DFT Analysis (5B)

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Frequency View of a DFT Matrix

	Row	Frequency	Cycles	Normalized Frequency
	$row\ \ 0$	$f = 0$	$(0\,cycle)$	0
$\frac{N}{2}-1$	$row\ \ 1$	$f = -1 * f_o$	$(-1\,cycle)$	$\frac{-1}{N}$
	$row\ \ 2$	$f = -2 * f_o$	$(-2\,cycles)$	$\frac{-2}{N}$
	$row\ \ (\frac{N}{2}-1)$	$f = -(\frac{N}{2}-1) * f_o$	$(-(\frac{N}{2}-1)\,cycles)$	$\frac{-1}{2} + \frac{1}{N}$
	$row\ \ (\frac{N}{2})$	$f = -(\frac{N}{2}) * f_o$	$(\frac{N}{2}\ \ cycles)$	$\frac{-1}{2}$
$\frac{N}{2}-1$	$row\ \ (\frac{N}{2}+1)$	$f = +(\frac{N}{2}-1) * f_o$	$(\frac{N}{2}-1\,cycles)$	$\frac{1}{2} - \frac{1}{N}$
	$row\ \ N-2$	$f = +2 * f_o$	$(2\,cycles)$	$\frac{2}{N}$
	$row\ \ N-1$	$f = +1 * f_o$	$(1\,cycle)$	$\frac{1}{N}$

