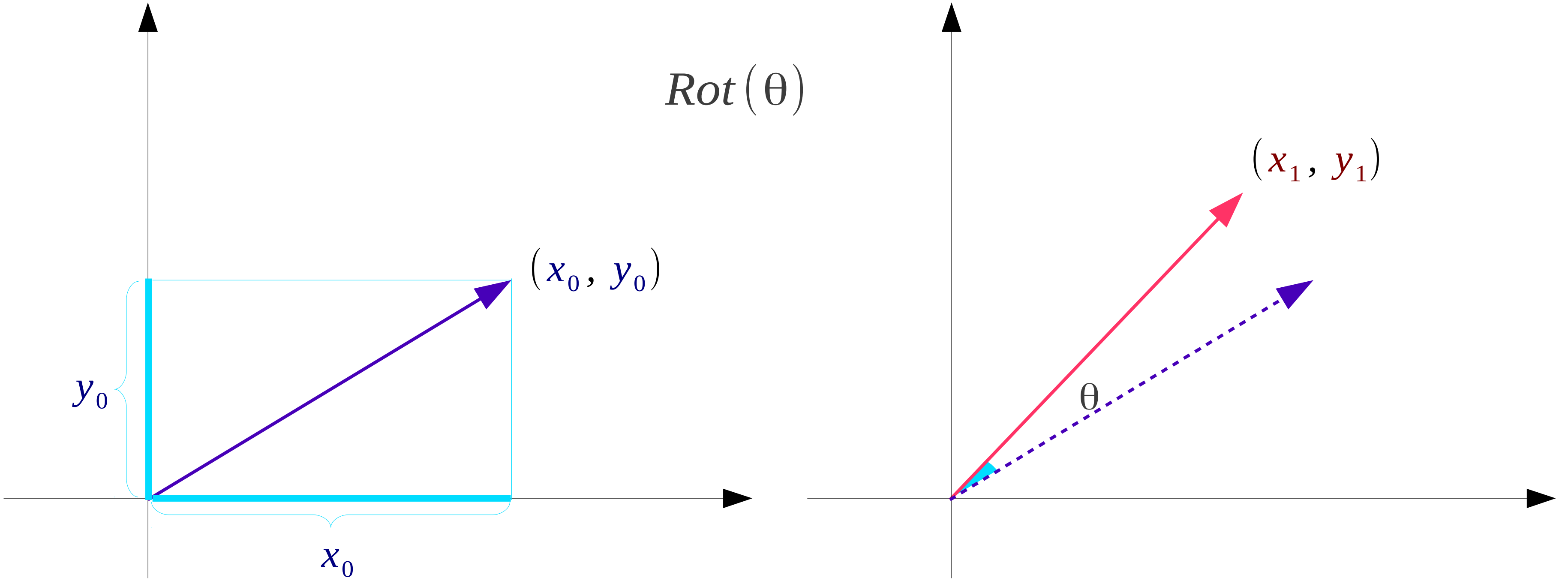
General Vector Space (2A)

Young Won Lim
11/4/12

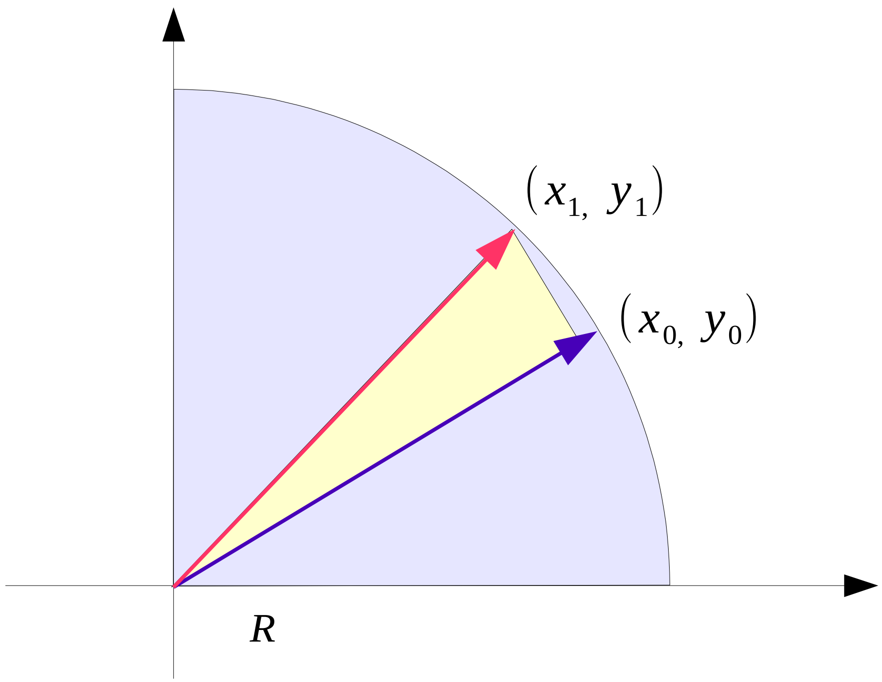
Please send corrections (or suggestions) to youngwlim@hotmail.com.

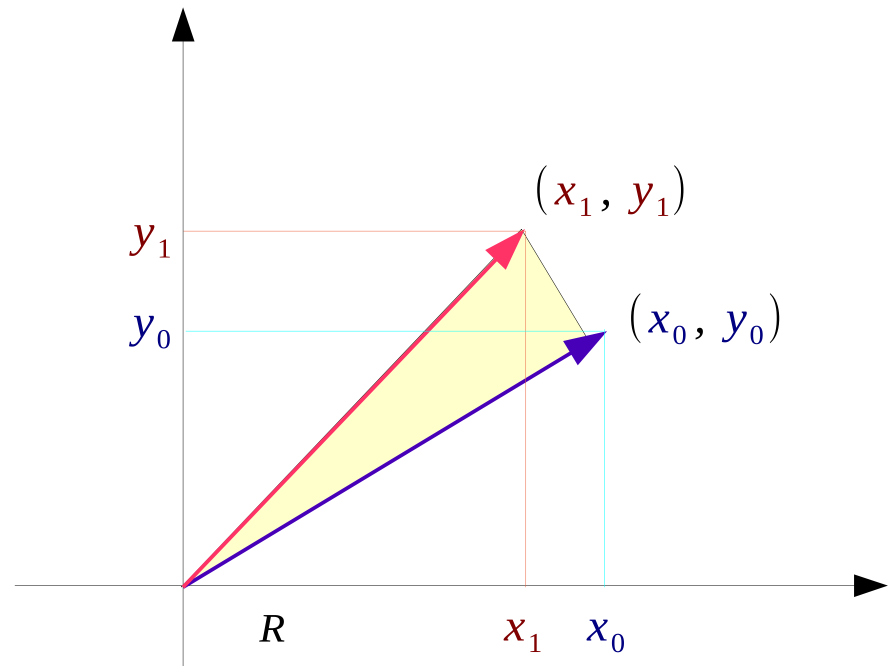
This document was produced by using OpenOffice and Octave.

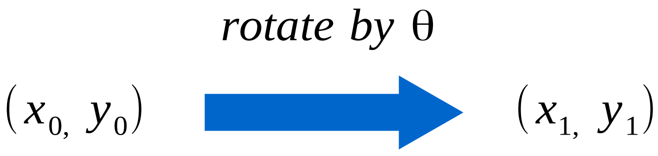
Vector Rotation (1)



Vector Rotation (2)



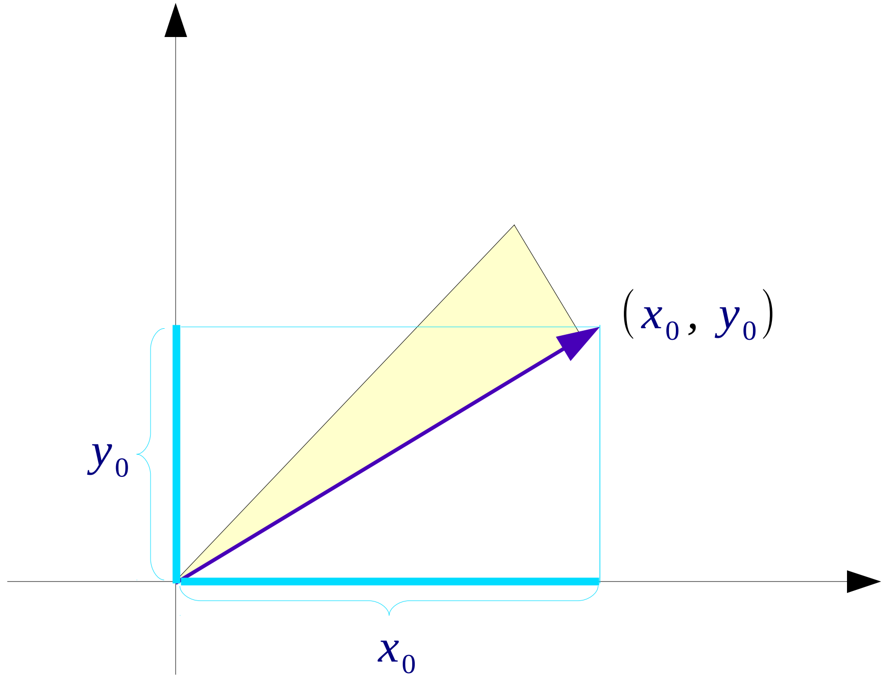


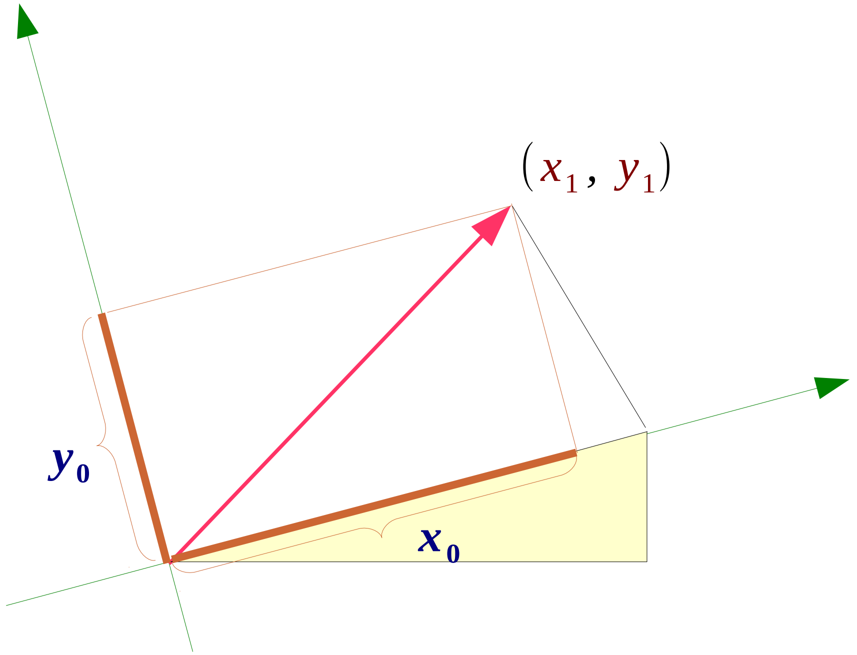


$$x_1 = x_0 \cos\theta - y_0 \sin\theta$$

$$y_1 = x_0 \sin\theta + y_0 \cos\theta$$

Vector Rotation (3)





