Determinant (5A)

Young Won Lim
09/18/2012

Please send corrections (or suggestions) to youngwlim@hotmail.com.

This document was produced by using OpenOffice and Octave.

Determinant

Determinant of order 2

$$\begin{bmatrix} a_1 & a_2 \\ b_1 & b_2 \end{bmatrix} \qquad \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} = a_1 b_2 - a_2 b_1$$

Determinant of order 3

$$\begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix} \qquad \begin{bmatrix} + & - & + \\ - & + & - \\ + & - & + \end{bmatrix}$$

$$\begin{bmatrix} a_1 & & \\ & b_2 & b_3 \\ & c_2 & c_3 \end{bmatrix} \qquad \begin{bmatrix} & a_2 & \\ b_1 & & b_3 \\ c_1 & & c_3 \end{bmatrix} \qquad \begin{bmatrix} & & a_3 \\ b_1 & b_2 & \\ c_1 & c_2 & \end{bmatrix}$$

Determinant

Determinant of order 3

$$\begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix} \qquad \begin{bmatrix} + & - & + \\ - & + & - \\ + & - & + \end{bmatrix}$$

$$\begin{bmatrix} a_1 & & \\ & b_2 & b_3 \\ & c_2 & c_3 \end{bmatrix} \qquad \begin{bmatrix} & a_2 & \\ b_1 & & b_3 \\ c_1 & & c_3 \end{bmatrix} \qquad \begin{bmatrix} & & a_3 \\ b_1 & b_2 & \\ c_1 & c_2 & \end{bmatrix}$$

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = + a_1 \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - a_2 \begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + a_3 \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}$$

Determinant – Rule of Sarrus

Determinant of order 3

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \qquad \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

Recursive Method

$$= a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

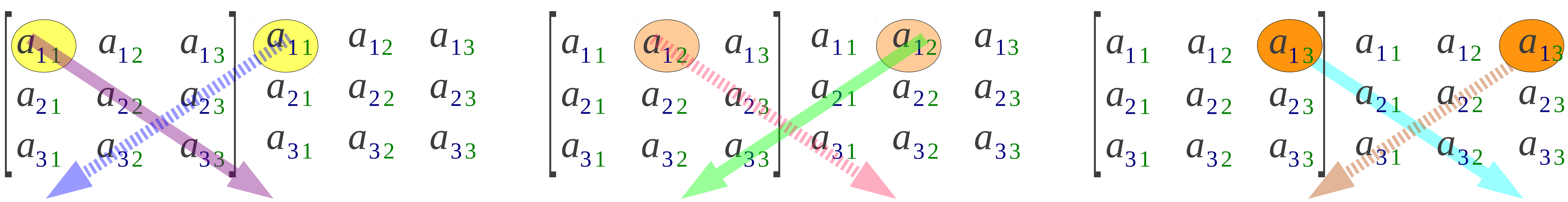
$$= a_{11}(a_{22}a_{33} - a_{23}a_{32})$$

$$- a_{12}(a_{21}a_{33} - a_{23}a_{31})$$

$$+ a_{13}(a_{21}a_{32} - a_{22}a_{31})$$

Determinant of order 3 only

Rule of Sarrus



Minor

The **minor** of entry a_{ij} $\quad M_{ij}$

The determinant of the submatrix
that remains after deleting i-th row and j-th column

Given matrix

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$n \times n$$

$$\begin{bmatrix} & & \\ & a_{22} & a_{23} \\ & a_{32} & a_{33} \end{bmatrix}$$

Sub-matrix

$$\begin{bmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{bmatrix}$$

$$(n-1) \times (n-1)$$

Minor

$$M_{11} = \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}$$

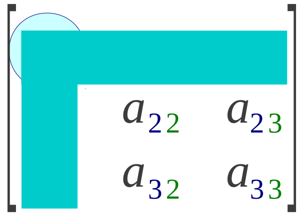
Minor

The **minor** of entry a_{ij} $\quad M_{ij}$

The determinant of the submatrix
that remains after deleting i-th row and j-th column

Given matrix

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$



$n \times n$

Sub-matrix

$$\begin{bmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{bmatrix}$$

$(n-1)\times(n-1)$

Minor

$$M_{11} = \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}$$

Cofactor

The **cofactor** of entry a_{ij} $\quad C_{ij} = (-1)^{i+j} M_{ij}$

The **minor** of entry a_{ij} $\quad M_{ij}$

The determinant of the submatrix
that remains after deleting i-th row and j-th column