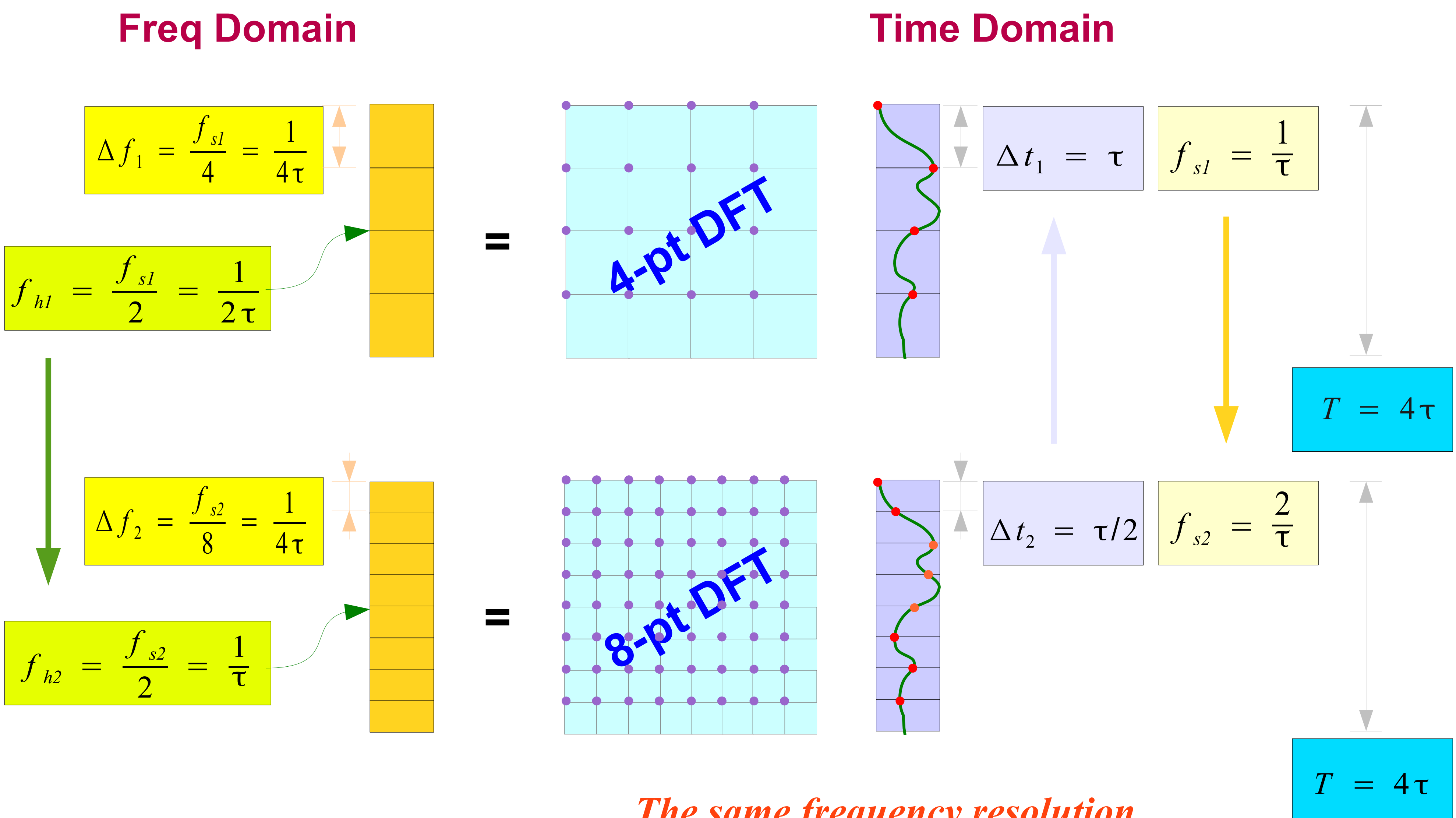
Normalized Frequency

$$f_o = \frac{f_s}{N}$$

Frequency View of a X[i] Vector

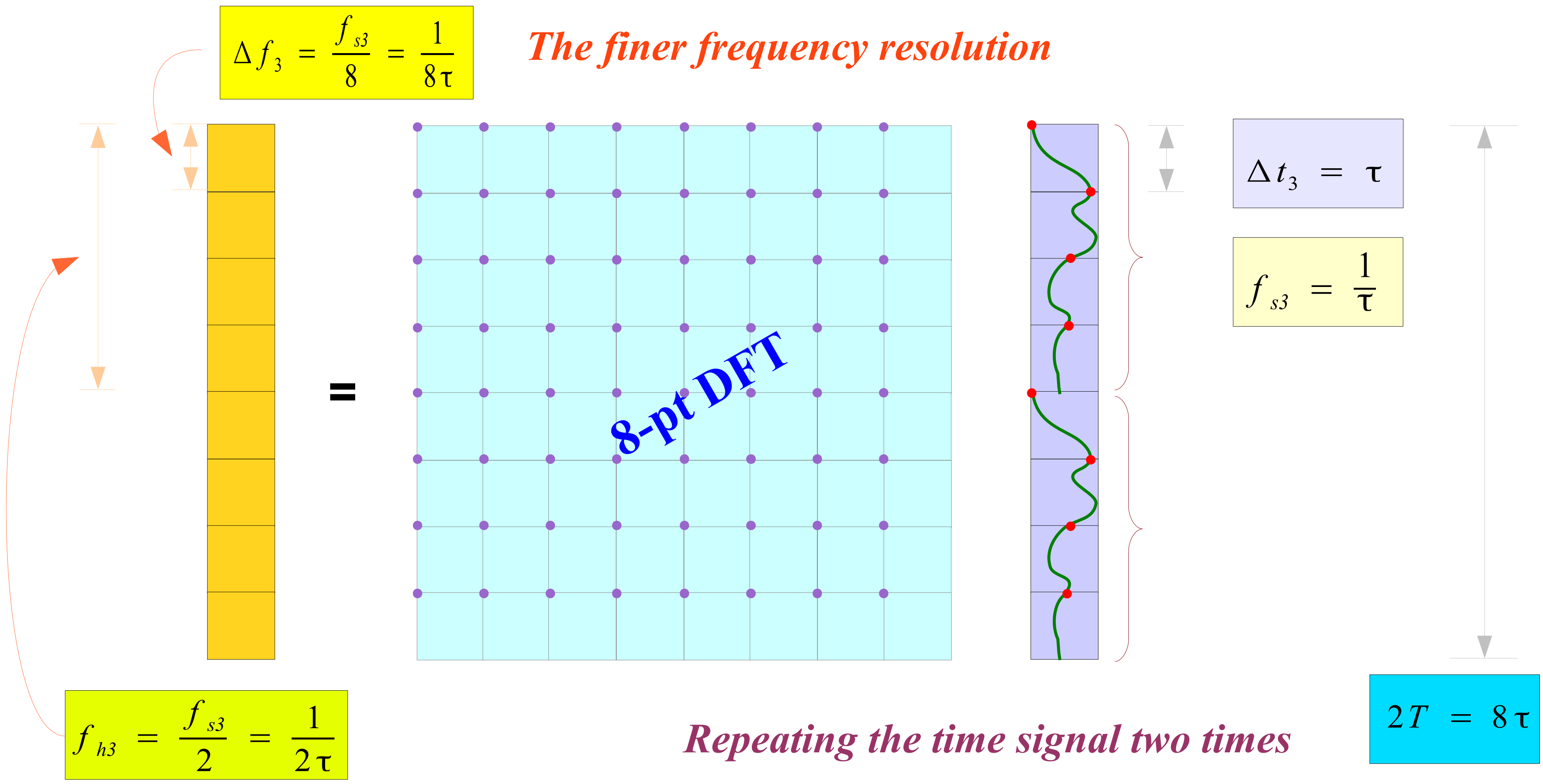
				Normalized Frequency	
	$X[0]$	$f = 0$	0	0	$(0\,cycle)$
	$X[1]$	$f = +1 * f_o$	$\frac{1}{N} f_s$	$\frac{1}{N}$	$(1\,cycle)$
	$X[2]$	$f = +2 * f_o$	$\frac{2}{N} f_s$	$\frac{2}{N}$	$(2\,cycles)$
$\frac{N}{2} - 1$					
	$X[\frac{N}{2}-1]$	$f = +(\frac{N}{2} - 1) * f_o$	$\frac{(\frac{N}{2} - 1)}{N} f_s$	$\frac{1}{2} - \frac{1}{N}$	$(\frac{N}{2} - 1\,cycles)$
	$X[\frac{N}{2}]$	$f = +(\frac{N}{2}) * f_o$	$\frac{1}{2} f_s$	$\frac{1}{2}$	$(\frac{N}{2}\,cycles)$
	$X[\frac{N}{2}+1]$	$f = -(\frac{N}{2} - 1) * f_o$	$-\frac{(\frac{N}{2} - 1)}{N} f_s$	$\frac{-1}{2} + \frac{1}{N}$	$(-(\frac{N}{2} - 1)\,cycles)$
$\frac{N}{2} - 1$					
	$X[N-2]$	$f = -2 * f_o$	$-\frac{2}{N} f_s$	$\frac{-2}{N}$	$(-2\,cycles)$
	$X[N-1]$	$f = -1 * f_o$	$-\frac{1}{N} f_s$	$\frac{-1}{N}$	$(-1\,cycle)$