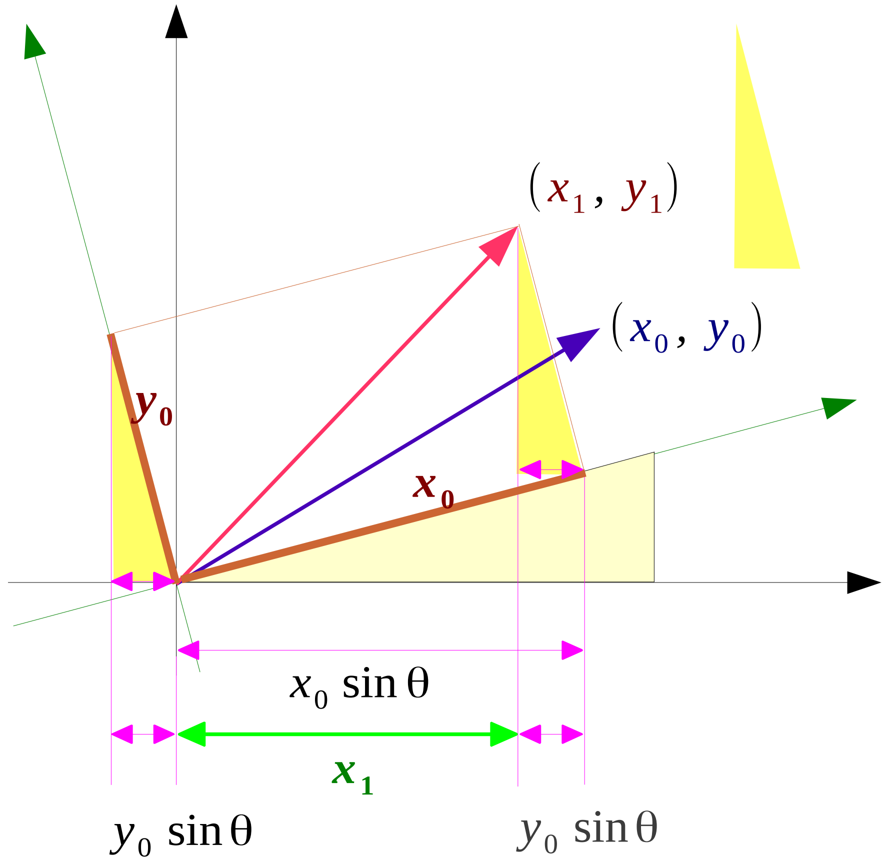
In the rotated coordinate

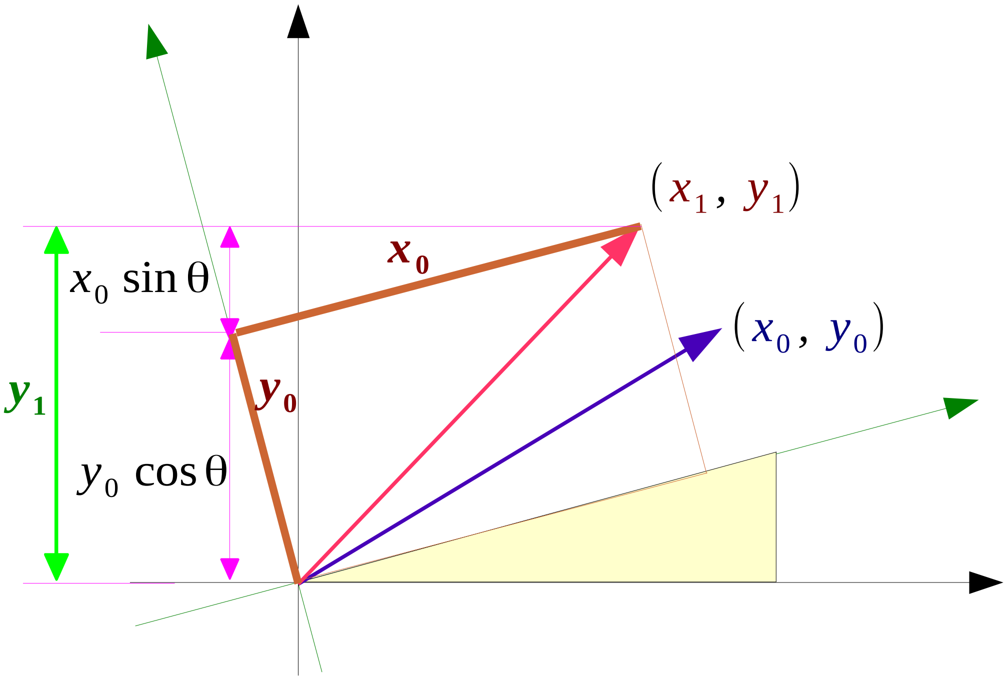
invariant length x_0, y_0

Vector Rotation (4)

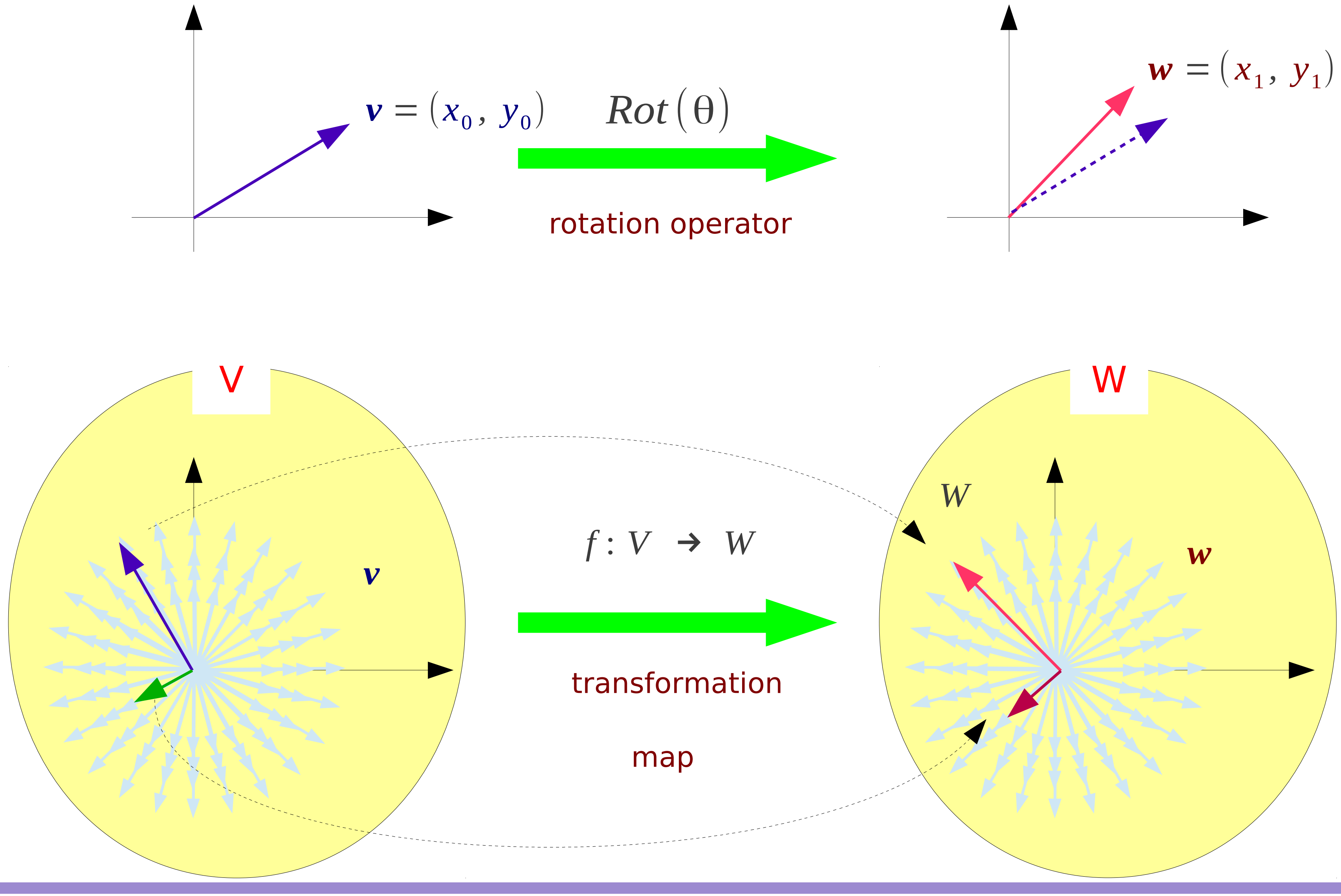
$$x_1 = x_0 \cos\theta - y_0 \sin\theta$$

