Sampler Spectra (8B)

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6/14/12

Please send corrections (or suggestions) to youngwlim@hotmail.com.

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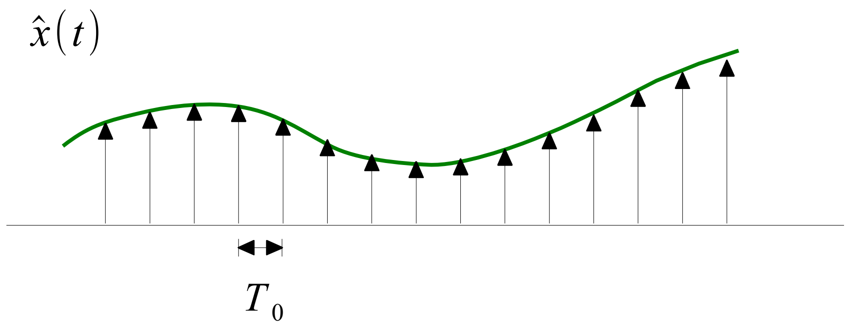
Sampler

Ideal Sampling

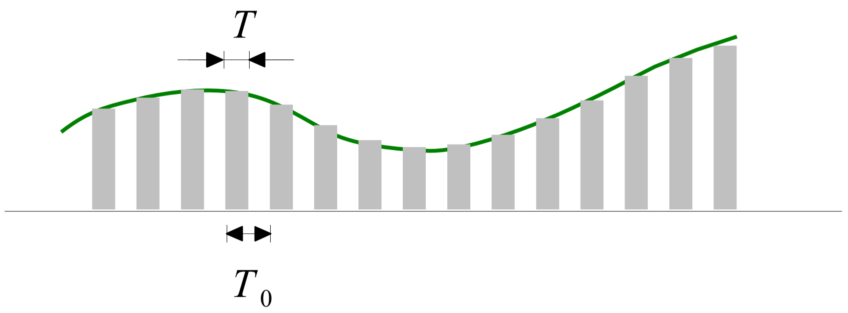


$$\hat{x}(t) = \sum_{n=-\infty}^{+\infty} x(nT_0)\,\delta(t - nT_0)$$

CTFT

?

Practical Sampling

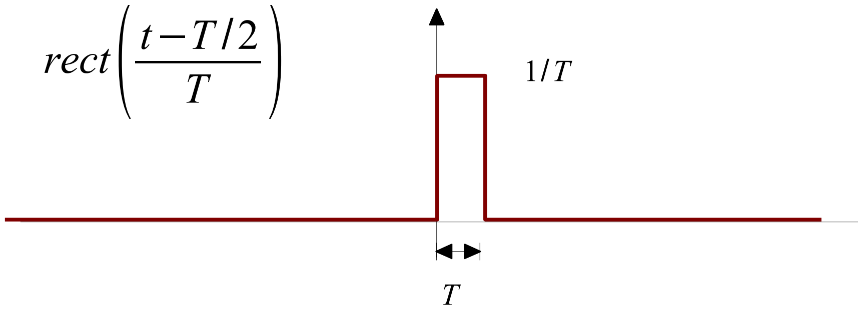


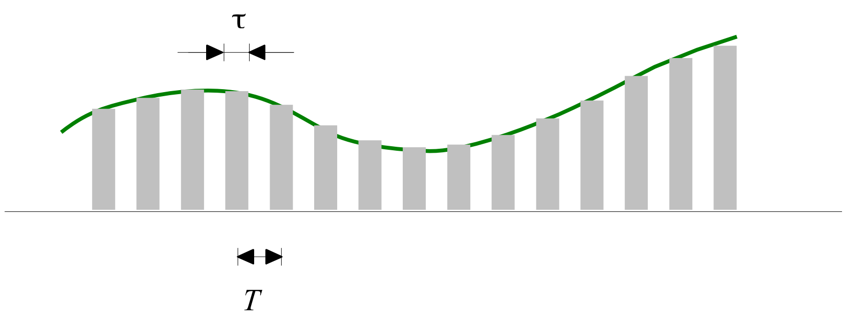
$$\hat{x}(t) \approx \sum_{n=-\infty}^{+\infty} x(nT_0)\,p(t - nT_0)$$

CTFT

?

Zero Order Hold (ZOH)





$$x_{ZOH}(t) = \sum_{n=-\infty}^{+\infty} x[n] \cdot rect\left(\frac{t-T/2-nT}{T}\right)$$

Square Wave CTFS (1)

Continuous Time Fourier Series

$$C_k = \frac{1}{T}\int_0^T x(t)\, e^{-jk\omega_0 t}\, dt \quad \Longleftrightarrow \quad x(t) = \sum_{n=0}^{\infty} C_k\, e^{+jk\omega_0 t}$$

$$\omega_0 = \frac{2\pi}{T_0}$$ Fundamental Frequency

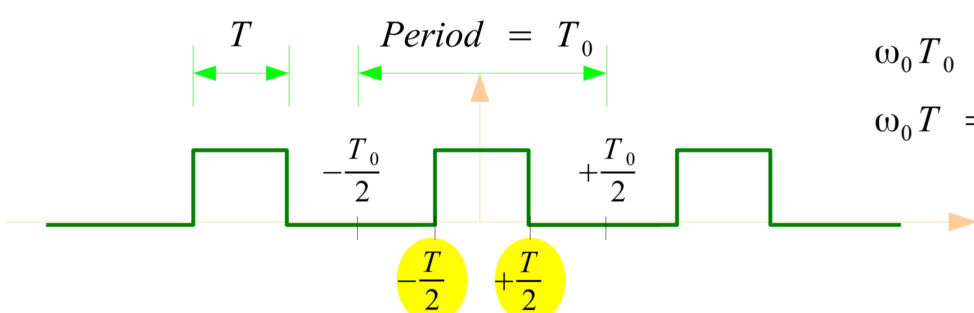
$$C_k = \frac{1}{T_0}\int_{-T_0/2}^{+T_0/2} x_{T_0}(t)\, e^{-jk\omega_0 t}\, dt$$

$$C_k T_0 = \int_{-T_0/2}^{+T_0/2} x_{T_0}(t)\, e^{-jk\omega_0 t}\, dt$$

$$\omega_0 T_0 = 2\pi \Rightarrow T/T_0 = 1/2 \Rightarrow \omega_0 T = \pi$$

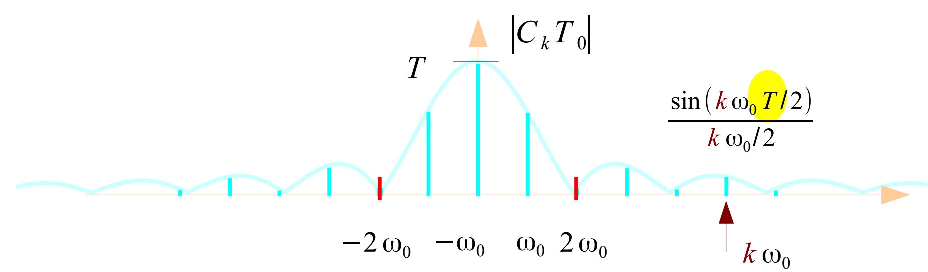
$$= \int_{-T_0/2}^{+T_0/2} e^{-jk\omega_0 t}\, dt = \left[\frac{-1}{jk\omega_0} e^{-jk\omega_0 t}\right]_{-T/2}^{+T/2}$$

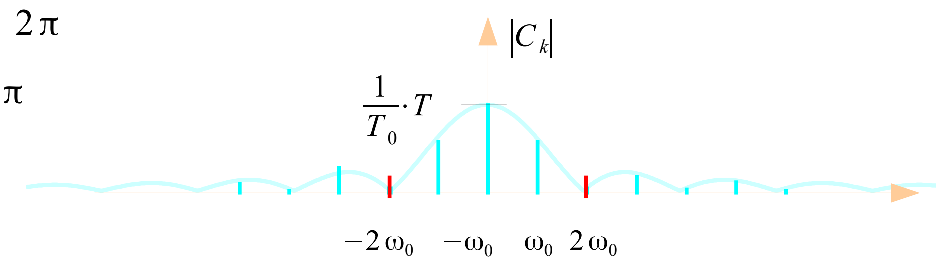
$$= -\frac{e^{-jk\omega_0 T/2} - e^{+jk\omega_0 T/2}}{jk\omega_0} = \frac{\sin(k\omega_0 T/2)}{k\omega_0/2}$$



$$\omega_0 T_0 = 2\pi$$

$$\omega_0 T = \pi$$





Square Wave CTFS (2)

$$C_k = \frac{1}{T_0} \cdot \frac{\sin(k\omega_0 T/2)}{k\omega_0/2} = \frac{1}{T_0} \cdot \frac{\sin(T k\omega_0/2)}{k\omega_0/2}$$

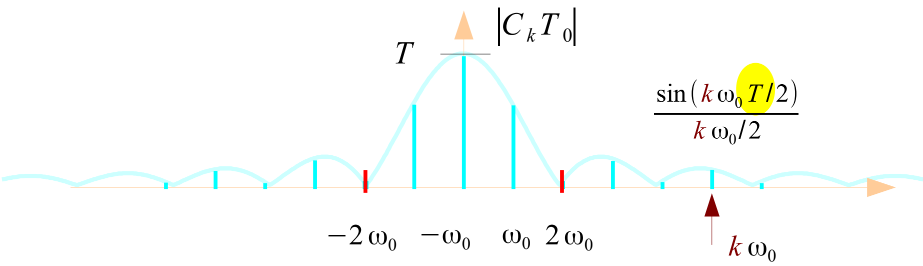
$$\omega_0 = \frac{2\pi}{T_0}$$ Fundamental Frequency

$$\sin(k\omega_0 T/2) = 0 \qquad T/T_0 = 1/2$$

$$\sin\left(k \frac{2\pi}{T_0}\frac{T}{2}\right) = 0 \quad \Rightarrow \quad \sin(k\pi/2) = 0$$

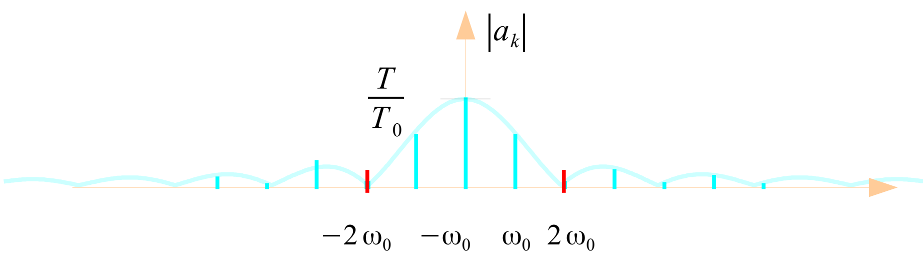
$$C_k = 0 \qquad k = \pm 2, \pm 4, \pm 6, \cdots$$

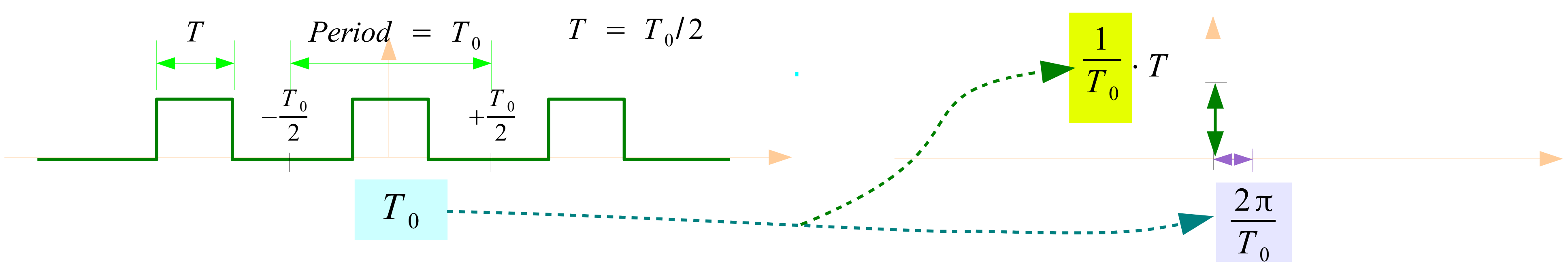
$$\Rightarrow \quad \omega = \pm 2\omega_0, \pm 4\omega_0, \pm 6\omega_0, \cdots$$



$$C_0 = \lim_{k\to 0} \frac{1}{T_0} \cdot \frac{\sin(T k\omega_0/2)}{k\omega_0/2}$$

$$C_0 = \lim_{k\to 0} \frac{1}{T_0} \cdot \frac{(T\omega_0/2)\cos(T k\omega_0/2)}{\omega_0/2} = \frac{T}{T_0}$$