Frequency and Time Interval (1)

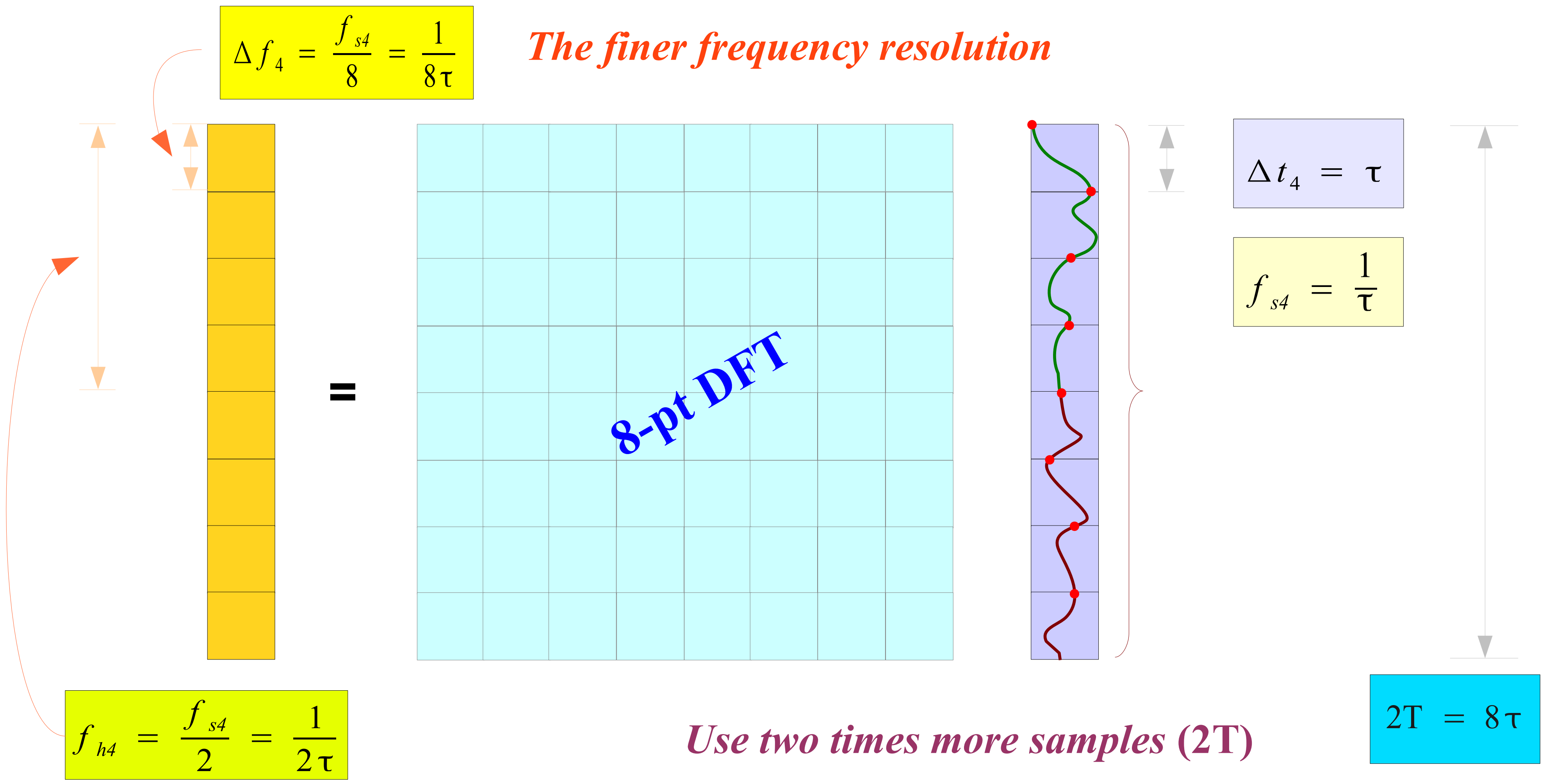


The same frequency resolution

Frequency and Time Interval (2)