$$y_1 = x_0 \sin\theta + y_0 \cos\theta$$

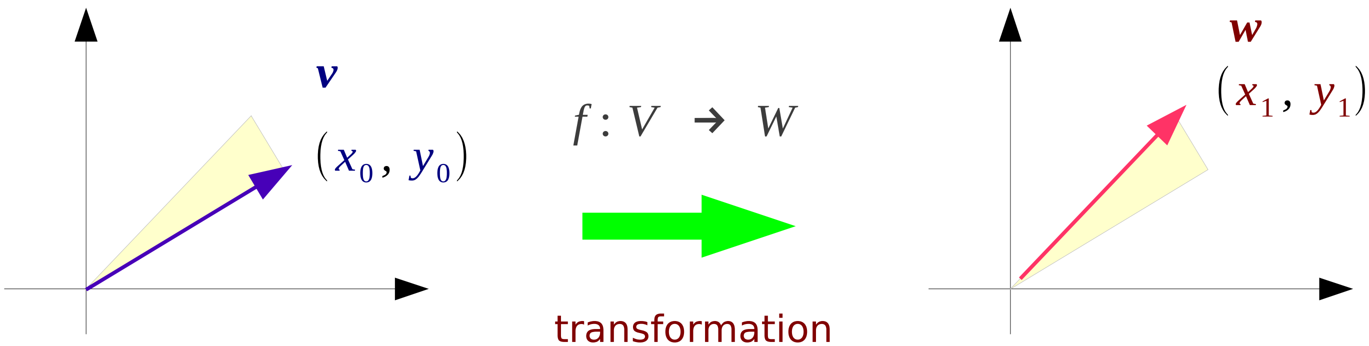




Transformation



Matrix Transformation



$$x_1 = x_0 \cos\theta - y_0 \sin\theta$$
$$y_1 = x_0 \sin\theta + y_0 \cos\theta$$

$$\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x_0 \\ y_1 \end{bmatrix}$$

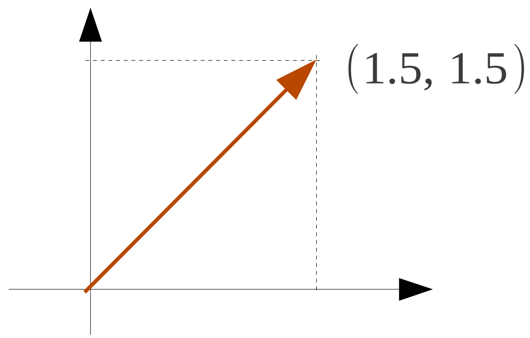
$$\boldsymbol{A} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

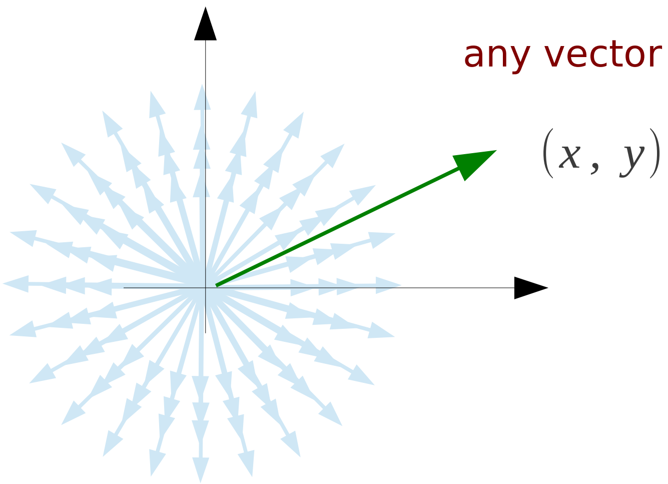
$$\boldsymbol{w} = \boldsymbol{A} \quad \boldsymbol{x}$$
$$\boldsymbol{w} = T_{\boldsymbol{A}}(\boldsymbol{x})$$

$$\boldsymbol{x} \xrightarrow{T_{\boldsymbol{A}}} \boldsymbol{w}$$

Coordinates and Coordinates Systems

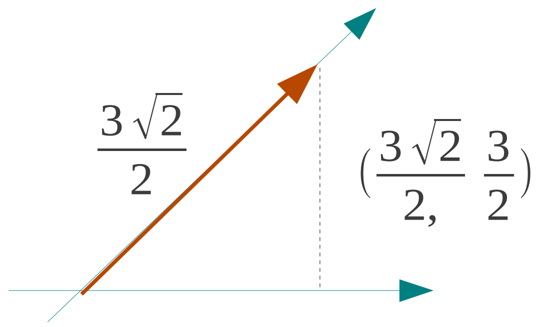
Rectangular Coordinate System

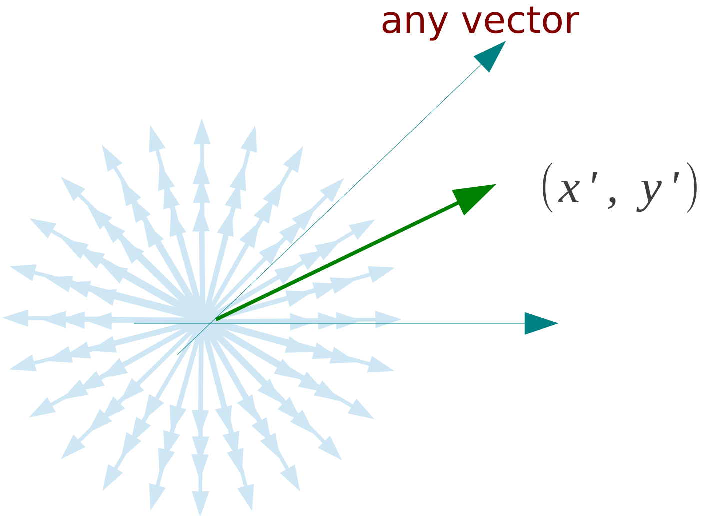




coordinates $(1.5,\ 1.5)$

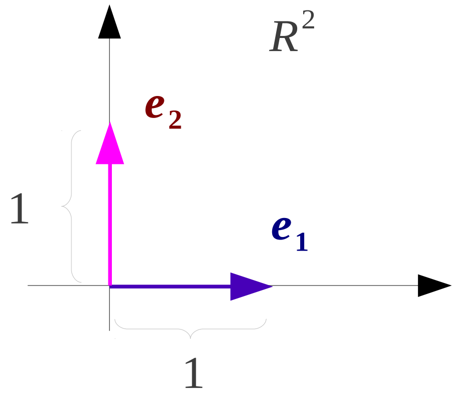
Non-Rectangular Coordinate System





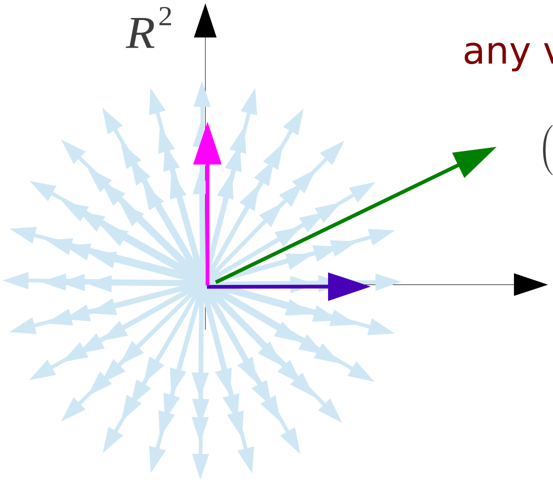
coordinates $(\frac{3\sqrt{2}}{2},\ \frac{3}{2})$

Standard Basis

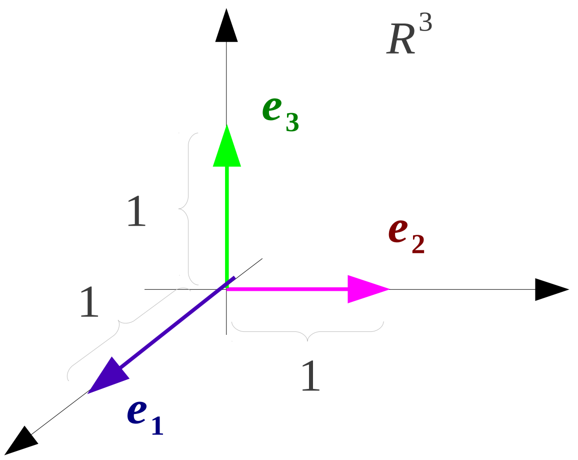


$e_1 = (1,\ 0)$

$e_2 = (0,\ 1)$



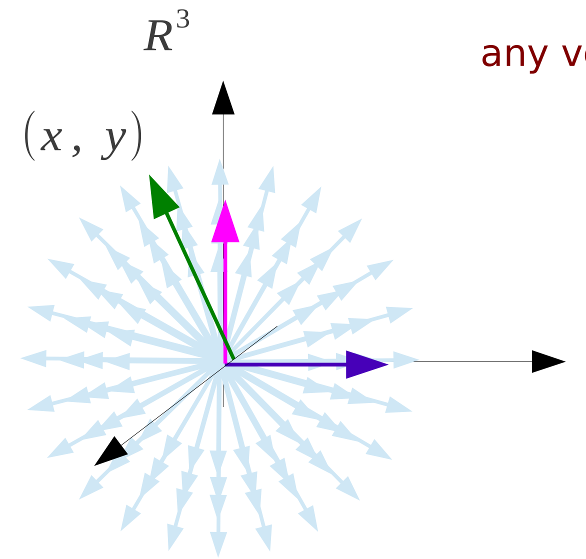
standard basis $\{e_1,\ e_2\}$ **spans** R^2