CTFT of a Rect(t/T) function (1)

Continuous Time Fourier Transform

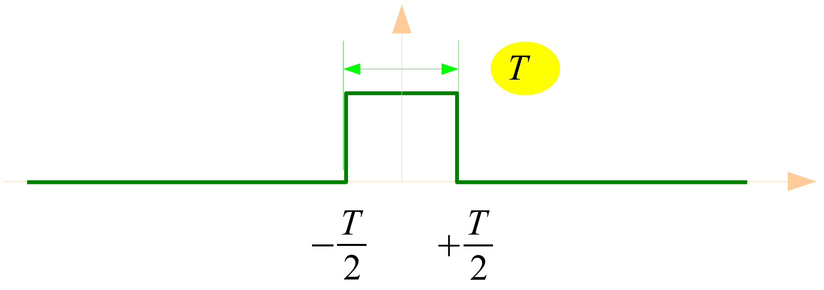
Aperiodic Continuous Time Signal

$$X(j\omega) = \int_{-\infty}^{+\infty} x(t)\, e^{-j\omega t}\, dt \quad \Leftrightarrow \quad x(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} X(j\omega)\, e^{+j\omega t}\, d\omega$$

$$X(j\omega) = \int_{-T/2}^{+T/2} e^{-j\omega t}\, dt$$

$$= \left[\frac{-1}{j\omega} e^{-j\omega t}\right]_{-T/2}^{+T/2} = -\frac{e^{-j\omega T/2} - e^{+j\omega T/2}}{j\omega}$$

$$= \frac{\sin(\omega T/2)}{\omega/2}$$

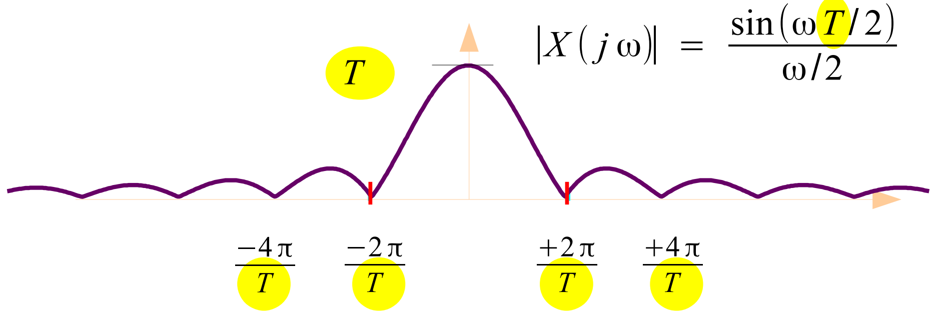


$$X(j0) = \lim_{\omega\to 0}\frac{\sin(\omega T/2)}{\omega/2} = \lim_{\omega\to 0}\frac{T}{2}\frac{\cos(\omega T/2)}{1/2} = T$$

$$\sin(\omega T/2) = 0 \quad \Rightarrow \quad \omega T/2 = \pi k$$

$$\Rightarrow \quad \omega = \frac{2\pi k}{T}$$

$$\Rightarrow \quad \omega = \pm\frac{2\pi}{T}, \pm\frac{4\pi}{T}, \pm\frac{6\pi}{T}, \cdots$$

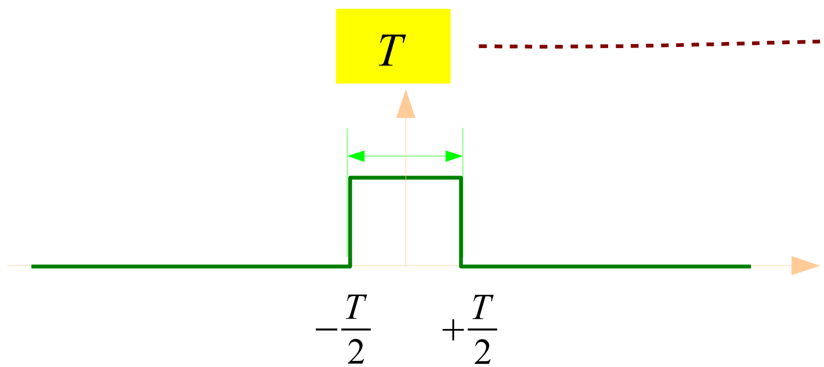


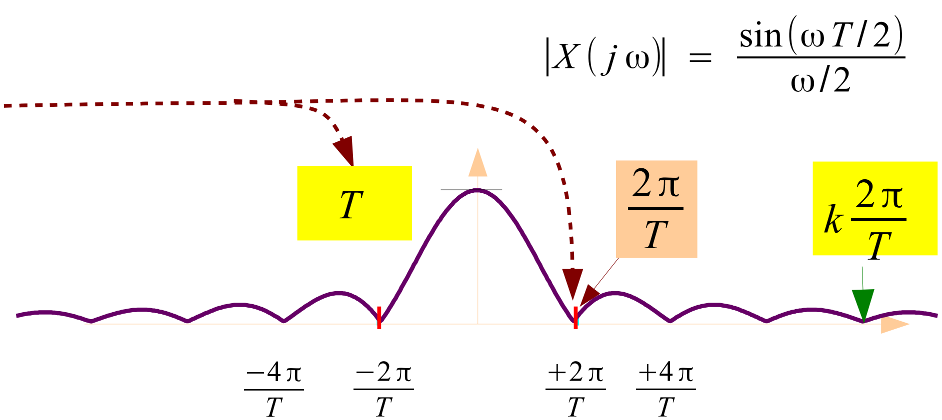
CTFT of a Rect(t/T) function (2)

Continuous Time Fourier Transform Aperiodic Continuous Time Signal

$$X(j\omega) = \int_{-\infty}^{+\infty} x(t)\, e^{-j\omega t}\, dt \quad \Longleftrightarrow \quad x(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} X(j\omega)\, e^{+j\omega t}\, d\omega$$

$$X(j\omega) = \int_{-T/2}^{+T/2} e^{-j\omega t}\, dt = \frac{\sin(\omega T/2)}{\omega/2}$$





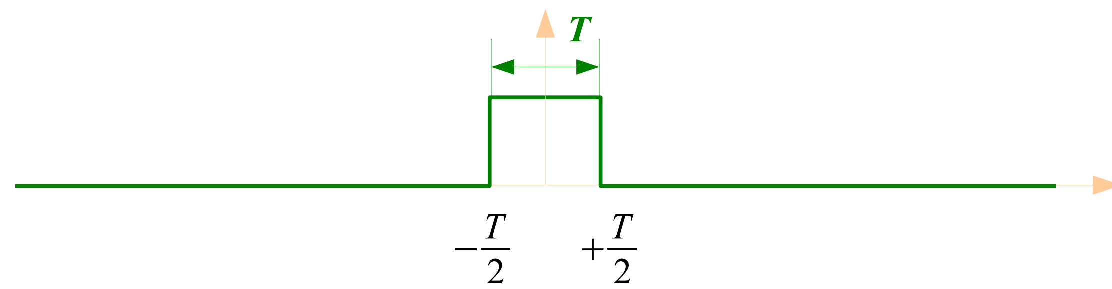
CTFT and CTFS

Continuous Time Fourier **Transform**

Aperiodic Continuous Time Signal

$$X(j\omega) = \int_{-\infty}^{+\infty} x(t)\, e^{-j\omega t}\, dt \quad \Longleftrightarrow \quad x(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} X(j\omega)\, e^{+j\omega t}\, d\omega$$

$$X(j\omega) = \frac{\sin(\omega T/2)}{\omega/2}$$

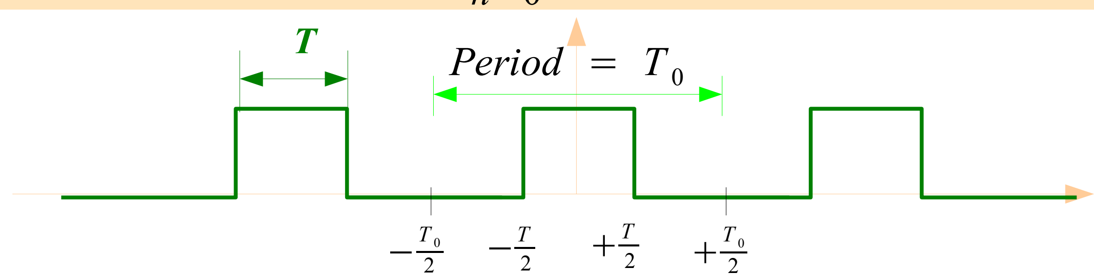


Continuous Time Fourier **Series**

Periodic Continuous Time Signal

$$C_k = \frac{1}{T} \int_0^T x(t)\, e^{-jk\omega_0 t}\, dt \quad \Longleftrightarrow \quad x(t) = \sum_{n=0}^{\infty} C_k\, e^{+jk\omega_0 t}$$

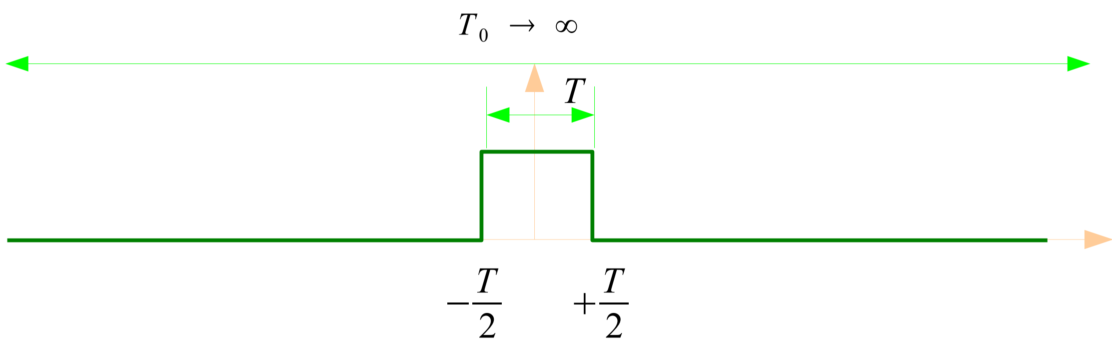
$$C_k = \frac{1}{T_0} \frac{\sin(k\omega_0 T/2)}{k\omega_0/2}$$