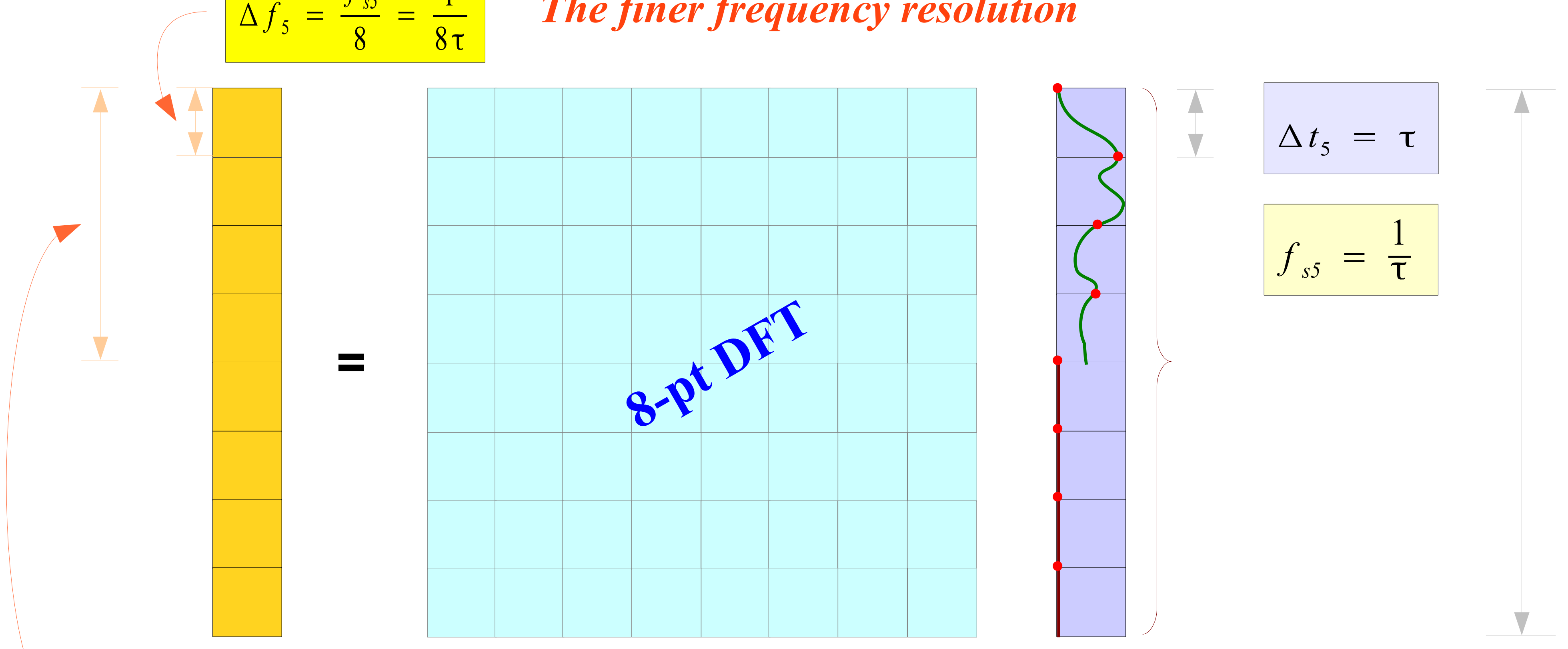
Frequency and Time Interval (3)



Frequency and Time Interval (4)