$e_1 = (1,\ 0,\ 0)$

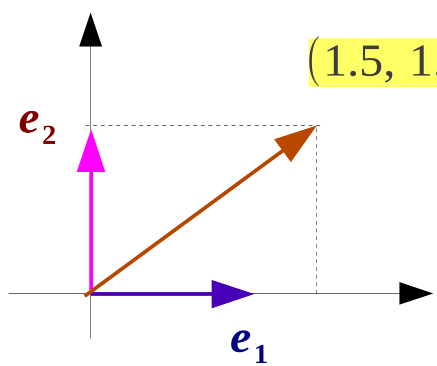
$e_2 = (0,\ 1,\ 0)$

$e_3 = (0,\ 0,\ 1)$



standard basis $\{e_1,\ e_2,\ e_3\}$ **spans** R^3

Basis and Coordinates



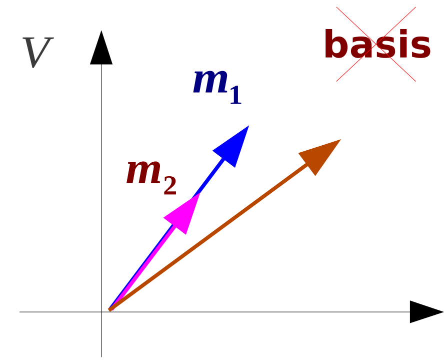
$$(1.5,\ 1.0) = 1.5\,\boldsymbol{e}_1 + 1.0\,\boldsymbol{e}_2$$

$$= 1.5\begin{bmatrix}1\\0\end{bmatrix} + 1.0\begin{bmatrix}0\\1\end{bmatrix}$$

$$= \begin{bmatrix}1.5 & 1.0\end{bmatrix}\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}$$

basis $\{\boldsymbol{e}_1,\ \boldsymbol{e}_2\}$

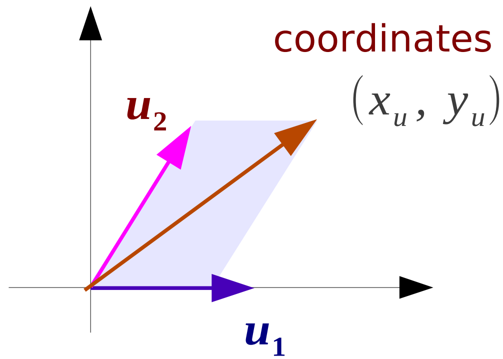
coordinates $(1.5,\ 1.0)$



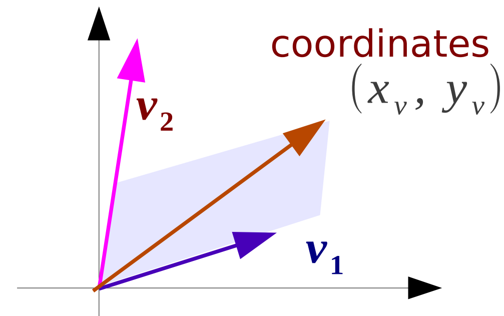
collinear vectors → linearly dependent vectors

many bases but the same number of basis vectors

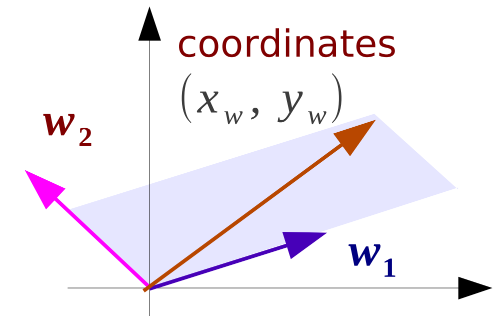
basis $\{\boldsymbol{u}_1,\ \boldsymbol{u}_2\}$



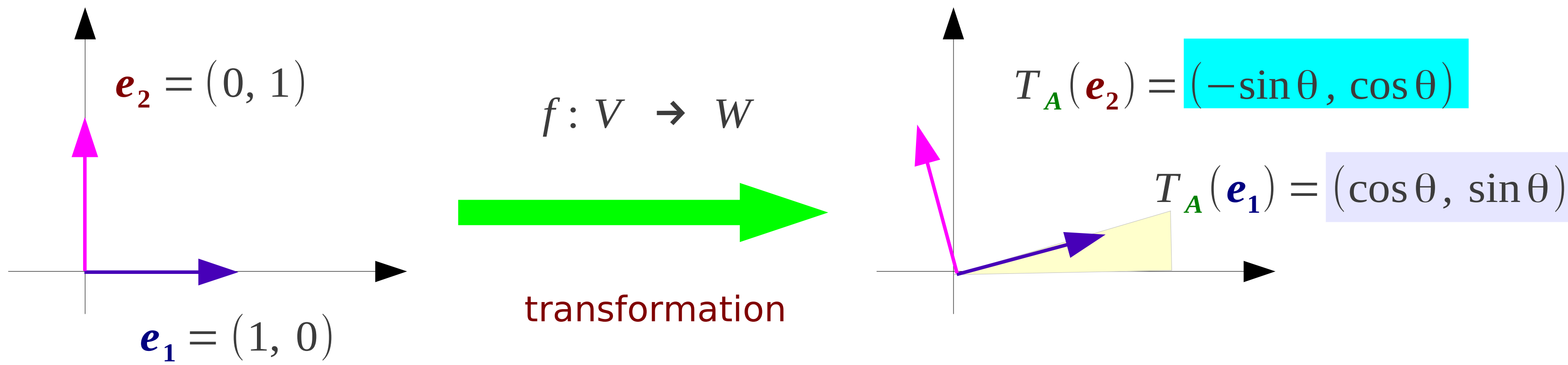
basis $\{\boldsymbol{v}_1,\ \boldsymbol{v}_2\}$



basis $\{\boldsymbol{w}_1,\ \boldsymbol{w}_2\}$



Standard Basis & Standard Matrix



standard basis

$$\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x_0 \\ y_1 \end{bmatrix}$$

$$\boldsymbol{w} = \boldsymbol{A} \qquad \boldsymbol{x}$$

$$\boldsymbol{w} = T_A(\boldsymbol{x})$$

$$\boldsymbol{x} \xrightarrow{T_A} \boldsymbol{w}$$

standard matrix

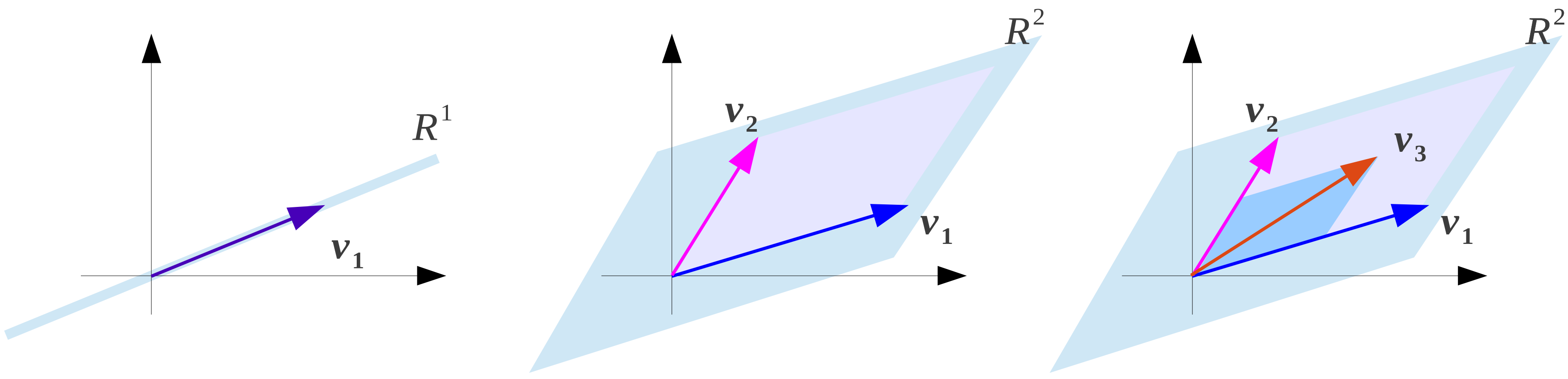
$$\boldsymbol{A} = \begin{pmatrix} T_A(\boldsymbol{e}_1) & T_A(\boldsymbol{e}_2) & & T_A(\boldsymbol{e}_n) \end{pmatrix}$$

Dimension

In vector space R^2

any one vector	line	R^1	linearly independent
any two non-collinear vectors	plane	R^2	linearly independent
any three or more vectors			linearly dependent

$$\boldsymbol{v}_3 = k_1 \boldsymbol{v}_1 + k_2 \boldsymbol{v}_2$$



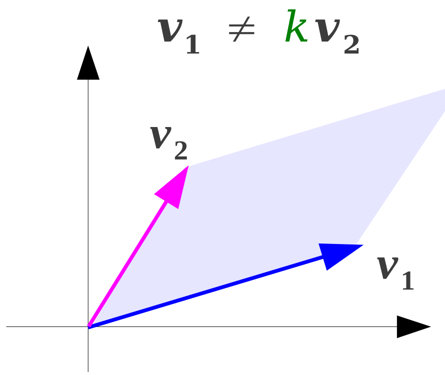
Basis

$S = \{\boldsymbol{v_1}, \boldsymbol{v_2}, \cdots, \boldsymbol{v_n}\}$ non-empty finite set of vectors in V