CTFT ← CTFS

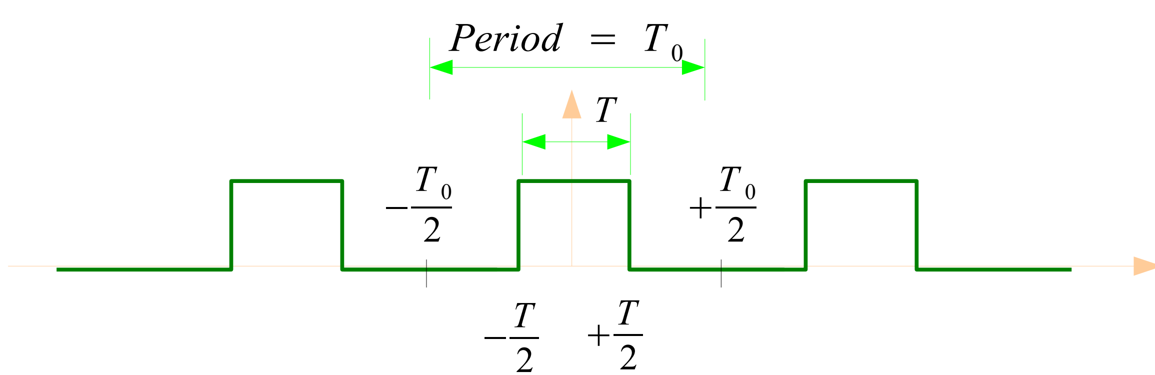
Aperiodic Continuous Time Signal

Continuous Time Fourier Transform



Periodic Continuous Time Signal

Continuous Time Fourier Series



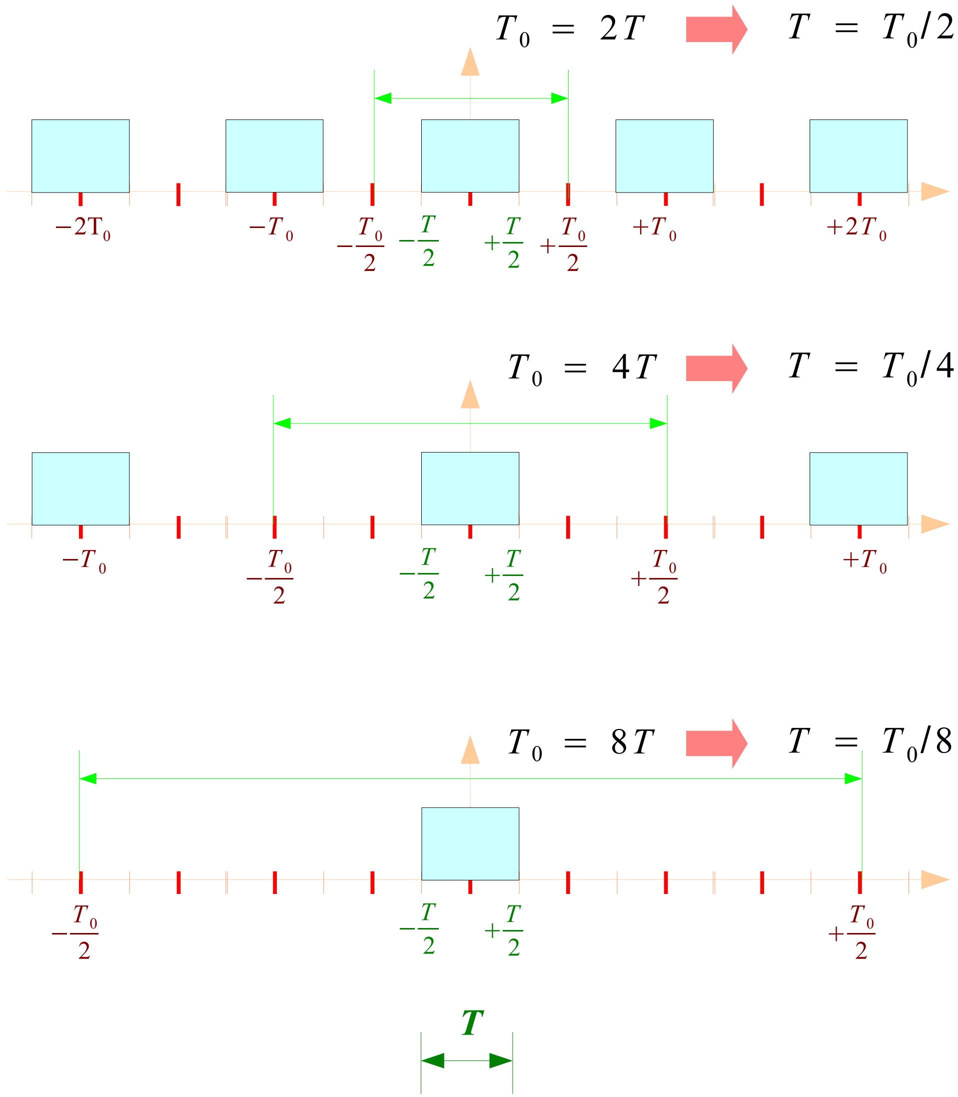
$$x(t)$$

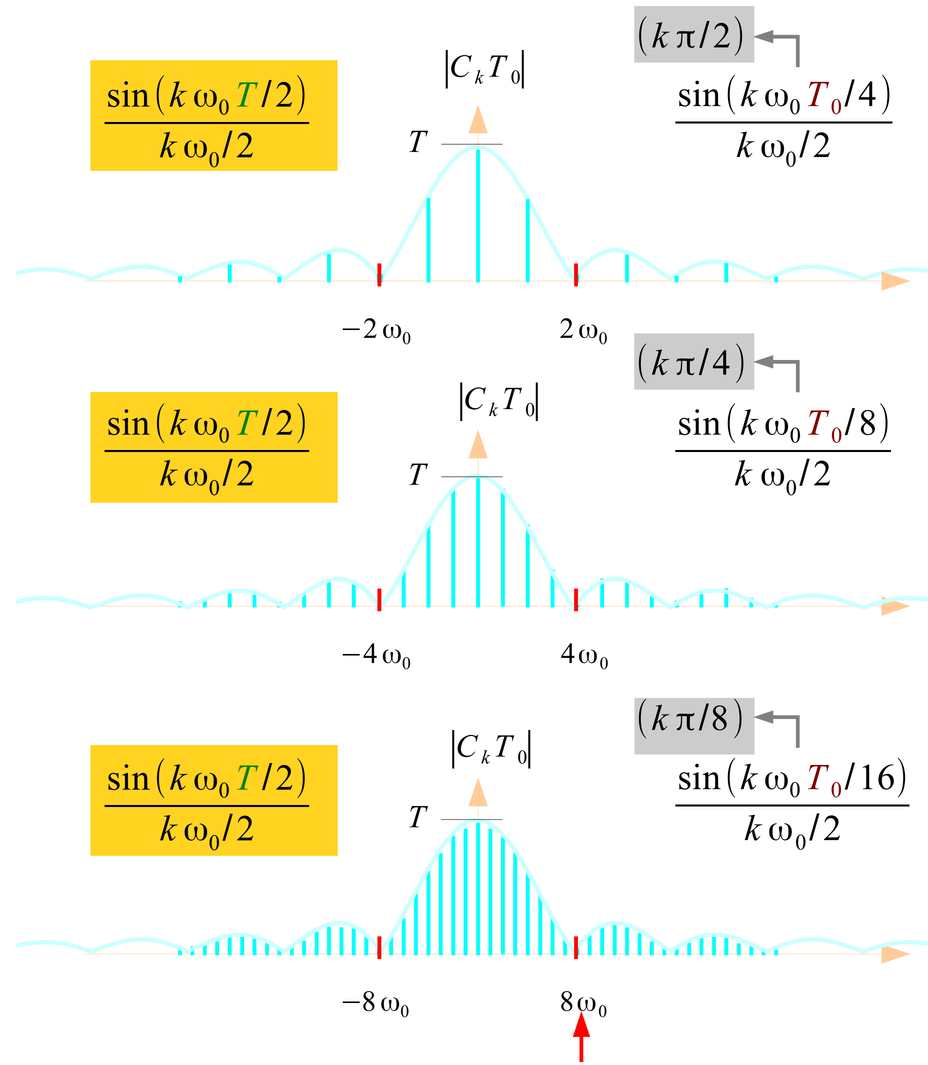
As $T_0 \to \infty$, $x_{T_0}(t) \to x(t)$

$$\omega_0 = \frac{2\pi}{T_0} \to 0$$

$$x_{T_0}(t) = \sum_{n=-\infty}^{+\infty} x(t - nT_0)$$

CTFT and CTFS as $T_0 \to \infty$ (1)





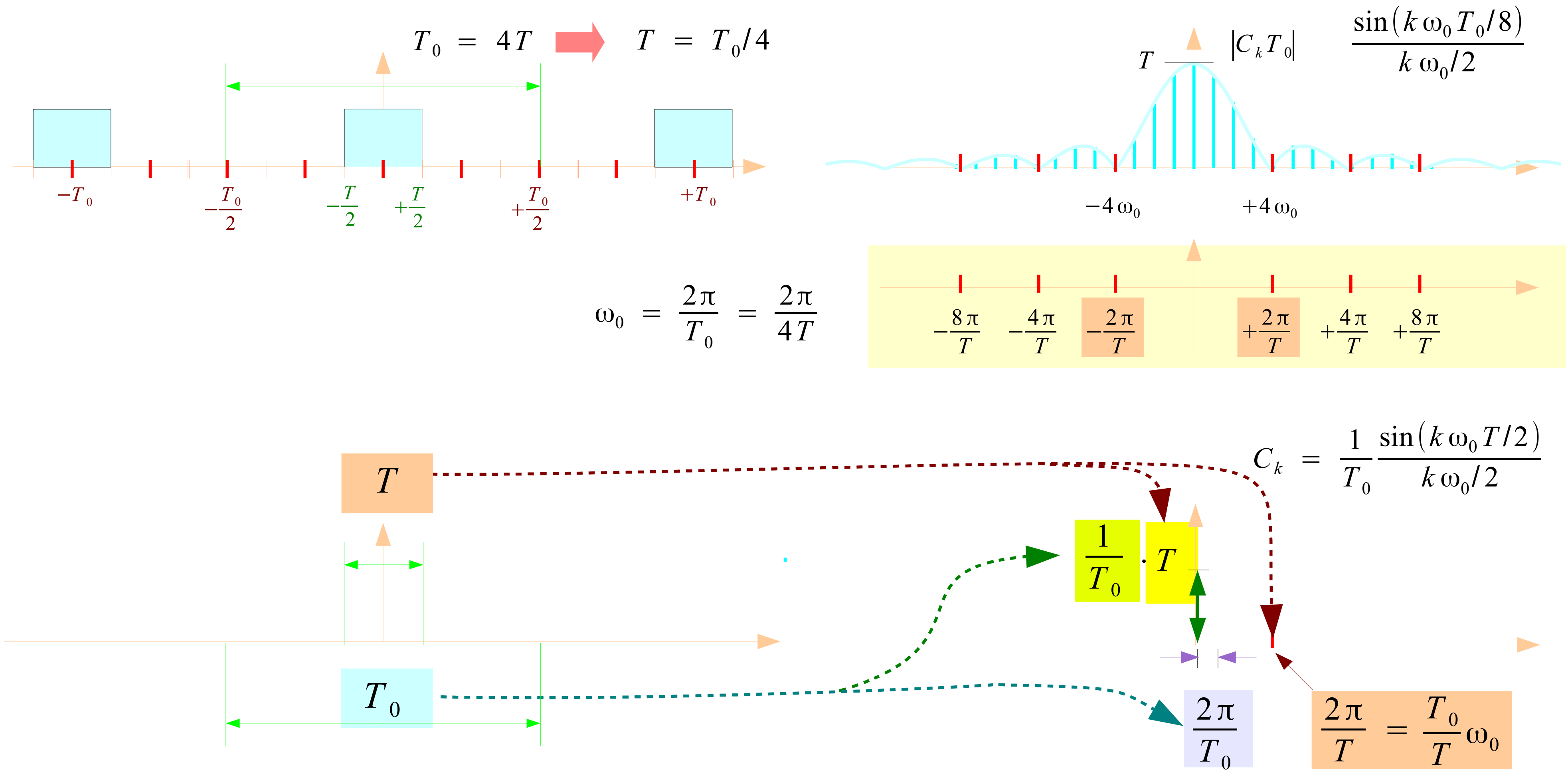
$$\frac{\sin(k\omega_0 T/2)}{k\omega_0/2} \qquad \frac{\sin(k\omega_0 T_0/4)}{k\omega_0/2}$$

$$\frac{\sin(k\omega_0 T/2)}{k\omega_0/2} \qquad \frac{\sin(k\omega_0 T_0/8)}{k\omega_0/2}$$

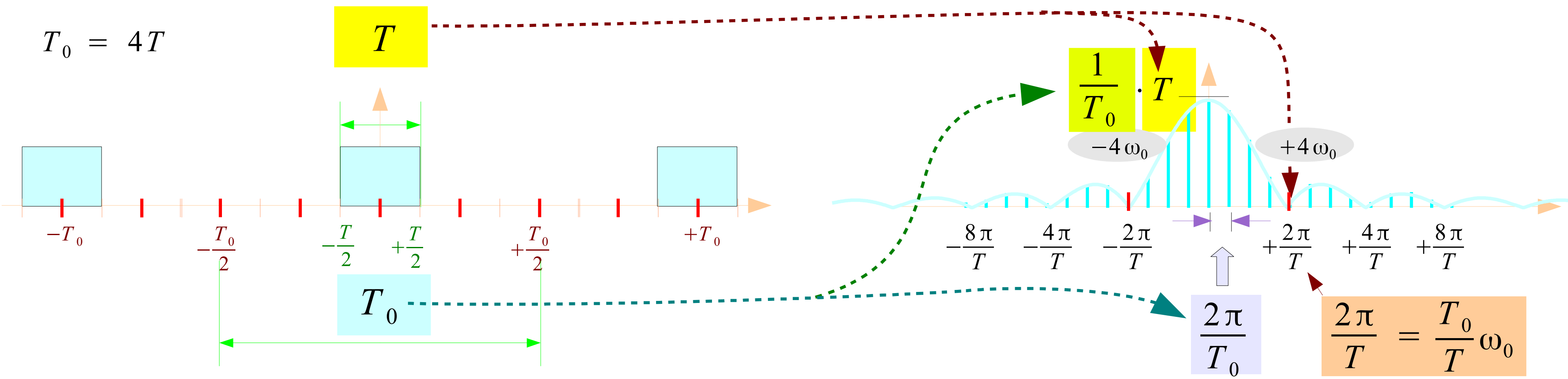
$$\frac{\sin(k\omega_0 T/2)}{k\omega_0/2} \qquad \frac{\sin(k\omega_0 T_0/16)}{k\omega_0/2}$$

$$\omega_0 = \frac{2\pi}{T_0} = \frac{2\pi}{8T} \quad \Rightarrow \quad \frac{T_0}{T}\omega_0 = \frac{2\pi}{T}$$