$$\Delta f_5 = \frac{f_{s5}}{8} = \frac{1}{8\tau}$$

The finer frequency resolution



$$\Delta t_5 = \tau$$

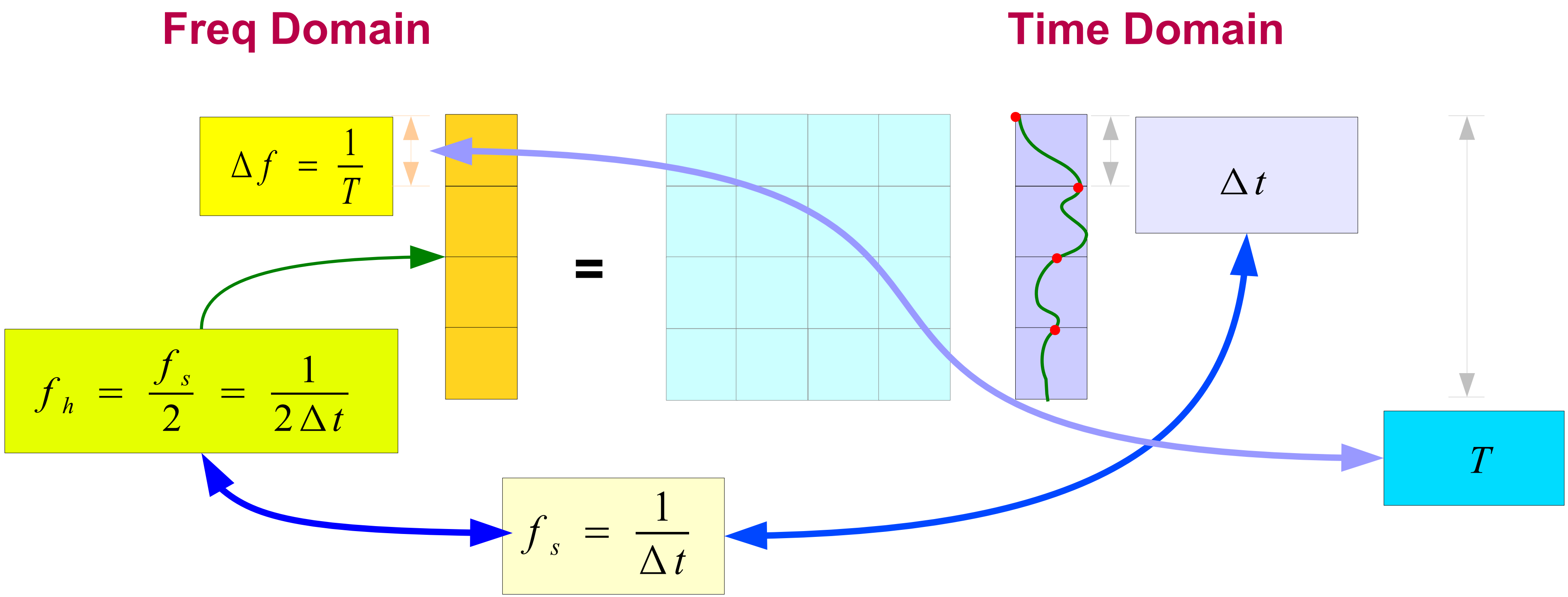
$$f_{s5} = \frac{1}{\tau}$$

$$f_{h5} = \frac{f_{s5}}{2} = \frac{1}{2\tau}$$

Use zero padding

$$2T = 8\tau$$

Frequency and Time Interval (5)



Single-Sided Spectrum

$$x(t) = a_0 + \sum_{k=1}^{\infty} \left(a_k \cos(k\omega_0 t) + b_k \sin(k\omega_0 t) \right)$$

$$a_0 = \frac{1}{T} \int_0^T x(t)\, dt$$

$$a_k = \frac{2}{T} \int_0^T x(t) \cos(k\omega_0 t)\, dt$$

$$b_k = \frac{2}{T} \int_0^T x(t) \sin(k\omega_0 t)\, dt$$

$\boldsymbol{k} = +1, +2, \ldots$

$$x(t) = g_0 + \sum_{k=1}^{\infty} g_k \cos(k\omega_0 t + \phi_k)$$

$$g_0 = a_0$$

$$g_k = \sqrt{a_k^2 + b_k^2}$$

$$\phi_k = \tan^{-1}\left(-\frac{b_k}{a_k} \right)$$

$\boldsymbol{k} = +1, +2, \ldots$

$$\cos(\alpha+\beta) = \cos(\alpha)\cos(\beta) - \sin(\alpha)\sin(\beta)$$

$$g_k \cos(k\omega_0 t + \phi_k) = g_k \cos(\phi_k)\cos(k\omega_0 t) - g_k \sin(\phi_k)\sin(k\omega_0 t)$$

$$a_k \cos(k\omega_0 t) + b_k \sin(k\omega_0 t)$$

$$a_k = g_k \cos(\phi_k)$$

$$-b_k = g_k \sin(\phi_k)$$

$$a_k^2 + b_k^2 = g_k^2$$

$$-\frac{b_k}{a_k} = \tan(\phi_k)$$

Two-Sided Spectrum

$$x(t) = \sum_{k=-\infty}^{+\infty} C_k \, e^{+jk\omega_0 t}$$

$$C_k = \frac{1}{T}\int_0^T x(t)\, e^{-jk\omega_0 t}\, dt$$

$$k = \ldots, -2, -1, 0, +1, +2, \ldots$$

$$C_k = \begin{cases} a_0 & (k = 0) \\ \frac{1}{2}(a_k - jb_k) & (k > 0) \\ \frac{1}{2}(a_k + jb_k) & (k < 0) \end{cases}$$

$$|C_k| = \begin{cases} a_0 & (k = 0) \\ \frac{1}{2}\sqrt{a_k^2 + b_k^2} & (k \neq 0) \end{cases}$$

$$Arg(C_k) = \begin{cases} \tan^{-1}(-b_k / a_k) & (k > 0) \\ \tan^{-1}(+b_k / a_k) & (k < 0) \end{cases}$$

