $\quad \pm\alpha_0, \ \pm\alpha_1, \ \pm\alpha_2, \ \pm\alpha_3, \ \cdots, \ \pm\alpha_L, \ \cdots \quad \left(K = \dfrac{1}{2^L}, \quad L = 0, 1, 2, \cdots \right)$

Phase and Magnitude of 1 + jK (1)

Cumulative Magnitude

L	$K = \dfrac{1}{2^L}$	$R = 1 + jK$	*Phase of R*	*Magnitude of R*	*CORDIC Gain*
0	1.0	1 + j1.0	45°	1.41421356	1.414213562
1	0.5	1 + j0.5	26.56505°	1.11803399	1.581138830
2	0.25	1 + j0.25	14.03624°	1.03077641	1.629800601
3	0.125	1 + j0.125	7.12502°	1.00778222	1.642484066
4	0.0625	1 + j0.0625	3.57633°	1.00195122	1.645688916
5	0.03125	1 + j0.03125	1.78991°	1.00048816	1.646492279
6	0.015625	1 + j0.015625	0.89517°	1.00012206	1.646693254
7	0.007813	1 + j0.007813	0.44761°	1.00003052	1.646743507
...	...	...	...	...	...
					1.647 ←

$$R = 1 + jK \xrightarrow{\substack{K = 1/2^L \\ L = 0, 1, 2, \cdots}} \sqrt{1^2 + K^2} > 1.0$$

References

[1]   http://en.wikipedia.org/
[2]   G Hampson, A VHDL Implementation of a CORDIC Arithmetic Processor Chip
       Monash University, Technical Report 94-9, 1994

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