CTFT and CTFS as $T_0 \to \infty$ (2)



CTFT of a Rect(t/T) function (3)



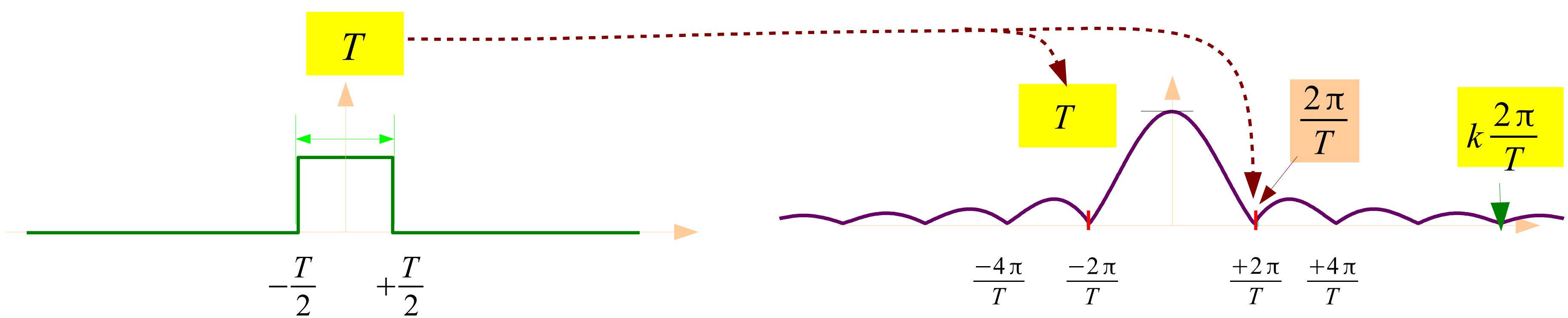
$$C_k T_0 = \frac{\sin(k\omega_0 T/2)}{k\omega_0/2}$$

$$C_k = \frac{1}{T_0}\frac{\sin(k\omega_0 T/2)}{k\omega_0/2}$$

$$X(j\omega) = \lim_{k\omega_0 \to \omega} \frac{\sin(k\omega_0 T/2)}{k\omega_0/2} = \frac{\sin(\omega T/2)}{\omega/2}$$

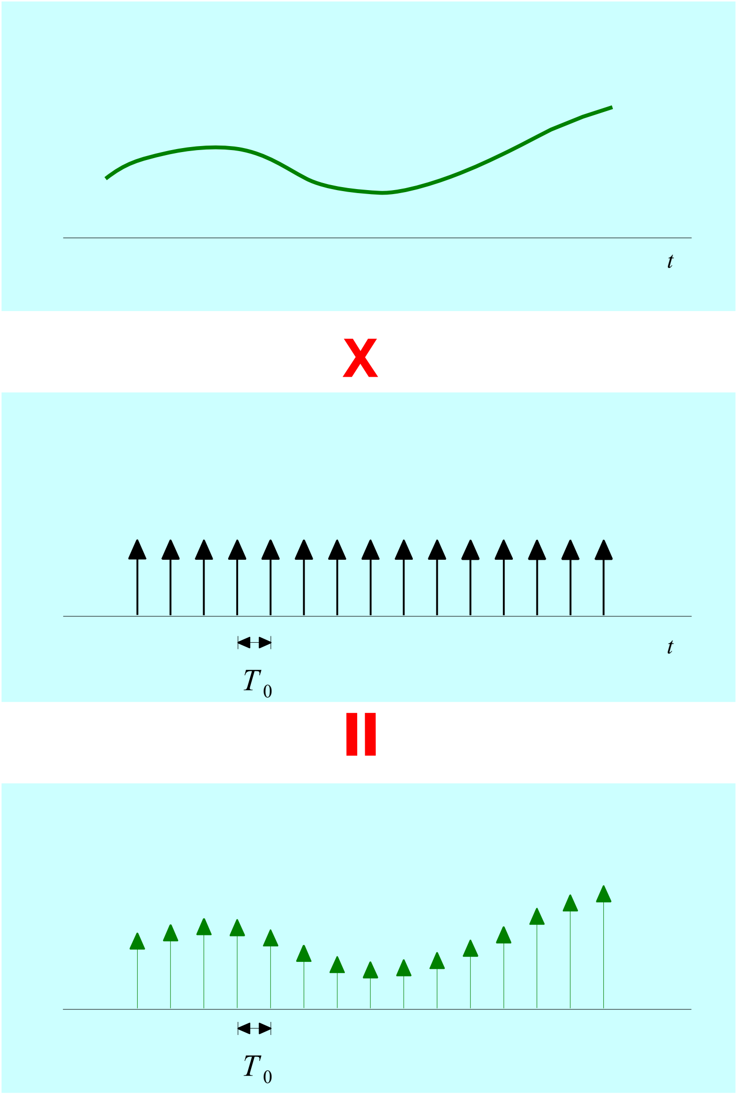
$$X(j\omega) = \frac{\sin(\omega T/2)}{\omega/2}$$

$$X(j\omega) = \frac{\sin(\omega T/2)}{\omega/2}$$

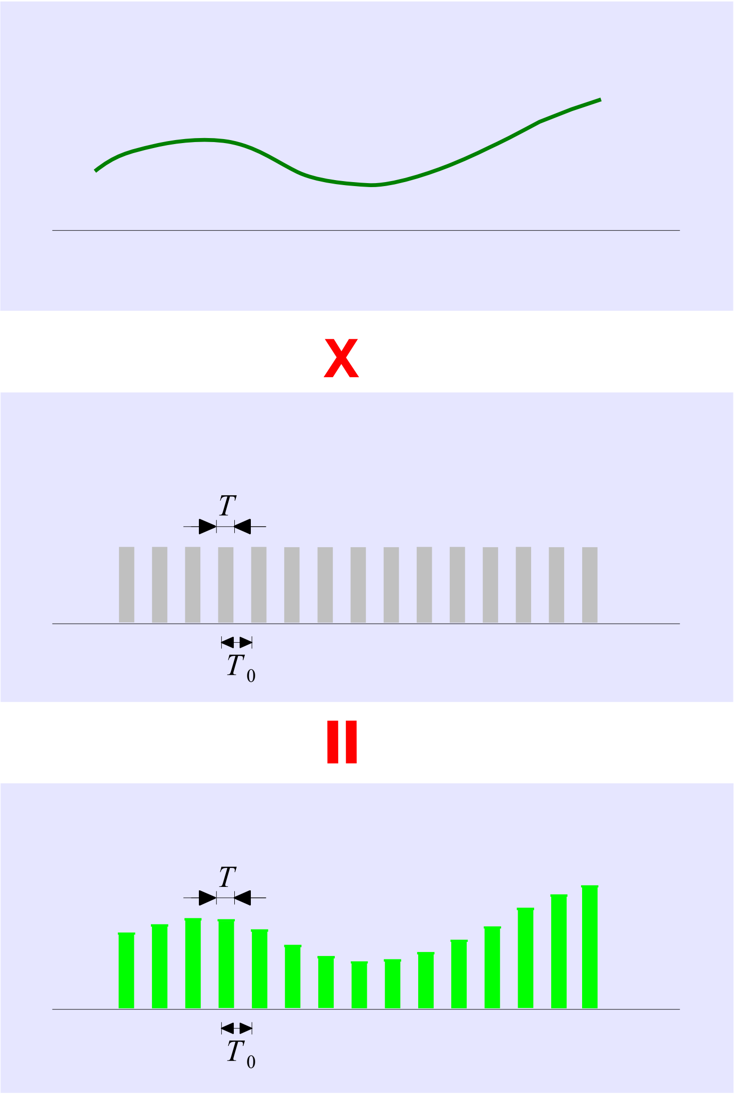


Sampling (1)