Power Spectrum *Two-Sided*

$$|C_k|^2 + |C_{-k}|^2 = \frac{1}{2}|g_k|^2 = \frac{1}{2}(a_k^2 + b_k^2)$$

$$x(t) = \sum_{k=-\infty}^{+\infty} C_k \, e^{+jk\omega_0 t}$$

$$C_k = \frac{1}{T}\int_0^T x(t)\, e^{-jk\omega_0 t}\, dt$$

$$k = \ldots, -2, -1, 0, +1, +2, \ldots$$

$$C_k = \begin{cases} a_0 & (k = 0) \\ \frac{1}{2} g_k \, e^{+j\phi_k} & (k > 0) \\ \frac{1}{2} g_k \, e^{-j\phi_k} & (k < 0) \end{cases}$$

$$|C_k| = \begin{cases} a_0 & (k = 0) \\ \frac{1}{2}|g_k| & (k \neq 0) \end{cases}$$

$$Arg(C_k) = \begin{cases} +\phi_k & (k > 0) \\ -\phi_k & (k < 0) \end{cases}$$

Periodogram *One-Sided*

$$2 \cdot |C_k| = |g_k| = \sqrt{a_k^2 + b_k^2}$$

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CTFS and DTFS (1)

$$x(t) = \sum_{k=-\infty}^{+\infty} C_k \, e^{+jk\omega_0 t}$$ **CTFS**

$$C_k = \frac{1}{T}\int_0^T x(t)\, e^{-jk\omega_0 t}\, dt$$

$$k = -2,\ -1,\ 0,\ +1,\ +2,\ \ldots$$

$$x(t) \approx \sum_{k=-M}^{+M} C_k \, e^{+jk\omega_0 t}$$

$$N = 2M + 1$$

$$jk\omega_0 t \rightarrow k\left(\frac{2\pi}{T}\right) n \left(\frac{T}{N}\right) = \left(\frac{2\pi}{T}\right) n k$$

$$x[n] = \sum_{k=-M}^{+M} \gamma_k \, e^{+j\left(\frac{2\pi}{N}\right)nk}$$

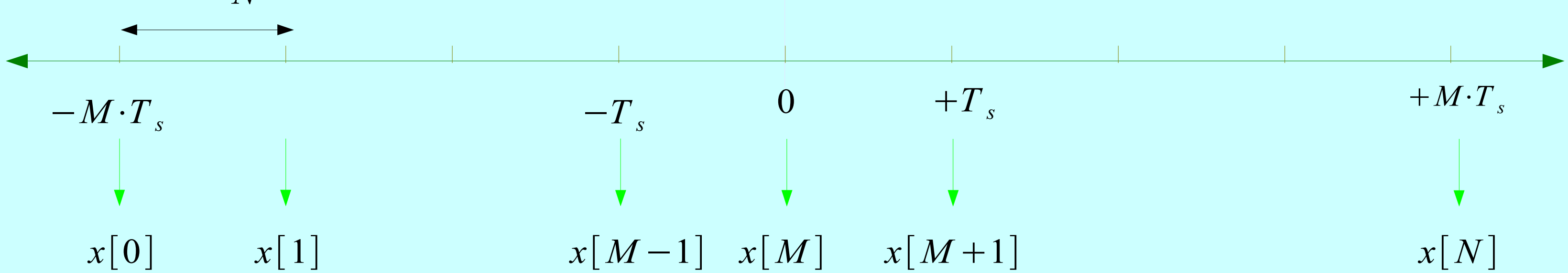
$$n = 0,\ 1,\ 2,\ \ldots,\ N-1,$$

$$\gamma_k = \frac{1}{N}\sum_{n=0}^{N-1} x[n]\, e^{-j\left(\frac{2\pi}{N}\right)nk}$$

$$k = -M,\ \ldots,\ 0,\ \ldots,\ +M$$

$$x_{FS}(t) = \sum_{k=-M}^{+M} \gamma_k \, e^{+jk\omega_0 t}$$ **DTFS**

$$T_s = \frac{T}{N}$$ **Time Samples**



$0 \le n < L$ $\qquad T_s \rightarrow 0$

Discrete Fourier Transform

$$X[k] = \sum_{n=0}^{N-1} x[n]\, e^{-j(2\pi/N)kn}$$

$$x[\boldsymbol{n}] = \sum_{\boldsymbol{k}=-M}^{+M} \gamma_{\boldsymbol{k}}\, e^{+j\left(\frac{2\pi}{N}\right)\boldsymbol{nk}}$$

$$\boldsymbol{n} = 0,\ 1,\ 2,\ \ldots\ ,\ N-1,$$

$$x[n] = \frac{1}{N} \sum_{k=0}^{N-1} X[k]\, e^{+j(2\pi/N)kn}$$

$$\gamma_{\boldsymbol{k}} = \frac{1}{N} \sum_{\boldsymbol{n}=0}^{N-1} x[\boldsymbol{n}]\, e^{-j\left(\frac{2\pi}{N}\right)\boldsymbol{nk}}$$

$$\boldsymbol{k} = -M,\ \ldots\ ,\ 0,\ \ldots\ ,\ +M$$

$$x(t) = \sum_{\boldsymbol{k}=-\infty}^{+\infty} C_{\boldsymbol{k}}\, e^{+j\boldsymbol{k}\omega_0 t}$$ **CTFS**