S is a basis ⟺

S linearly independent

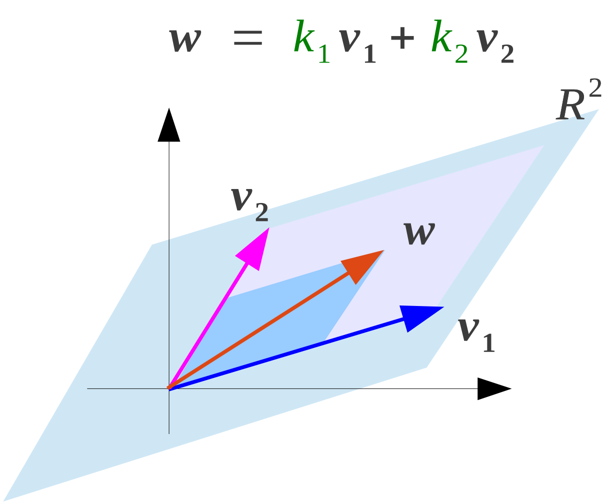
S spans V



span(S) $= span\{\boldsymbol{v_1}, \boldsymbol{v_2}, \cdots, \boldsymbol{v_n}\}$ ⟺

all possible linear combination of the vectors in in S

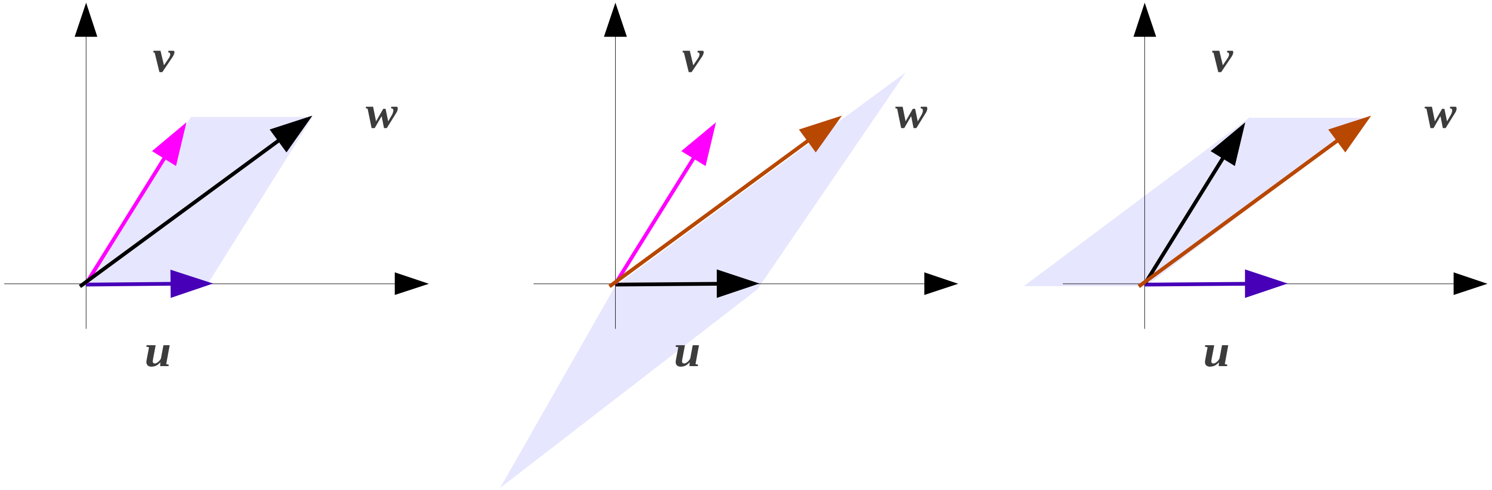
$\{ \boldsymbol{w} = k_1\boldsymbol{v_1} + k_2\boldsymbol{v_2} + \cdots + k_n\boldsymbol{v_n} \}$



Linear Dependent (1)

$\{\boldsymbol{u}\ ,\ \boldsymbol{v}\ ,\ \boldsymbol{w}\}$ linearly dependent

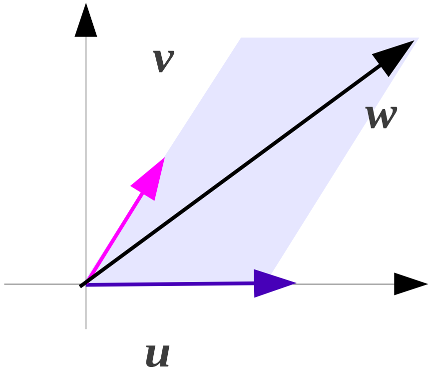
$\boldsymbol{w} = \boldsymbol{u} + \boldsymbol{v}$ $\boldsymbol{u} = \boldsymbol{w} - \boldsymbol{v}$ $\boldsymbol{v} = \boldsymbol{w} - \boldsymbol{u}$

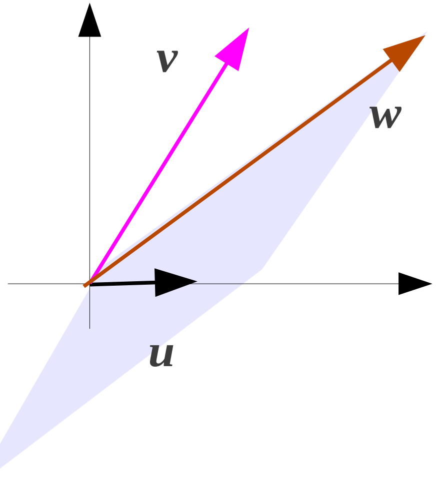


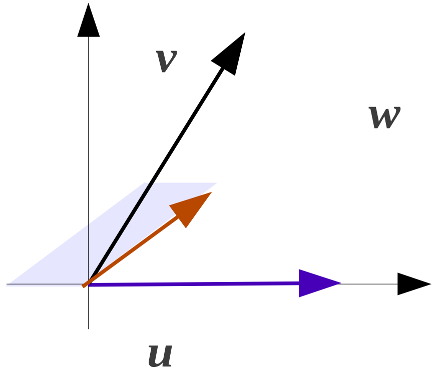
$\boldsymbol{u} + \boldsymbol{v} - \boldsymbol{w} = \mathbf{0}$ $\boldsymbol{u} + \boldsymbol{v} - \boldsymbol{w} = \mathbf{0}$ $\boldsymbol{u} + \boldsymbol{v} - \boldsymbol{w} = \mathbf{0}$

Linear Dependent (2)

$\{\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}\}$ linearly dependent







$$k_1\boldsymbol{u} + k_2\boldsymbol{v} + k_3\boldsymbol{w} = \boldsymbol{0}$$

$$(k_1 = 0)\wedge(k_2 = 0)\wedge(k_3 = 0)$$
$$(k_1 \neq 0)\vee(k_2 \neq 0)\vee(k_3 \neq 0)$$

$$m_1\boldsymbol{u} + m_2\boldsymbol{v} + m_3\boldsymbol{w} = \boldsymbol{0}$$

$$(m_1 = 0)\wedge(m_2 = 0)\wedge(m_3 = 0)$$
$$(m_1 \neq 0)\vee(m_2 \neq 0)\vee(m_3 \neq 0)$$

$$n_1\boldsymbol{u} + n_2\boldsymbol{v} + n_3\boldsymbol{w} = \boldsymbol{0}$$

$$(n_1 = 0)\wedge(n_2 = 0)\wedge(n_3 = 0)$$
$$(n_1 \neq 0)\vee(n_2 \neq 0)\vee(n_3 \neq 0)$$

Linear Independent (1)

$S = \{\boldsymbol{v_1}, \boldsymbol{v_2}, \cdots, \boldsymbol{v_n}\}$ non-empty set of vectors in V

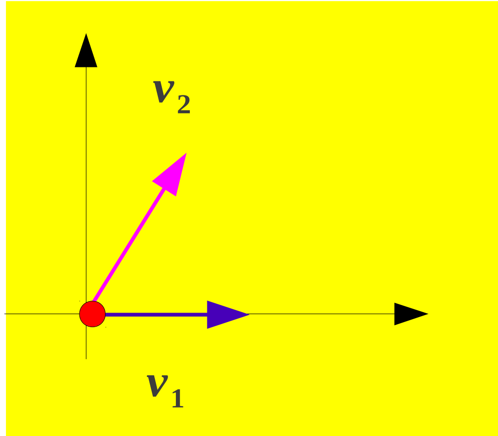
$$k_1\boldsymbol{v_1} + k_2\boldsymbol{v_2} + \cdots + k_n\boldsymbol{v_n} = \boldsymbol{0}$$

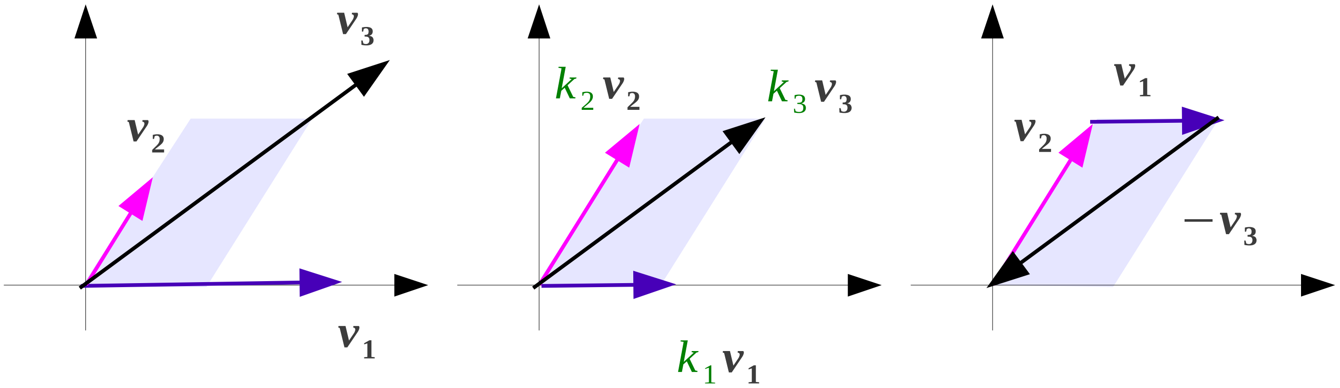
the solution of the above equation

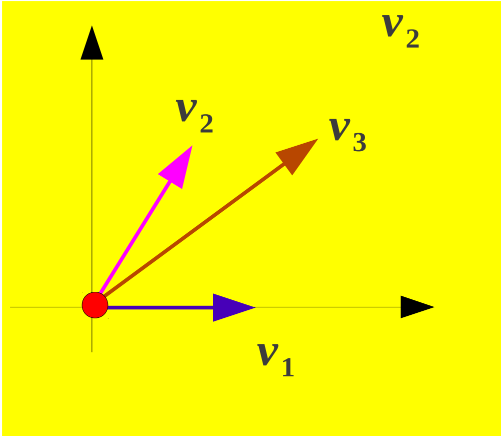
trivial solution: $k_1 = k_2 = \cdots = k_n = 0$

if other solution exists — S linearly dependent

if no other solution exists — S linearly independent







Linear Independent (2)