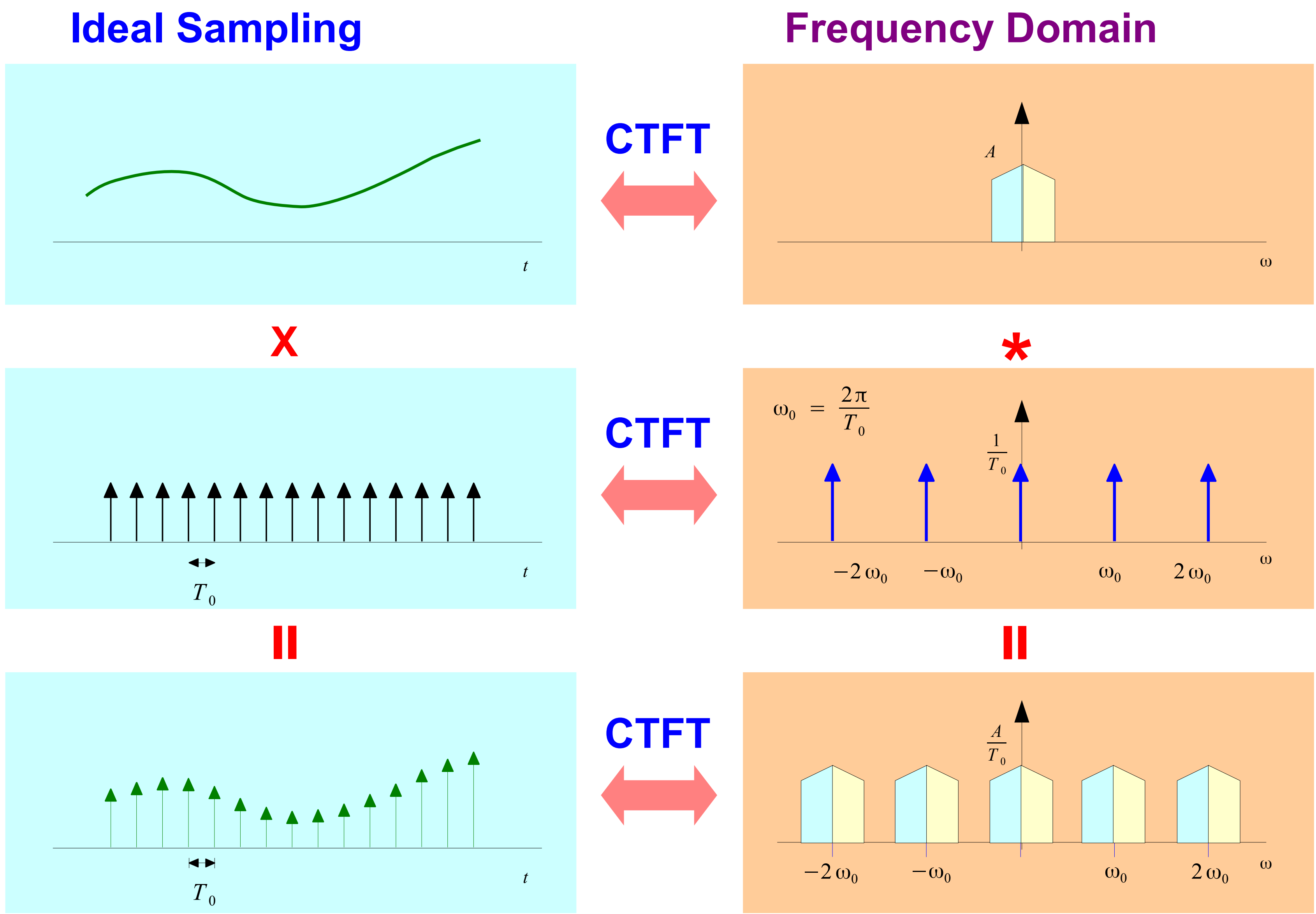
Ideal Sampling



Practical Sampling



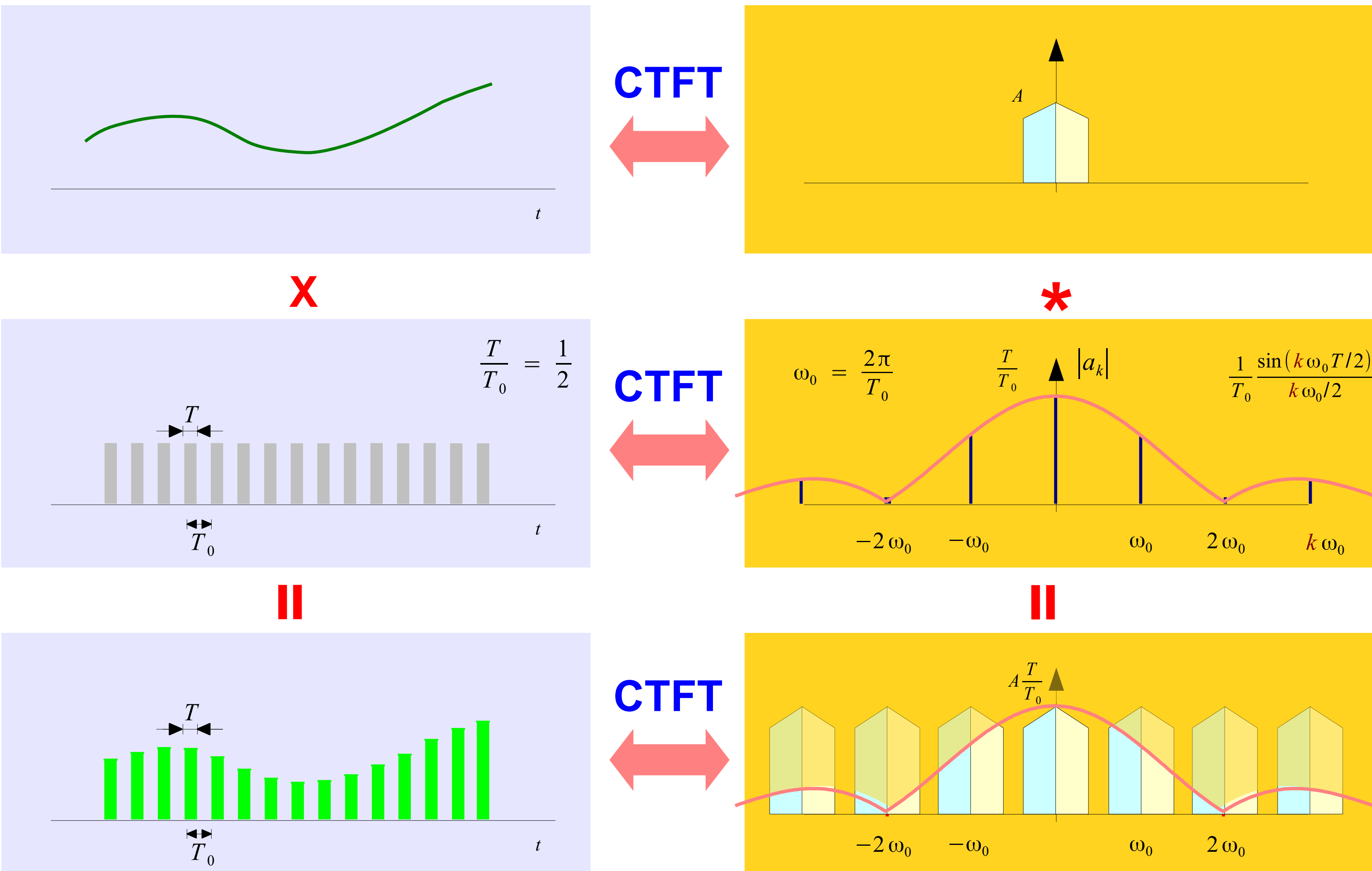
Sampling (2)



Ideal Sampling
Frequency Domain
CTFT
t
A
ω
X
*
CTFT
T 0
ω0 = 2π / T 0
1 / T 0
−2 ω0
−ω0
ω0
2 ω0
CTFT
A / T 0

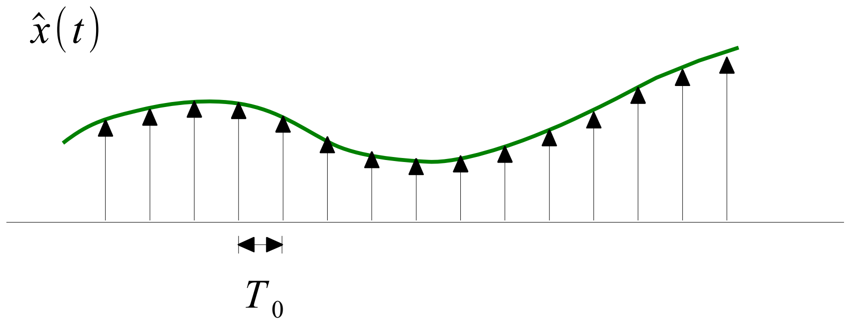
Sampling (3)