Truncate Fourier Coefficients

$$x_{FS}(t) = \sum_{\boldsymbol{k}=-M}^{+M} \gamma_{\boldsymbol{k}}\, e^{+j\boldsymbol{k}\omega_0 t}$$ **DTFS**

$$C_k \approx \gamma_{\boldsymbol{k}} = \frac{X[k]}{N}$$

Approximated Fourier Coefficients

fft and ifft

Discrete Fourier Transform

$$X[k] = \sum_{n=0}^{N-1} x[n]\, e^{-j(2\pi/N)kn}$$

`X = fft(x)`

$$x[n] = \frac{1}{N} \sum_{k=0}^{N-1} X[k]\, e^{+j(2\pi/N)kn}$$

`x = ifft(X)`

$$x(t) = \sum_{k=-\infty}^{+\infty} C_k\, e^{+jk\omega_0 t}$$ **CTFS**

Approximated Fourier Series Coefficients

$$x_{FS}(t) = \sum_{k=-M}^{+M} \gamma_k\, e^{+jk\omega_0 t}$$ **DTFS**

`fc = fft(x)/N = X/N`

$$C_k \approx \gamma_k = \frac{X[k]}{N}$$ **Approximated Fourier Coefficients**

`x = ifft(fc)*N`

$$|C_k|^2 \approx \frac{|X[k]|^2}{N}$$ **Approximated Periodogram**

	Periodic Signals	Aperiodic Signals	Random Signals	
Frequency Spacing	$\Delta f = \frac{1}{N \Delta t}$	$\Delta f = \frac{1}{N \Delta t}$	$\sum S \Delta f = \frac{1}{N \Delta t} \sum S$	$\frac{1}{N \Delta t} \sum x^2 \Delta t$
Two Sided	$\frac{1}{N} X(k)$	$\frac{\Delta t}{N} X(k)$	$S(k) = \frac{\Delta t}{N} \lvert X(k) \rvert^2$	$P = \sum_{k=0}^{N-1} S(k) \Delta f$
One Sided				
$k=0, \; \frac{N}{2}$	$\frac{1}{N} X(k)$	$\frac{\Delta t}{N} X(k)$	$S_1(k) = 2\mathrm{S}(k)$	$P = \sum_{k=0}^{N/2} S_1(k) \Delta f$
$k = 1, \cdots, \frac{N}{2} - 1$	$\frac{2}{N} X(k)$	$\frac{2 \Delta t}{N} X(k)$	$S_1(k) = S(k)$	
Frequency Scale	$k \Delta f$	$k \Delta f$	$k \Delta f$	

Periodic Signals

Frequency Spacing

$\Delta f = \frac{1}{N \Delta t}$

Two Sided Fourier Series Coefficient

$\frac{1}{N} X(k)$

One Sided Fourier Series Coefficient

$\frac{1}{N} X(k) \qquad k = 0, \ \frac{N}{2}$

$\frac{2}{N} X(k) \qquad k = 1, \cdots, \frac{N}{2} - 1$

Frequency Scale

$k \Delta f$

Aperiodic Signals

$\Delta f = \frac{1}{N \Delta t}$

Two Sided Fourier Series Coefficient

$\frac{\Delta t}{N} X(k)$

One Sided Fourier Series Coefficient

$\frac{\Delta t}{N} X(k) \qquad k = 0, \ \frac{N}{2}$

$\frac{2 \Delta t}{N} X(k) \qquad k = 1, \cdots, \frac{N}{2} - 1$

$k \Delta f$

Random Signals

One-sided Power Spectral Density

$$P = \sum_{k=0}^{N-1} S(k) \Delta f$$

One-sided Power Spectral Density

$$P = \sum_{k=0}^{N/2} S_1(k) \Delta f$$

$$S_1(k) = 2S(k) \qquad k = 1, \ldots, \tfrac{N}{2} - 1$$

$$S_1(k) = S(k) \qquad k = 0, \tfrac{N}{2}$$

Two Sided Fourier Series Coefficient

$$\frac{1}{N \Delta t} \sum x^2 \Delta t$$

$$\sum S \Delta f = \frac{1}{N \Delta t} \sum S$$

$$S(k) = \frac{\Delta t}{N} |X(k)|^2$$

$$k \Delta f$$

Amplitude Spectrum

$$A_k = \frac{1}{N}|X(k)| = \frac{1}{N}\sqrt{\Re^2(X(k))+\Im^2(X(k))}$$

$$k = 0,1,2,\cdots,N-1$$

Power Spectrum

$$P_k = \frac{1}{N^2}|X(k)|^2 = \frac{1}{N^2}\{\Re^2(X(k))+\Im^2(X(k))\}$$

$$k = 0,1,2,\cdots,N-1$$

One Sided Amplitude Spectrum

$$\bar{A}_k = \frac{1}{N}|X(0)| \qquad k=0$$

$$\bar{A}_k = \frac{2}{N}|X(0)| \qquad k=1,2,\cdots,N/2$$

One Sided Power Spectrum

$$\bar{P}_k = \frac{1}{N^2}|X(0)|^2 \qquad k=0$$

$$\bar{P}_k = \frac{2}{N^2}|X(0)|^2 \qquad k=1,2,\cdots,N/2$$

Frequency Bin

$$f=\frac{k f_s}{N}$$

Frequency Bin

$$f=\frac{k f_s}{N}$$

Phase Spectrum

$$\phi_k = \tan^{-1}\left(\frac{\Im(X(k))}{\Re(X(k))}\right) \qquad k=0,1,2,\cdots,N-1$$