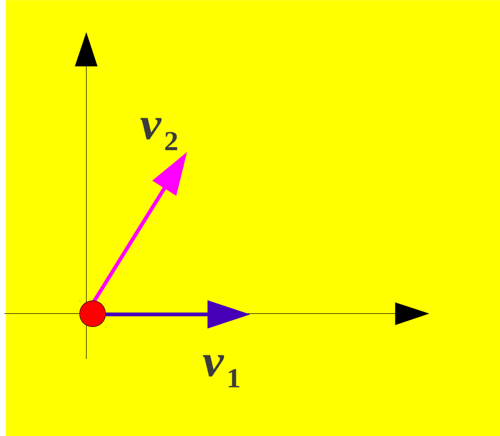
$S = \{\boldsymbol{v_1}, \boldsymbol{v_2}, \cdots, \boldsymbol{v_n}\}$ non-empty set of vectors in V

$$k_1\boldsymbol{v_1} + k_2\boldsymbol{v_2} + \cdots + k_n\boldsymbol{v_n} = \boldsymbol{0}$$

the solution of the above equation

$$k_1 = k_2 = \cdots = k_n = 0$$

- if other solution exists ⟺ S linearly dependent
- if no other solution exists ⟺ S linearly independent



at least one vector in S is a linear combination of the other vectors in S

$$\boldsymbol{v_i} = k_1\boldsymbol{v_1} + k_2\boldsymbol{v_2} + \cdots + k_{i-1}\boldsymbol{v_{i-1}} + k_{i+1}\boldsymbol{v_{i+1}} + \cdots + k_n\boldsymbol{v_n}$$

⟺ S linearly dependent

no vector in S is a linear combination of the other vectors in S

$$\boldsymbol{v_i} \neq k_1\boldsymbol{v_1} + k_2\boldsymbol{v_2} + \cdots + k_{i-1}\boldsymbol{v_{i-1}} + k_{i+1}\boldsymbol{v_{i+1}} + \cdots + k_n\boldsymbol{v_n}$$

⟺ S linearly independent

Linear Independent (3)

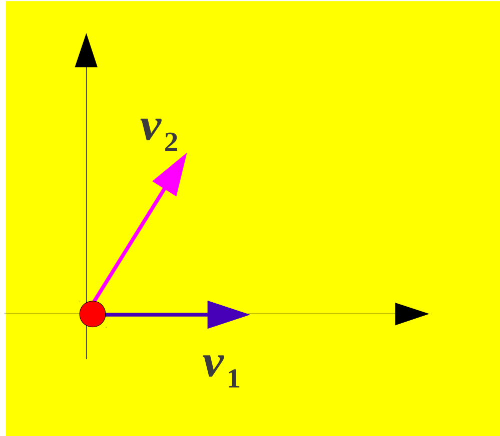
$S = \{\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_n\}$ non-empty set of vectors in V

$k_1\mathbf{v}_1 + k_2\mathbf{v}_2 + \cdots + k_n\mathbf{v}_n = \mathbf{0}$

the solution of the above equation

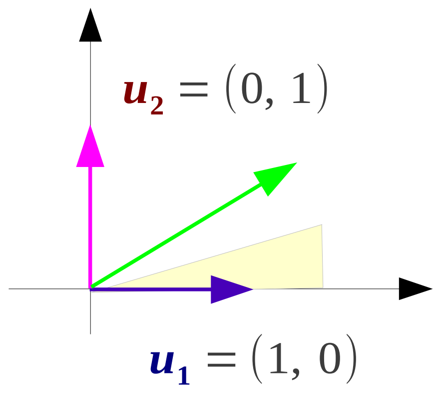
$$k_1 = k_2 = \cdots = k_n = 0$$

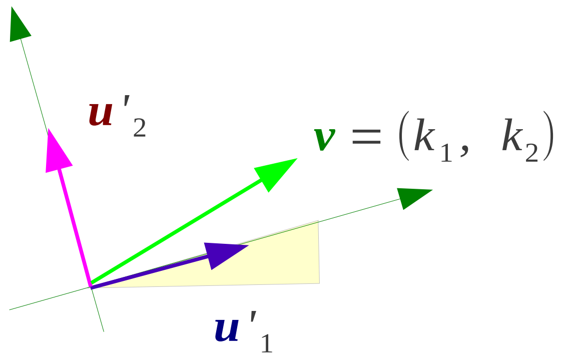
- if other solution exists ⟺ S linearly dependent
- if no other solution exists ⟺ S linearly independent



$S = \{\mathbf{0}\}$	linearly dependent
$S = \{\mathbf{v}_1\}$	linearly independent
$S = \{\mathbf{v}_1, \mathbf{v}_2\}$ $\mathbf{v}_1 \neq k\mathbf{v}_2$	linearly independent

Change of Basis





Old Basis $B = \{\boldsymbol{u}_1,\ \boldsymbol{u}_2\}$

New Basis $B' = \{\boldsymbol{u}'_1,\ \boldsymbol{u}'_2\}$

$$[\boldsymbol{u}'_1]_B = \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}$$

coordinate of **u'₁** with respect to B

$$[\boldsymbol{u}'_2]_B = \begin{bmatrix} -\sin\theta \\ \cos\theta \end{bmatrix}$$

coordinate of $\mathbf{u'_2}$ with respect to B

$$\boldsymbol{u}'_1 = \cos\theta\, \boldsymbol{u}_1 + \sin\theta\, \boldsymbol{u}_2$$

$$\boldsymbol{u}'_2 = -\sin\theta\, \boldsymbol{u}_1 + \cos\theta\, \boldsymbol{u}_2$$

$$\begin{aligned} \boldsymbol{v} &= k_1 \boldsymbol{u}'_1 + k_2 \boldsymbol{u}'_2 \\ &= k_1(\cos\theta\, \boldsymbol{u}_1 + \sin\theta\, \boldsymbol{u}_2) + k_2(-\sin\theta\, \boldsymbol{u}_1 + \cos\theta\, \boldsymbol{u}_2) \\ &= (k_1\cos\theta - k_2\sin\theta)\boldsymbol{u}_1 + (k_1\sin\theta + k_2\cos\theta)\boldsymbol{u}_2 \end{aligned}$$

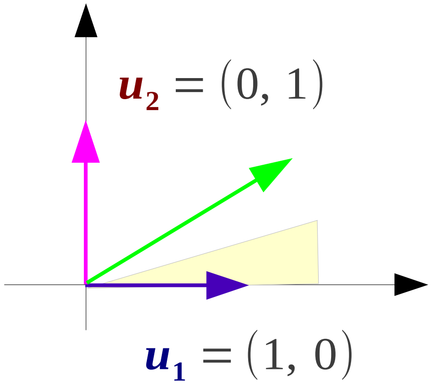
$$[\boldsymbol{v}]_{B'} = \begin{bmatrix} k_1 \\ k_2 \end{bmatrix}$$

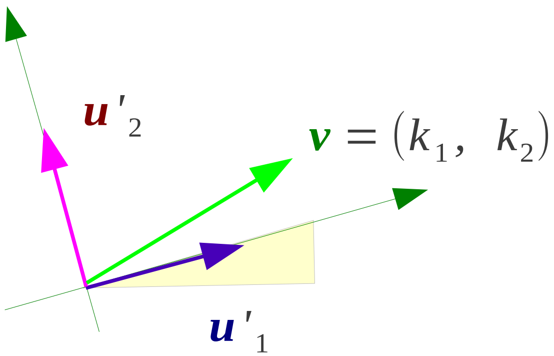
coordinate of **v** with respect to B'

$$[\boldsymbol{v}]_B = \begin{bmatrix} k_1\cos\theta - k_2\sin\theta \\ k_1\sin\theta + k_2\cos\theta \end{bmatrix}$$

coordinate of **v** with respect to B

Change of Basis





Old Basis $B = \{u_1,\ u_2\}$

New Basis $B' = \{u'_1,\ u'_2\}$

$$[v]_{B'} = \begin{bmatrix} k_1 \\ k_2 \end{bmatrix}$$

coordinate of **v**
with respect to B'

$$[v]_B = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \end{bmatrix}$$

coordinate of **v**
with respect to B

$$[v]_B = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} [v]_{B'}$$

$$[u'_1]_B = \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}$$

coordinate of $\mathbf{u'_1}$
with respect to B

$$[v]_B = P_{B' \to B}[v]_{B'}$$

$$[u'_2]_B = \begin{bmatrix} -\sin\theta \\ \cos\theta \end{bmatrix}$$

coordinate of $\mathbf{u'_2}$
with respect to B

$$P_{B' \to B} = \begin{bmatrix} [u'_1]_B & [u'_2]_B \end{bmatrix}$$

Transition Matrix

$$P_{B' \to B} = \left[\; [\boldsymbol{u}'_1]_B \quad [\boldsymbol{u}'_2]_B \quad \cdots \quad [\boldsymbol{u}'_n]_B \; \right]$$

$$[\boldsymbol{v}]_B = P_{B' \to B} [\boldsymbol{v}]_{B'}$$

$[\boldsymbol{u}'_1]_B$ coordinate of $\mathbf{u'}_1$ with respect to B

$[\boldsymbol{u}'_2]_B$ coordinate of $\mathbf{u'}_2$ with respect to B

$[\boldsymbol{v}]_{B'}$ coordinate of $\mathbf{v}$ with respect to B'