Practical Sampling | **Frequency Domain**



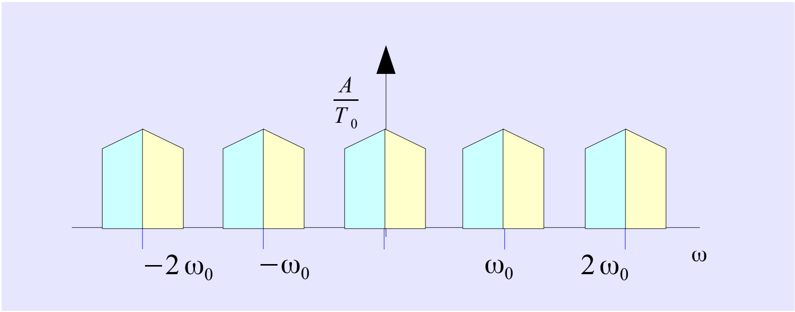
Sampling CTFT

Ideal Sampling

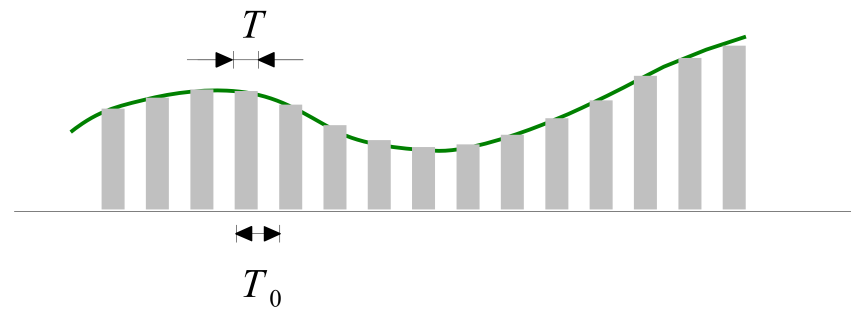


$$\hat{x}(t) = \sum_{n=-\infty}^{+\infty} x(nT_0)\,\delta(t - nT_0)$$

CTFT

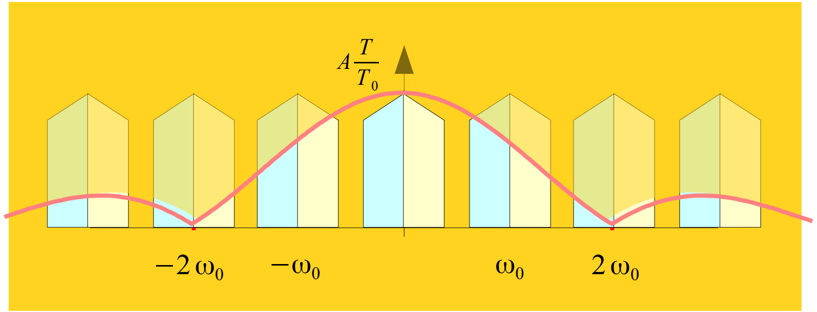


Practical Sampling



$$\hat{x}(t) \approx \sum_{n=-\infty}^{+\infty} x(nT_0)\,p(t - nT_0)$$

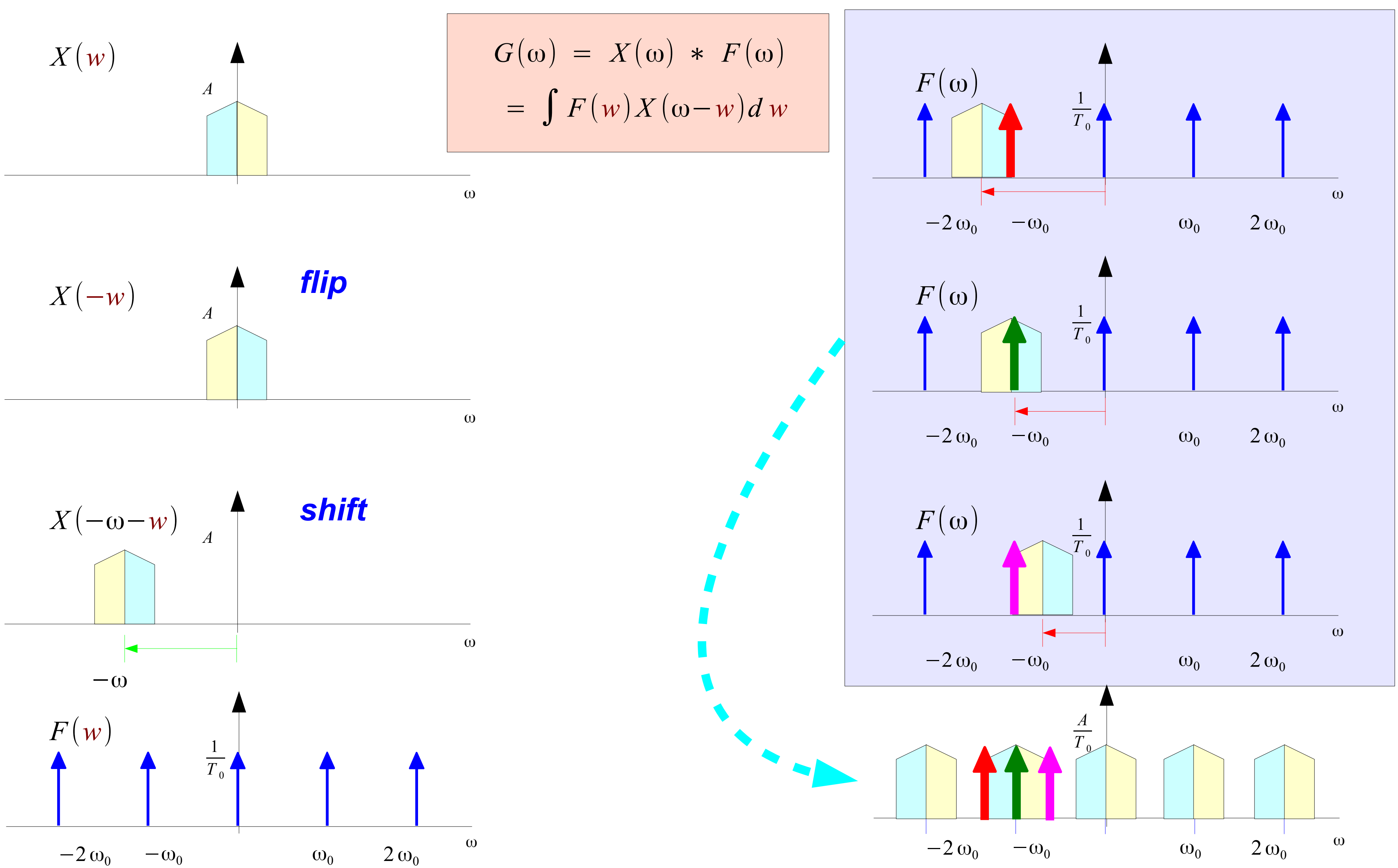
CTFT



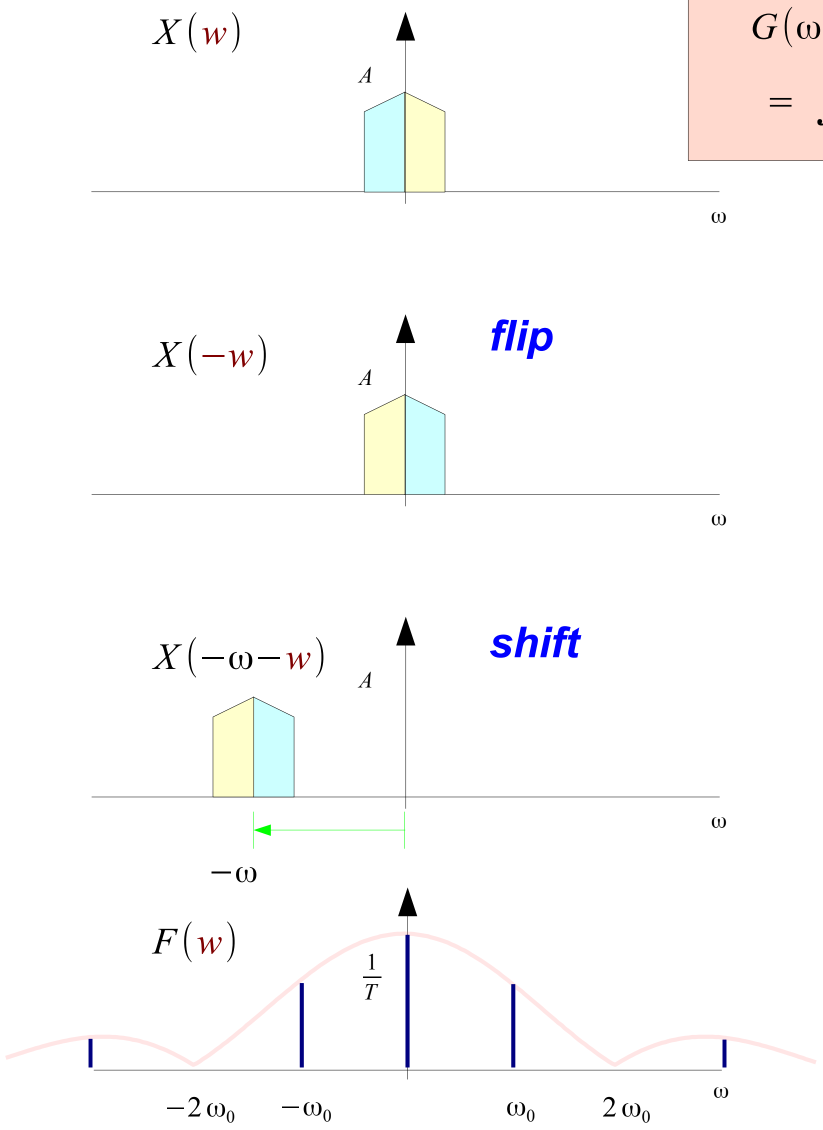
Convolution with Impulse Train

$$G(\omega) = X(\omega) * F(\omega)$$

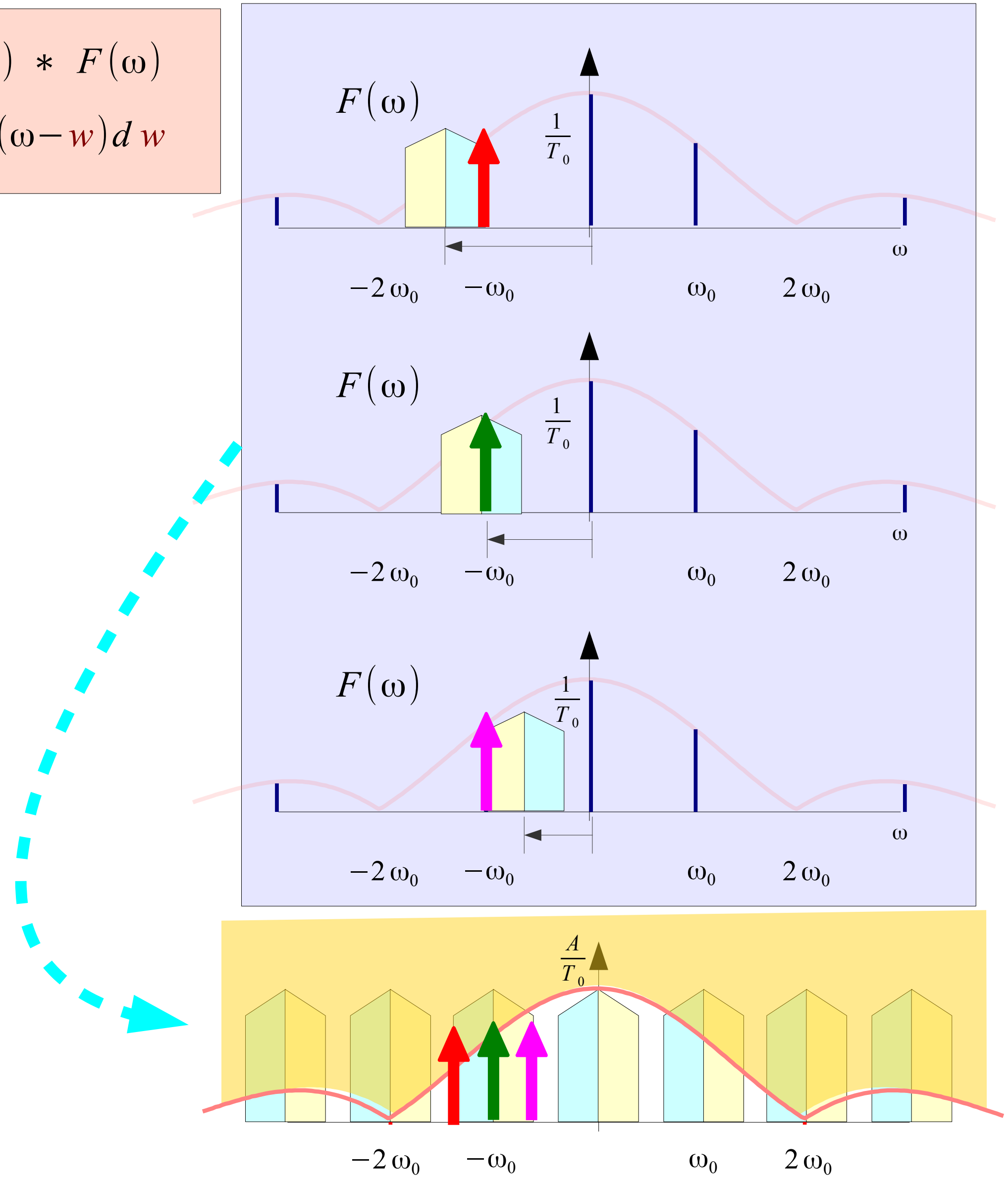
$$= \int F(w) X(\omega - w) \, dw$$



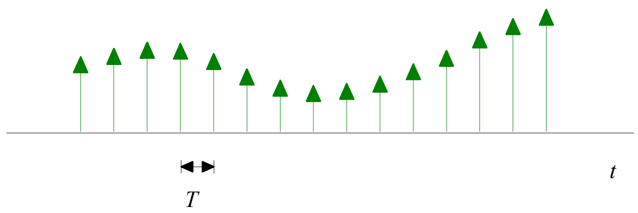
Convolution with Sinc Impulse Train

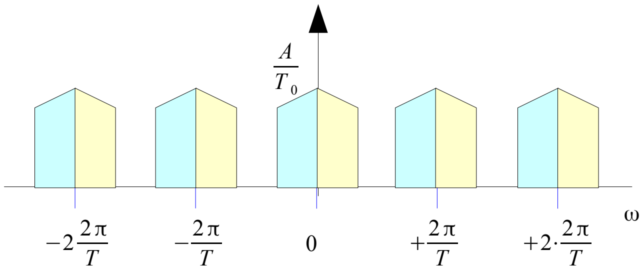


$$G(\omega) = X(\omega) * F(\omega)$$
$$= \int F(w) X(\omega - w)\, d w$$



CTFT of Sampled Signal





$$\hat{x}(t) = \sum_{n=-\infty}^{+\infty} x(nT)\,\delta(t-nT)$$

CTFT ⇒

$$\hat{X}(f) = \int_{-\infty}^{+\infty} \hat{x}(t)\, e^{-j2\pi f t}\, dt$$

$$= \int_{-\infty}^{+\infty} \sum_{n=-\infty}^{+\infty} x(nT)\,\delta(t-nT)\, e^{-j2\pi f t}\, dt$$

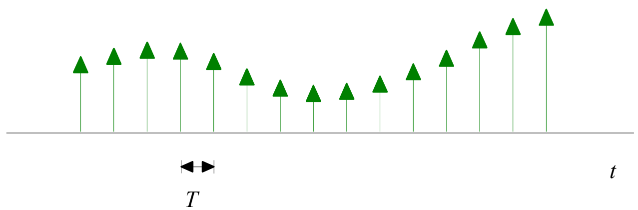
$$= \sum_{n=-\infty}^{+\infty} x(nT)\, e^{-j2\pi f T n}\, dt$$

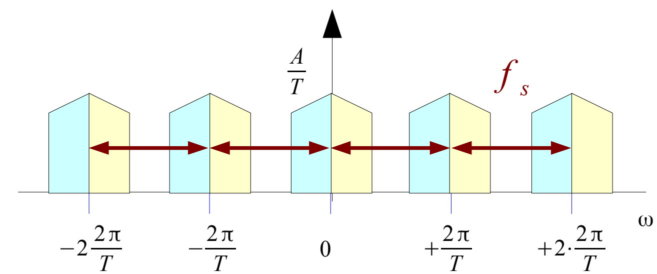
$$\hat{x}(t) = \sum_{n=-\infty}^{+\infty} x(nT)\,\delta(t-nT)$$

CTFT ⇒

$$\hat{X}(f) = \sum_{n=-\infty}^{+\infty} x(nT)\, e^{-j2\pi f T n}$$

Periodicity in Frequency





$$f_s = \frac{1}{T}$$

$$2\pi f_s = \frac{2\pi}{T} = \omega_0$$

$$2\pi f = \omega$$

$$\hat{x}(t) = \sum_{n=-\infty}^{+\infty} x(nT)\,\delta(t - nT)$$

CTFT

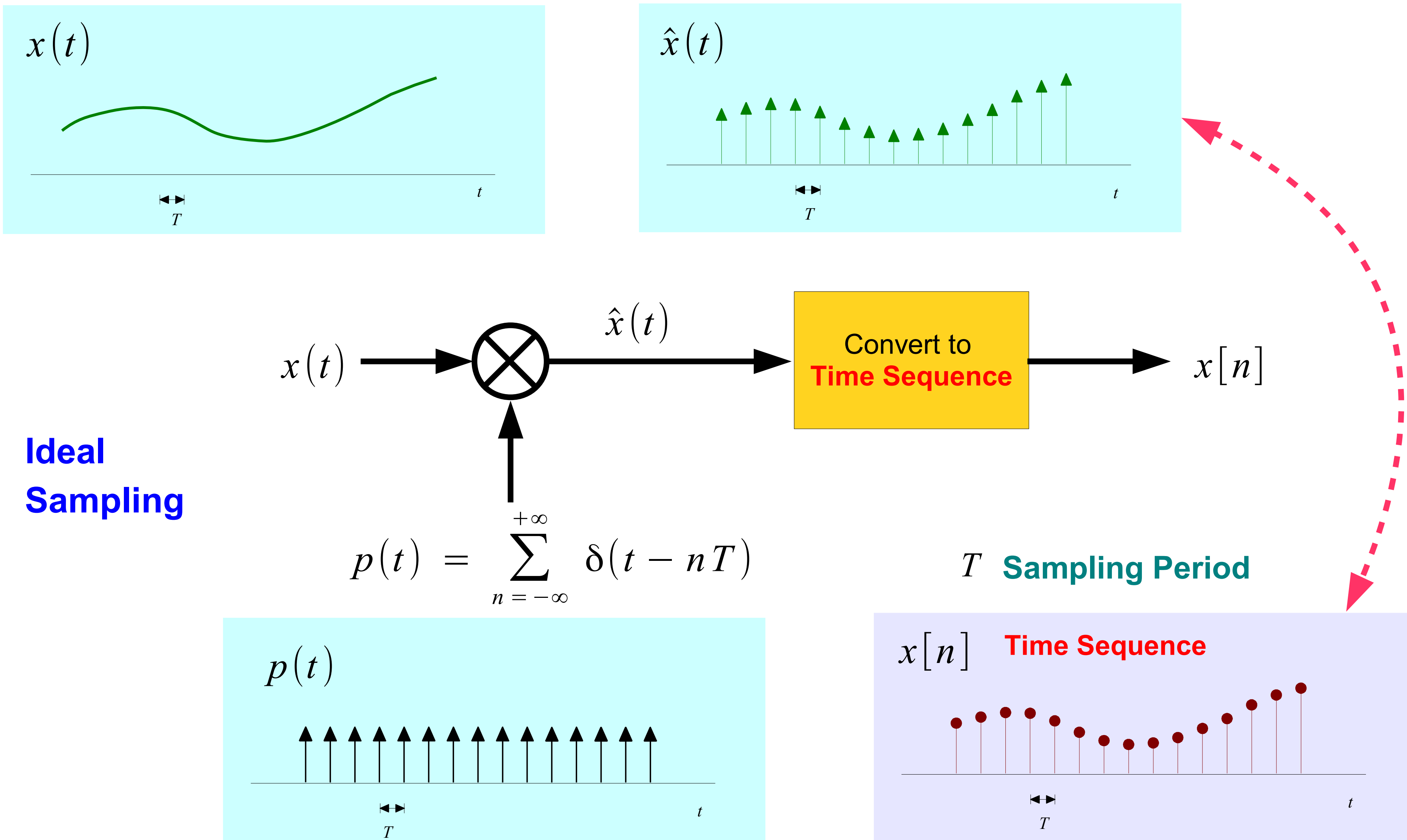
$$\hat{X}(f) = \sum_{n=-\infty}^{+\infty} x(nT)\, e^{-j2\pi f T n}$$