Given matrix

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$n \times n$

$$\begin{bmatrix} & & \\ & a_{22} & a_{23} \\ & a_{32} & a_{33} \end{bmatrix}$$

Sub-matrix

$$\begin{bmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{bmatrix}$$

$(n-1) \times (n-1)$

Minor

$$M_{11} = \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}$$

Cofactor

$$C_{11} = (-1)^{1+1} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}$$

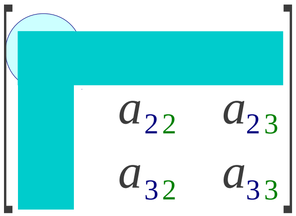
$$\begin{bmatrix} +1 & -1 & +1 \\ -1 & +1 & -1 \\ +1 & -1 & +1 \end{bmatrix}$$

$$\begin{bmatrix} (-1)^{1+1} & (-1)^{1+2} & (-1)^{1+3} \\ (-1)^{2+1} & (-1)^{2+2} & (-1)^{2+3} \\ (-1)^{3+1} & (-1)^{3+2} & (-1)^{3+3} \end{bmatrix} = \begin{bmatrix} (-1)^{2} & (-1)^{3} & (-1)^{4} \\ (-1)^{3} & (-1)^{4} & (-1)^{5} \\ (-1)^{4} & (-1)^{5} & (-1)^{6} \end{bmatrix}$$

Minor Example (1)

Given matrix | Sub-matrix | Minor

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

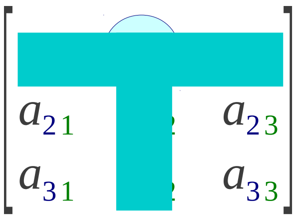


$$\begin{bmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{bmatrix}$$

$$M_{11} = \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}$$

$$C_{11} = +M_{11}$$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

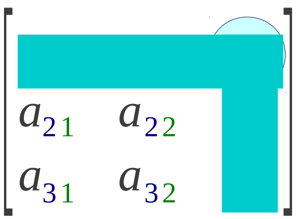


$$\begin{bmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{bmatrix}$$

$$M_{12} = \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix}$$

$$C_{12} = -M_{12}$$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$



$$\begin{bmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix}$$

$$M_{13} = \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

$$C_{13} = +M_{13}$$

Minor Example (2)

Given matrix | Sub-matrix | Minor

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \quad \begin{bmatrix} & a_{12} & a_{13} \\ & & \\ & a_{32} & a_{33} \end{bmatrix} \quad \begin{bmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{bmatrix}$$

$$M_{21} = \begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix}$$

$$C_{21} = -M_{21}$$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \quad \begin{bmatrix} a_{11} & & a_{13} \\ & & \\ a_{31} & & a_{33} \end{bmatrix} \quad \begin{bmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{bmatrix}$$

$$M_{22} = \begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix}$$

$$C_{22} = +M_{22}$$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \quad \begin{bmatrix} a_{11} & a_{12} & \\ & & \\ a_{31} & a_{32} & \end{bmatrix} \quad \begin{bmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{bmatrix}$$

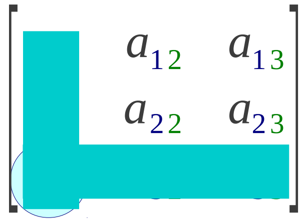
$$M_{23} = \begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix}$$

$$C_{23} = -M_{23}$$

Minor Example (2)

Given matrix | Sub-matrix | Minor / Cofactor

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

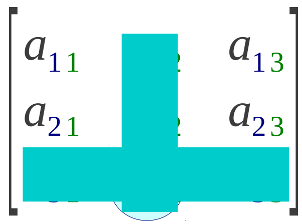


$$\begin{bmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{bmatrix}$$

$$M_{31} = \begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix}$$

$$C_{31} = +M_{31}$$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

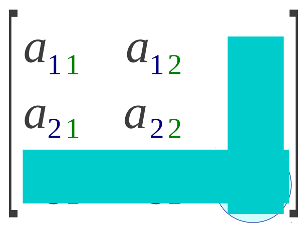


$$\begin{bmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{bmatrix}$$

$$M_{32} = \begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix}$$

$$C_{32} = -M_{32}$$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$



$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

$$M_{33} = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$$

$$C_{33} = +M_{33}$$

Determinant

The **determinant** of an $n \times n$ matrix $\boldsymbol{A}$ $det(\boldsymbol{A})$