↓

$[\boldsymbol{v}]_B$ coordinate of $\mathbf{v}$ with respect to B

$$P_{B \to B'} = \left[\; [\boldsymbol{u}_1]_{B'} \quad [\boldsymbol{u}_2]_{B'} \quad \cdots \quad [\boldsymbol{u}_n]_{B'} \; \right]$$

$$[\boldsymbol{v}]_{B'} = P_{B \to B'} [\boldsymbol{v}]_B$$

$[\boldsymbol{u}_1]_{B'}$ coordinate of $\mathbf{u}_1$ with respect to B'

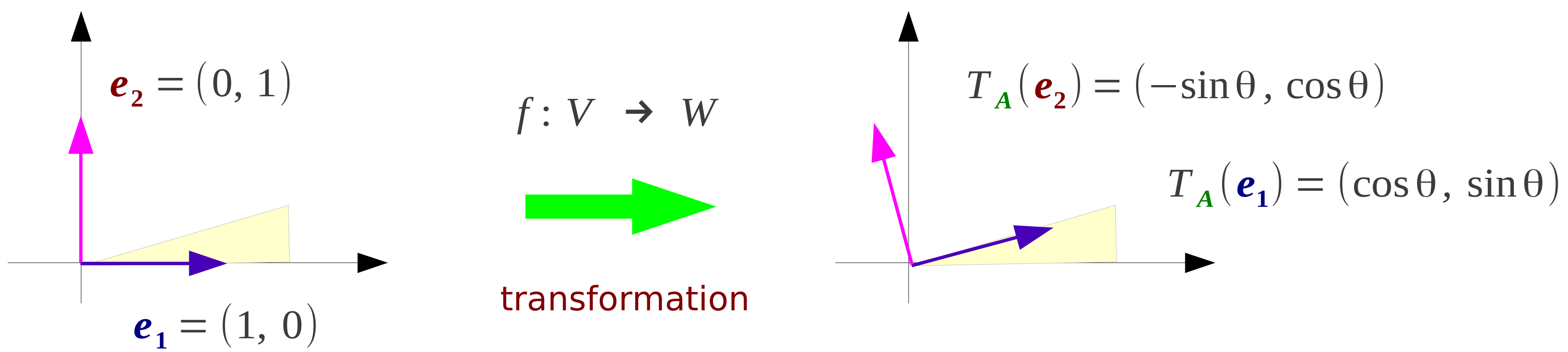
$[\boldsymbol{u}_2]_{B'}$ coordinate of $\mathbf{u}_2$ with respect to B'

$[\boldsymbol{v}]_B$ coordinate of $\mathbf{v}$ with respect to B

↓

$[\boldsymbol{v}]_{B'}$ coordinate of $\mathbf{v}$ with respect to B'

Change of Basis



Old Basis

New Basis

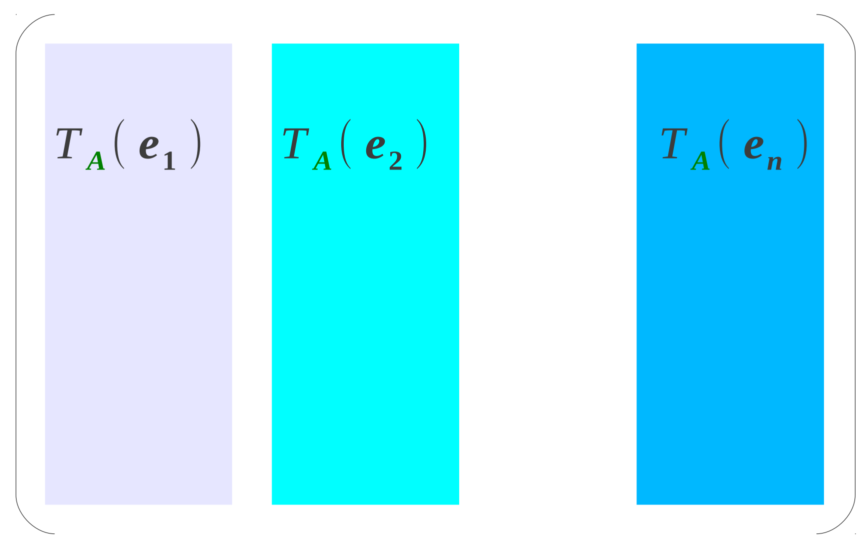
$$\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x_0 \\ y_1 \end{bmatrix}$$