Data Truncation
Frequency Resolution
Zero Padding
Periodogram
Spectral Plot

Amplitude spectrum in quantity peak
Phase spectrum in radians
Amplitude spectrum in volts rms
Phase spectrum in degrees
Power spectrum

Signals without discontinuity
Signals with discontinuity

Sampling frequency is not an integer multiple of the FFT length

Leakage

$$[0, \quad \frac{f_s}{2}]$$

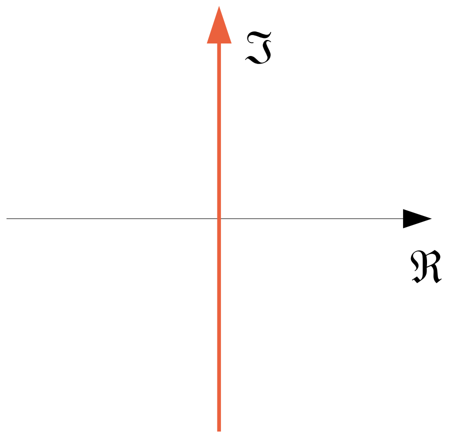
Fourier Transform

$f(t)$ A continuous sum of weighted exponential functions :

$$f(t)\, e^{-j\omega t}$$

$$-\infty < \omega < +\infty$$

Not so useful in transient analysis

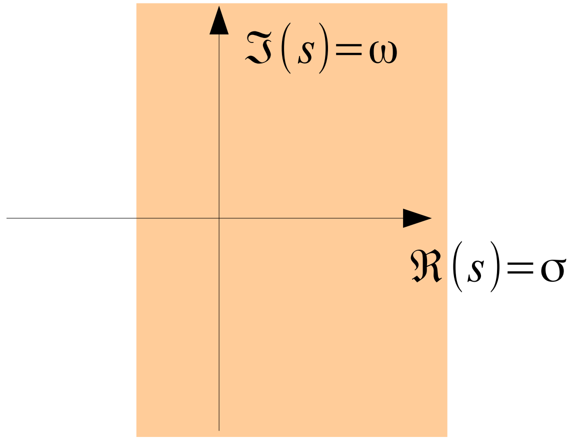


Laplace Transform

$$f(t)\, e^{-st} = f(t) e^{-(\sigma + j\omega)t}$$

Linear Time Domain Analysis

Initial Condition



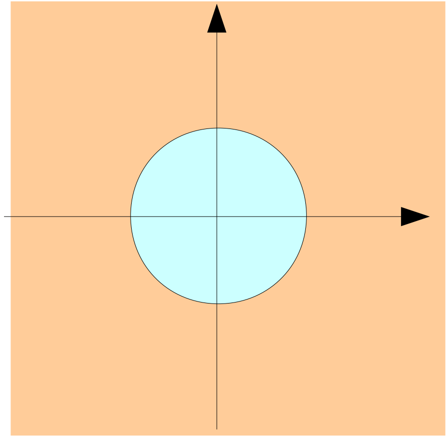
z Transform

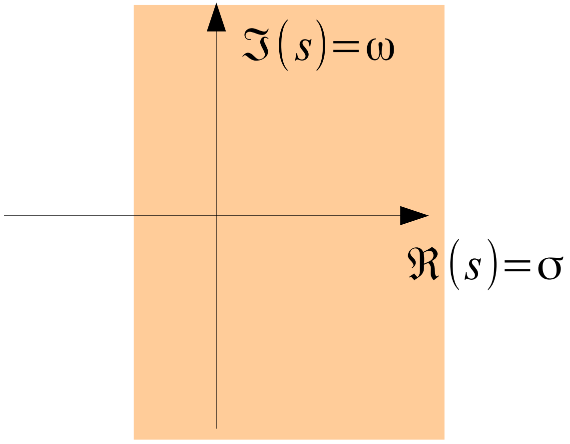
$$f[n]\, z^{-n}$$

Discrete Time System

Difference Equation

$$z = e^{sT} = e^{\sigma T}\, e^{j\omega T}$$





References

[1] http://en.wikipedia.org/
[2] J.H. McClellan, et al., Signal Processing First, Pearson Prentice Hall, 2003
[3] A “graphical interpretation” of the DFT and FFT, by Steve Mann