$$e^{-j2\pi(f+f_s)Tn} = e^{-j2\pi(f)Tn} \quad \Leftarrow \quad f_s T = 1$$

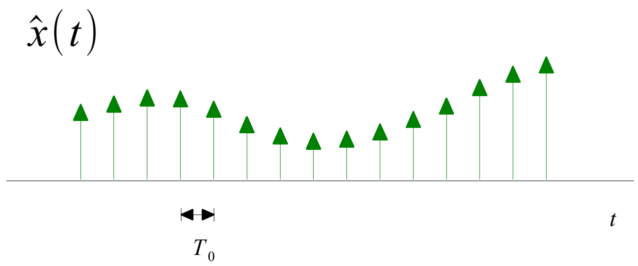
Period = Sampling Frequency f_s

$$\hat{X}(f) = \hat{X}(f + f_s)$$

Time Sequence

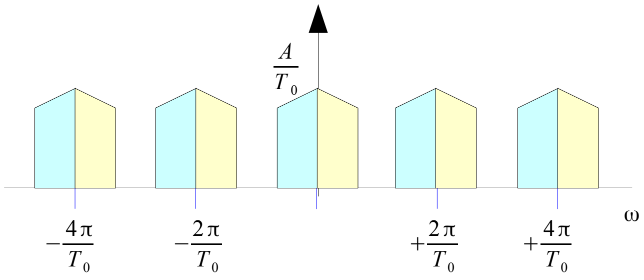


DTFT of a Time Sequence

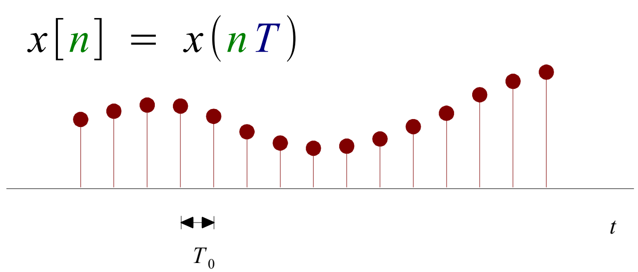


$$\hat{x}(t) = \sum_{n=-\infty}^{+\infty} x(nT_0)\,\delta(t-nT_0)$$

CTFT

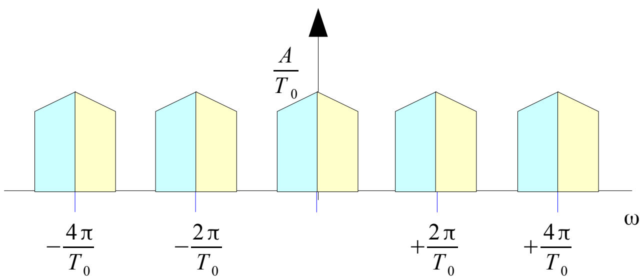


$$\hat{X}(f) = \sum_{n=-\infty}^{+\infty} x(nT_0)\, e^{-j2\pi f T_0 n}$$



$x[n]$, Sampling Period T_0

DTFT

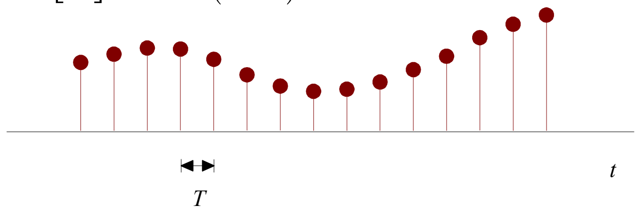


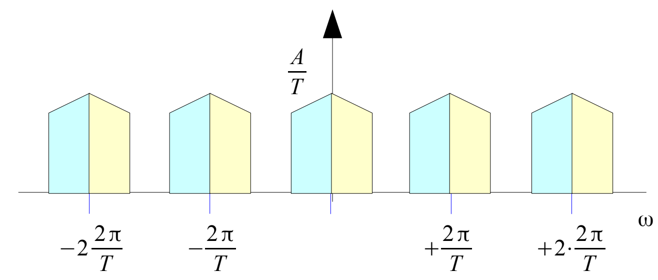
$$X(f) = \sum_{n=-\infty}^{+\infty} x[n]\, e^{-j2\pi f T_0 n}$$

Here, $X(f)$ does not denote the CTFT of $x(t)$

Discrete Time Fourier Transform (1)

$$x[n] = x(nT)$$





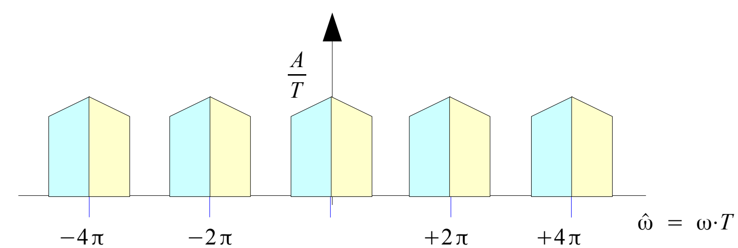
$x[n]$, Sampling Period T

DTFT

$$X(f) = \sum_{n=-\infty}^{+\infty} x[n]\, e^{-j 2\pi f T n}$$

Normalized Angular Frequency

$$2\pi f T = \frac{2\pi f}{1/T} = 2\pi \frac{f}{f_s} = \hat{\omega}$$

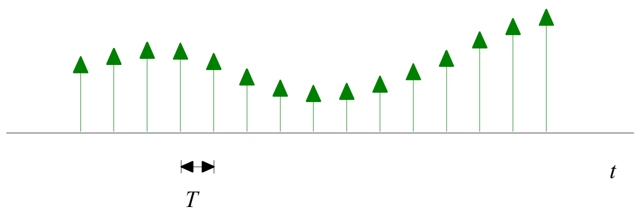


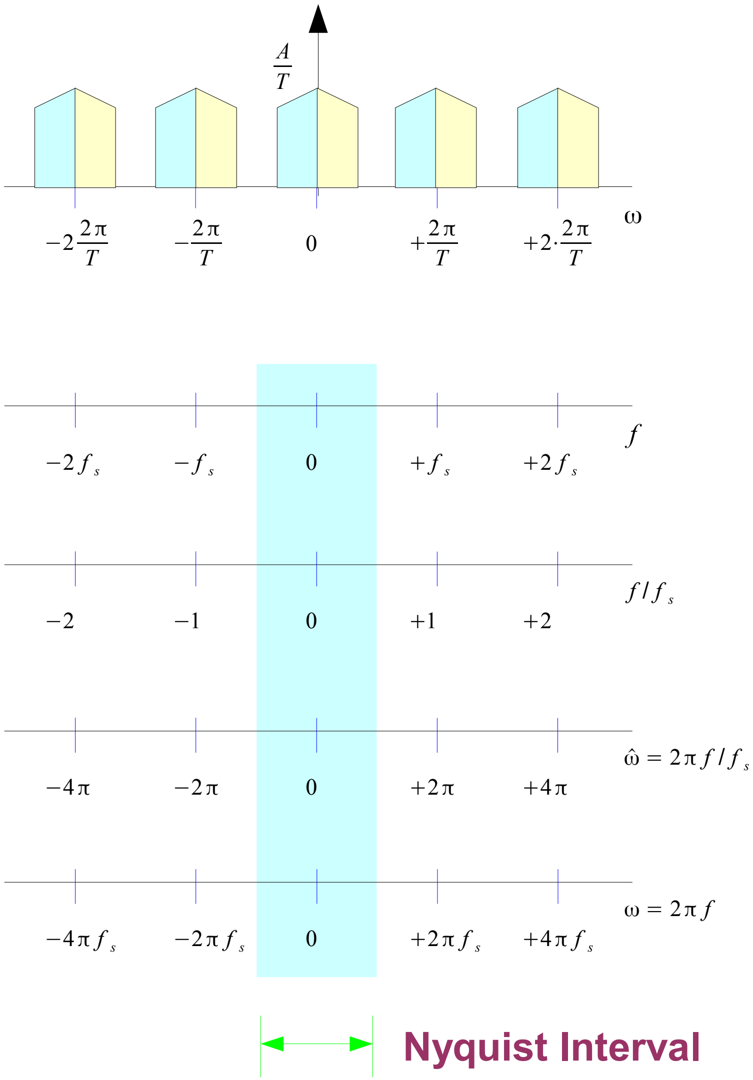
$x[n]$, Sampling Period T

DTFT

$$X(e^{j\hat{\omega}}) = \sum_{n=-\infty}^{+\infty} x[n]\, e^{-j\hat{\omega} n}$$

Discrete Time Fourier Transform (2)





$$f_s = \frac{1}{T}$$

$$2\pi f_s = \frac{2\pi}{T} = \omega_0$$

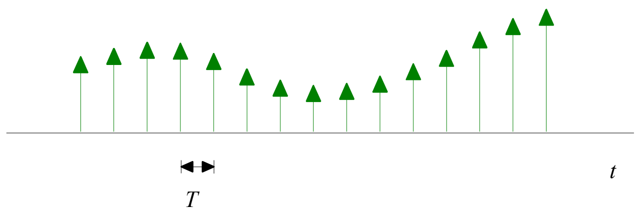
$$X(f) = \sum_{n=-\infty}^{+\infty} x[n]\, e^{-j2\pi f T n}$$

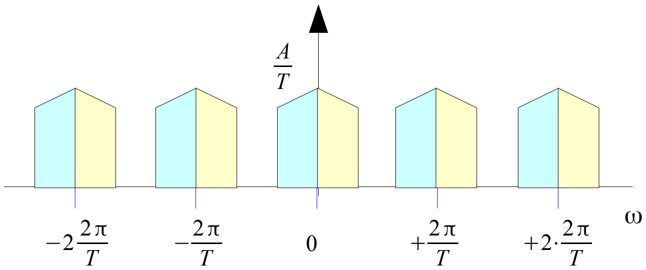
Normalized Angular Frequency

$$2\pi f T = \frac{2\pi f}{1/T} = 2\pi \frac{f}{f_s} = \hat{\omega}$$

$$X(e^{j\hat{\omega}}) = \sum_{n=-\infty}^{+\infty} x[n]\, e^{-j\hat{\omega} n}$$

Discrete Time Fourier Transform (3)



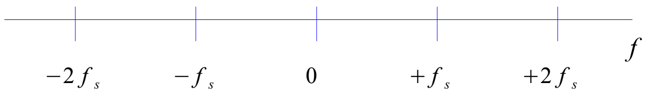


$$f_s = \frac{1}{T}$$

$$2\pi f_s = \frac{2\pi}{T} = \omega_0$$

$$X(f) = \sum_{n=-\infty}^{+\infty} x[n]\, e^{-j2\pi f T n}$$

$\hat{X}(f)$ **Absolute Frequency**



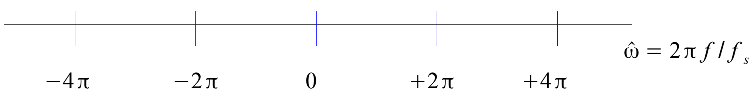
Normalized Angular Frequency

$$2\pi f T = \frac{2\pi f}{1/T} = 2\pi \frac{f}{f_s} = \hat{\omega}$$

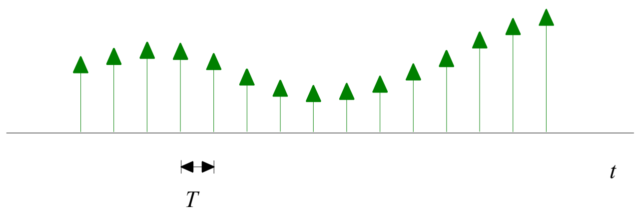
$\hat{X}(e^{j\hat{\omega}})$ **Normalized Angular Frequency**