$$\mathbf{w} = \mathbf{A} \quad \mathbf{x}$$

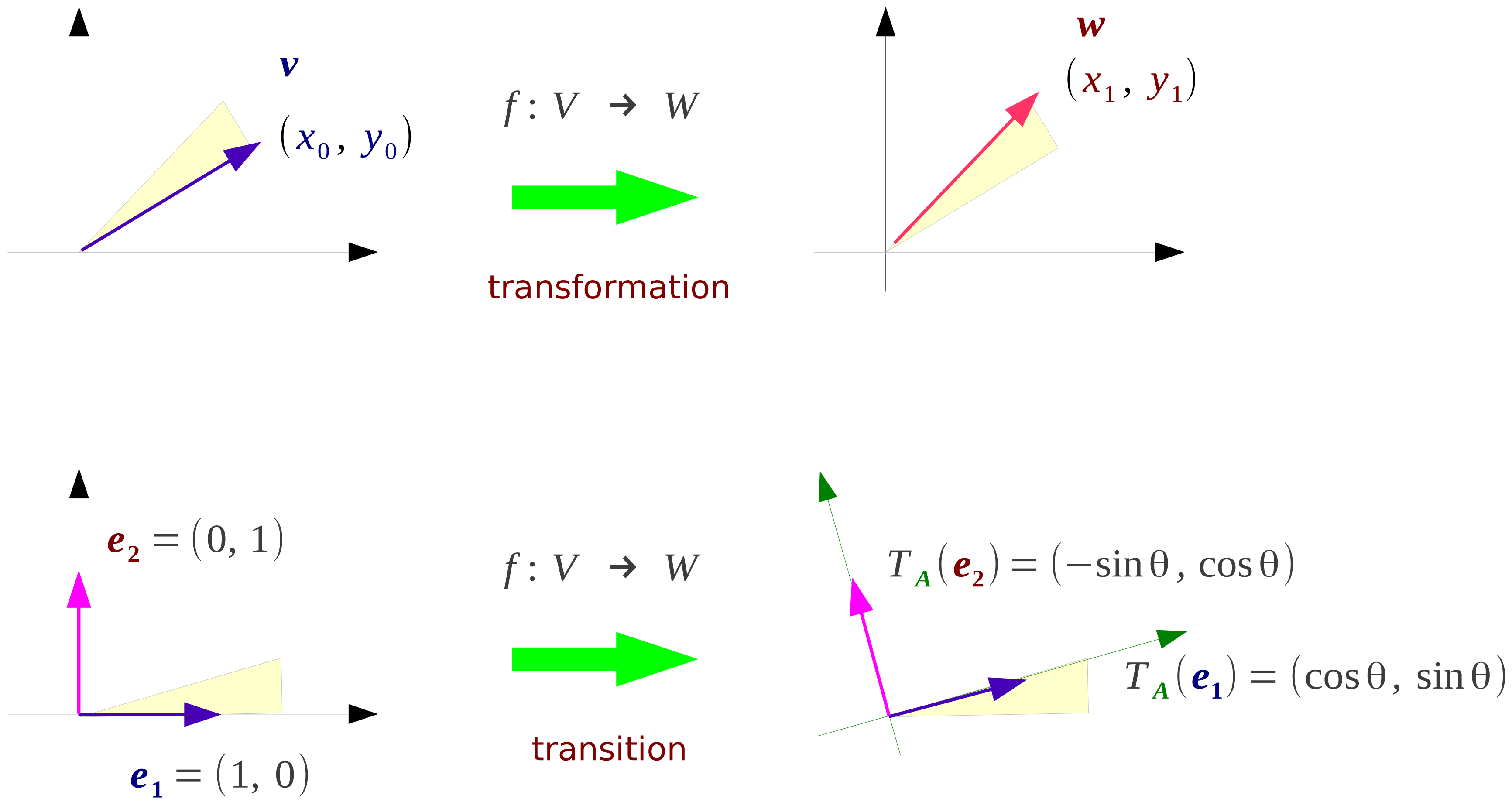
$$\mathbf{w} = T_A(\mathbf{x})$$

$$\mathbf{x} \xrightarrow{T_A} \mathbf{w}$$

$\mathbf{A} =$

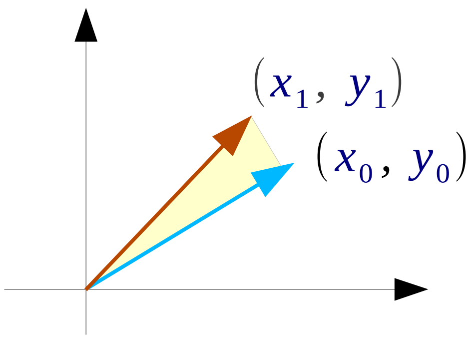


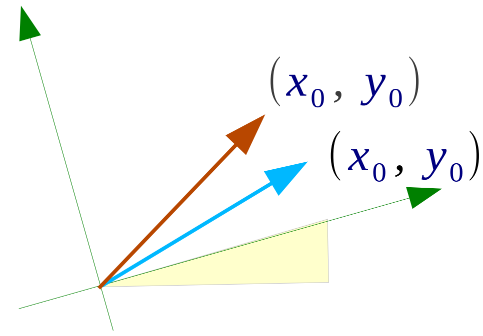
Transformation

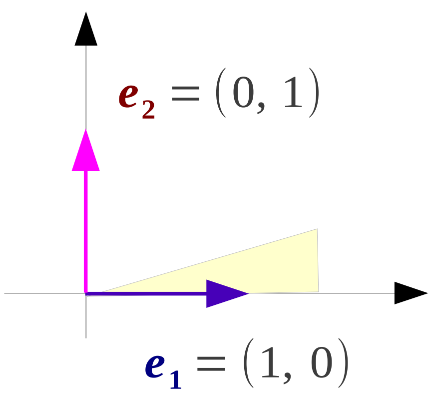


Old Basis **New Basis**

Transformation







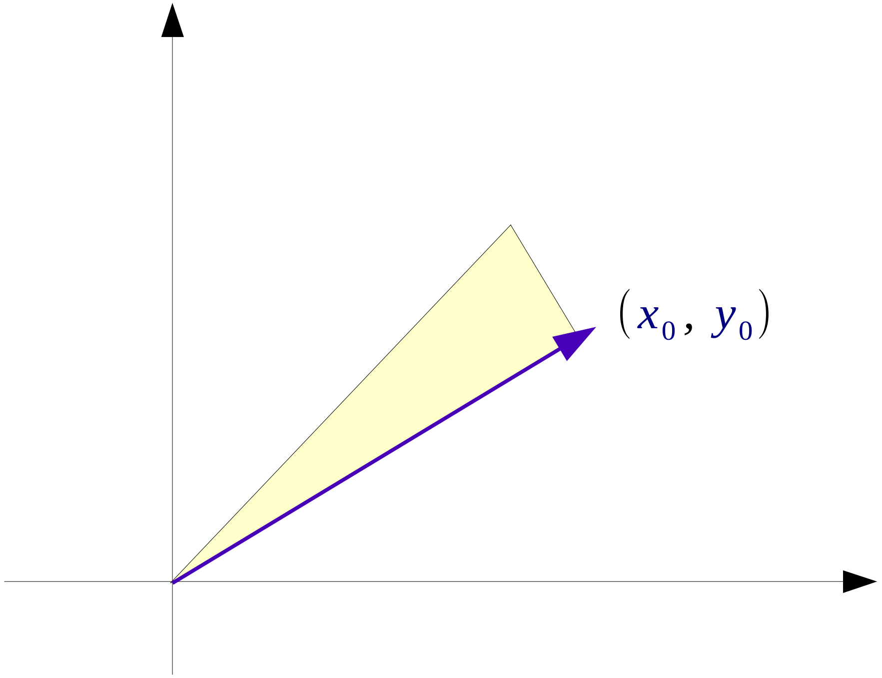
$f : V \rightarrow W$

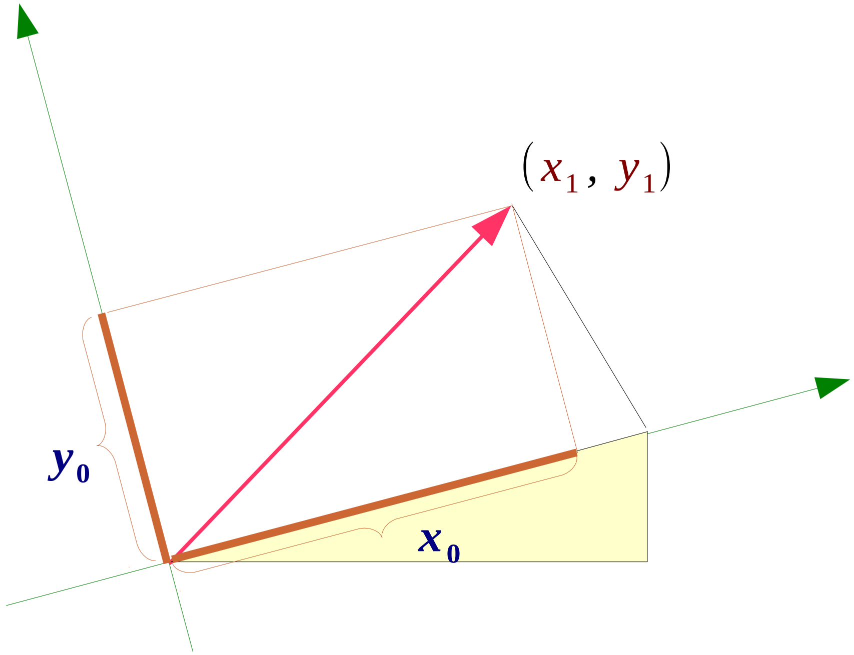
transition

Old Basis

New Basis

Vector Rotation (2)





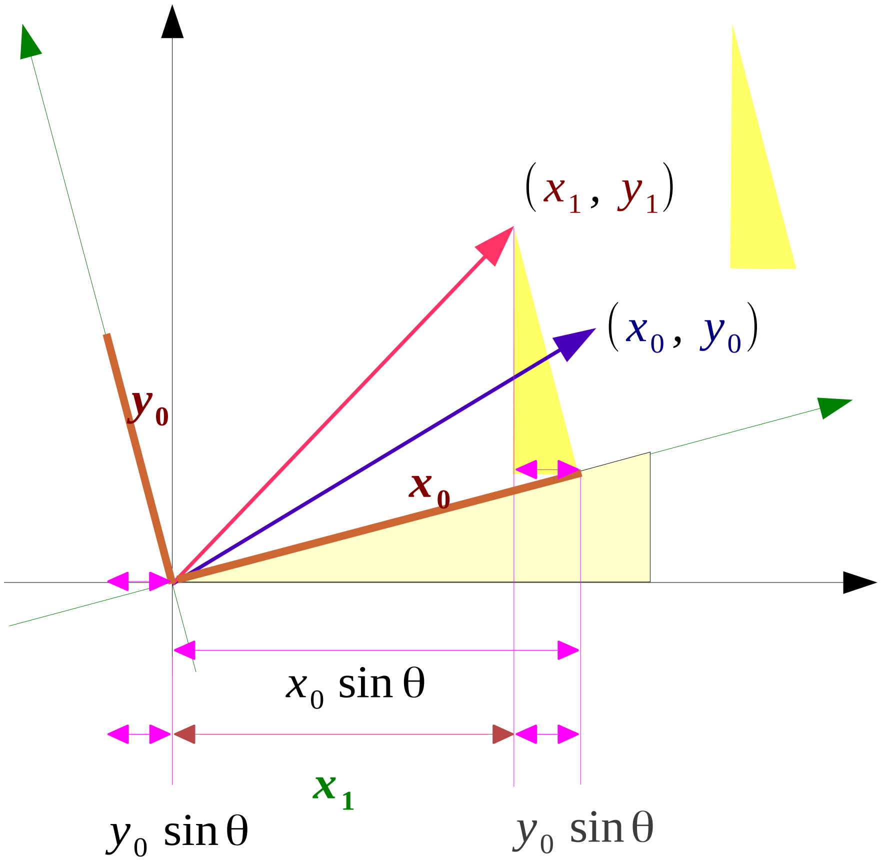
In the rotated coordinate

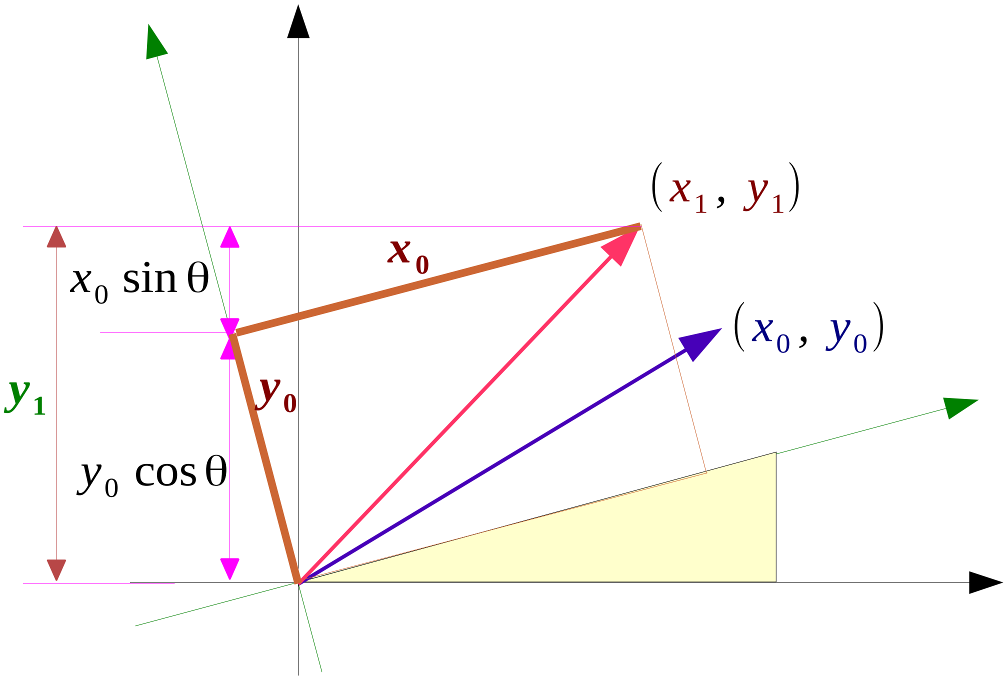
invariant length x_0, y_0

Trasformation

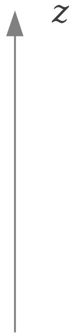
$$x_1 = x_0 \cos\theta - y_0 \sin\theta$$

$$y_1 = x_0 \sin\theta + y_0 \cos\theta$$





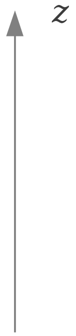
Normal Vector & 3 Points



$$\boldsymbol{a} \times \boldsymbol{b} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = \boldsymbol{i}\begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} - \boldsymbol{j}\begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} + \boldsymbol{k}\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}$$

Normal Vector & 3 Points

Non-collinear
3 points



References

[1] http://en.wikipedia.org/
[2] http://planetmath.org/
[3] M.L. Boas, “Mathematical Methods in the Physical Sciences”
[4] D.G. Zill, “Advanced Engineering Mathematics”