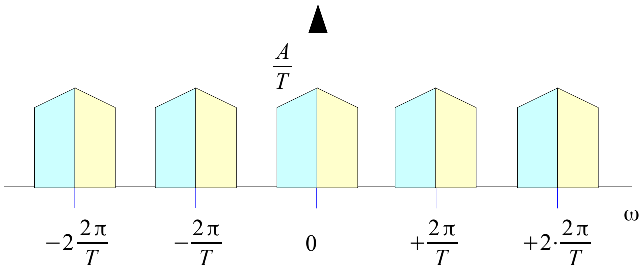
unit circle →
emphasize the periodic nature

$$X(e^{j\hat{\omega}}) = \sum_{n=-\infty}^{+\infty} x[n]\, e^{-j\hat{\omega} n}$$



Fourier Series Interpretation





$$\hat{x}(t) = \sum_{n=-\infty}^{+\infty} x(nT)\,\delta(t-nT)$$

CTFT

$$\hat{X}(f) = \sum_{n=-\infty}^{+\infty} x(nT)\, e^{-j2\pi f T n}$$

$$x(nT) = \frac{1}{f_s}\int_{+f_s/2}^{-f_s/2} \hat{X}(f)\, e^{+j2\pi f T n}\, df$$

$$= \int_{-\pi}^{+\pi} \hat{X}(\omega)\, e^{+j\omega n}\, \frac{d\omega}{2\pi}$$

$$\omega = 2\pi f / f_s \qquad \frac{df}{f_s} = \frac{d\omega}{2\pi}$$

CTFS

$$\hat{X}(f) = \sum_{n=-\infty}^{+\infty} x(nT)\, e^{-j2\pi f T n}$$

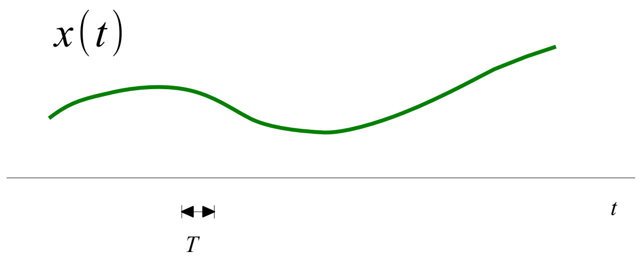
Fourier Series Coefficients $x(nT)$

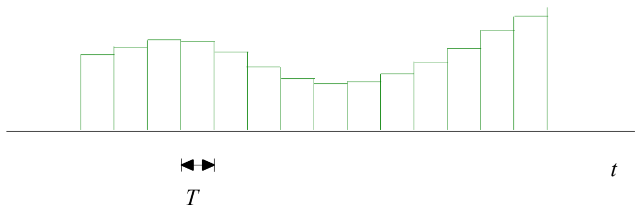
$\hat{X}(f)$ **Continuous Periodic Function**

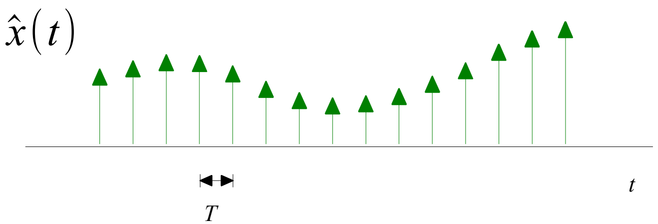
View as a Fourier Series Expansion

Numerical Approximation

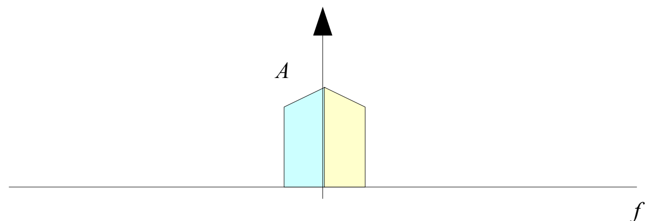
$$X(f) = \lim_{T \to 0} T\,\hat{X}(f)$$







$$\hat{x}(t) = \sum_{n=-\infty}^{+\infty} x(nT)\,\delta(t-nT)$$

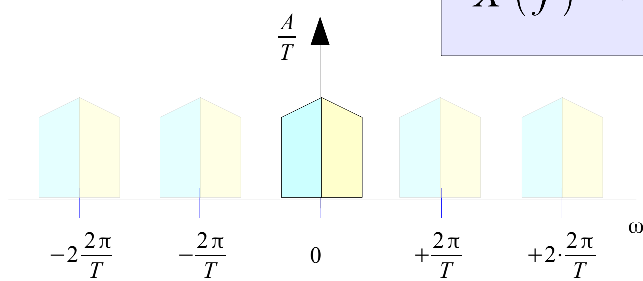


CTFT

$$X(f) = \int_{-\infty}^{+\infty} x(t)\, e^{+j2\pi f t}\, dt$$

$$\approx \sum_{n=-\infty}^{+\infty} x(nT)\, e^{-j2\pi f T n} \cdot T$$

$$X(f) \approx T\,\hat{X}(f)$$

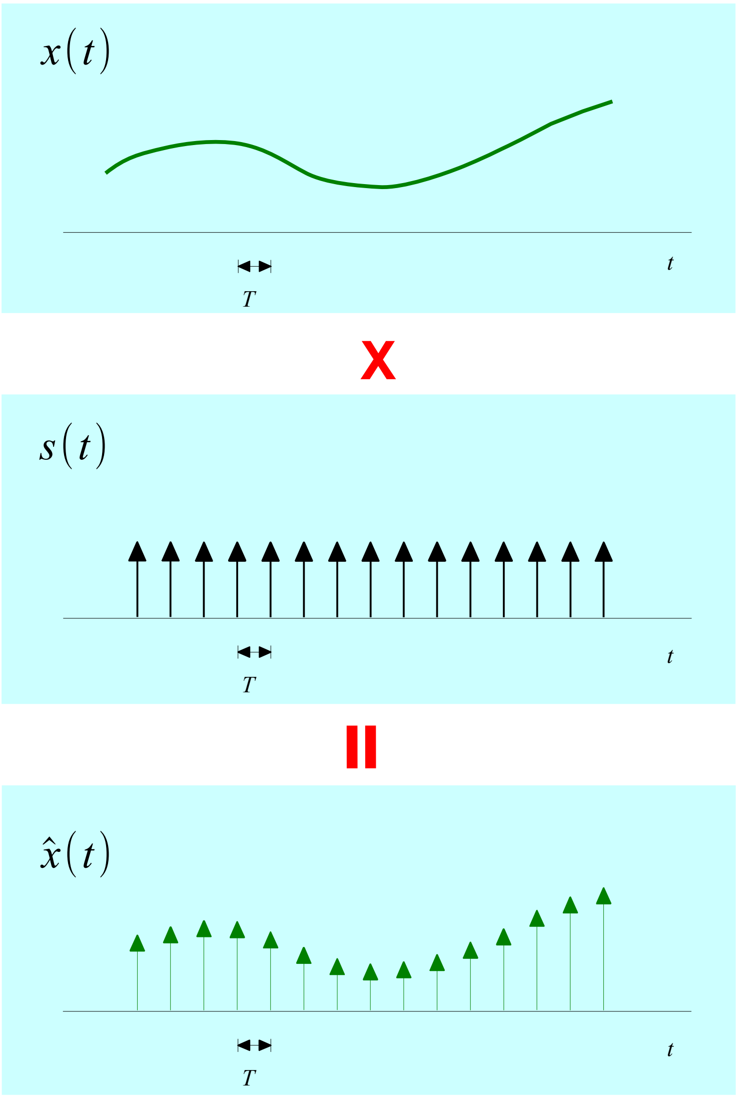


CTFT

$$\hat{X}(f) = \sum_{n=-\infty}^{+\infty} x(nT)\, e^{-j2\pi f T n}$$

Spectrum Replication (1)

Ideal Sampling



$$\hat{x}(t) = \sum_{n=-\infty}^{+\infty} x(nT)\,\delta(t-nT)$$

$$s(t) = \sum_{n=-\infty}^{+\infty} \delta(t-nT)$$

$$= \frac{1}{T} \sum_{m=-\infty}^{+\infty} e^{+j2\pi m f_s t}$$

$$\hat{x}(t) = \frac{1}{T} \sum_{n=-\infty}^{+\infty} x(t)\, e^{+j2\pi m f_s t}$$

Shift Property

$$\hat{X}(f) = \frac{1}{T} \sum_{n=-\infty}^{+\infty} X(f-m f_s)$$

Spectrum Replication (2)

$$S(f) = \frac{1}{T} \sum_{m=-\infty}^{+\infty} \delta(f - m f_s)$$

Convolution in Frequency

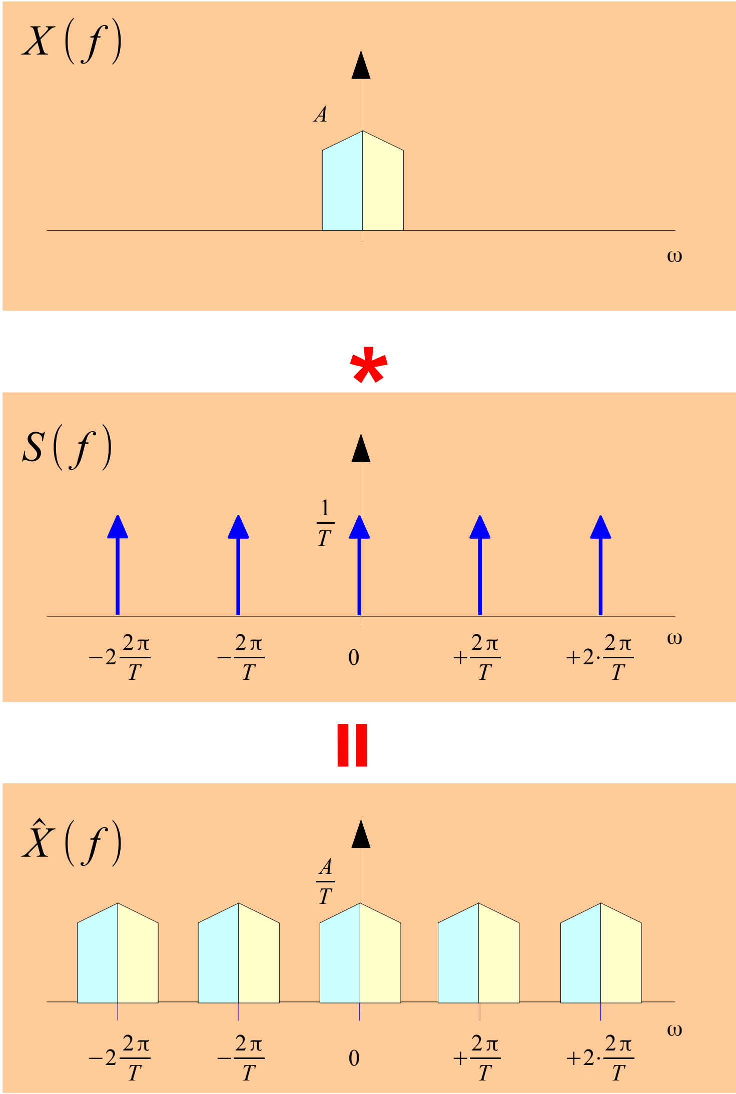
$$\hat{X}(f) = X(f) * S(f)$$

$$= \int_{-\infty}^{+\infty} X(f - f')S(f')\, d f'$$

$$= \frac{1}{T} \sum_{m=-\infty}^{+\infty} \int_{-\infty}^{+\infty} X(f - f')\delta(f' - m f_s)\, d f'$$

$$\hat{X}(f) = \frac{1}{T} \sum_{n=-\infty}^{+\infty} X(f - m f_s)$$

Frequency Domain



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