Cofactor expansion along the i-th row

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$\begin{aligned} det(\boldsymbol{A}) &= a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13} \\ &= a_{21}C_{21} + a_{22}C_{22} + a_{23}C_{23} \\ &= a_{31}C_{31} + a_{32}C_{32} + a_{33}C_{33} \end{aligned}$$

Cofactor expansion along the j-th column

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$\begin{aligned} det(\boldsymbol{A}) &= a_{11}C_{11} + a_{21}C_{21} + a_{31}C_{31} \\ &= a_{12}C_{12} + a_{22}C_{22} + a_{32}C_{32} \\ &= a_{13}C_{13} + a_{23}C_{23} + a_{33}C_{33} \end{aligned}$$

Adjoint

The **cofactor** of entry a_{ij} $\qquad C_{ij} = (-1)^{i+j} M_{ij}$

The **minor** of entry a_{ij} $\qquad M_{ij}$

The determinant of the submatrix
that remains after deleting i-th row and j-th column

Given matrix

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$n \times n$

$$\begin{matrix} a_{11} \Leftrightarrow C_{11} & a_{12} \Leftrightarrow C_{12} & a_{13} \Leftrightarrow C_{13} \\ a_{21} \Leftrightarrow C_{21} & a_{22} \Leftrightarrow C_{22} & a_{23} \Leftrightarrow C_{23} \\ a_{31} \Leftrightarrow C_{31} & a_{32} \Leftrightarrow C_{32} & a_{33} \Leftrightarrow C_{33} \end{matrix}$$

Matrix of Cofactors

$$\begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix}$$

$n \times n$

transpose →

Adjoint

$$\begin{bmatrix} C_{11} & C_{21} & C_{31} \\ C_{12} & C_{22} & C_{32} \\ C_{13} & C_{23} & C_{33} \end{bmatrix}$$

$n \times n$

Inverse Matrix

The **cofactor** of entry a_{ij} $\qquad C_{ij} = (-1)^{i+j} M_{ij}$

The **minor** of entry a_{ij} $\qquad M_{ij}$

The determinant of the submatrix
that remains after deleting i-th row and j-th column

Given matrix

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$n \times n$

Matrix of Cofactors

$$\begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix}$$

$n \times n$

transpose →

Adjoint

$$\begin{bmatrix} C_{11} & C_{21} & C_{31} \\ C_{12} & C_{22} & C_{32} \\ C_{13} & C_{23} & C_{33} \end{bmatrix}$$

$n \times n$

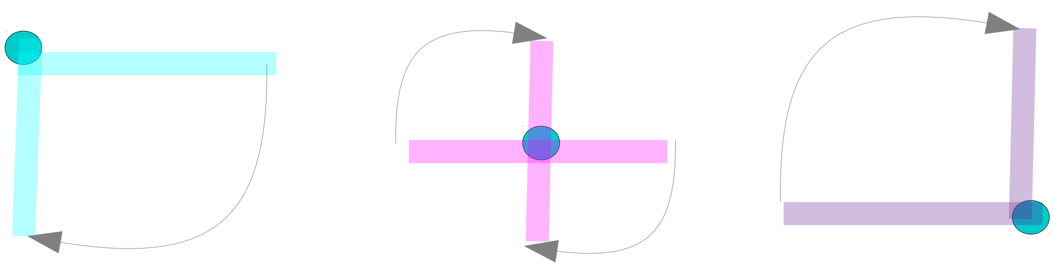
$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}^{-1}$$

$$\boldsymbol{A}^{-1} = \frac{1}{det(\boldsymbol{A})} adj(\boldsymbol{A}) = \frac{1}{det(\boldsymbol{A})} \begin{bmatrix} C_{11} & C_{21} & C_{31} \\ C_{12} & C_{22} & C_{32} \\ C_{13} & C_{23} & C_{33} \end{bmatrix}$$

Matrix Transpose

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \qquad A^T = \begin{bmatrix} a_{11} & a_{21} & a_{31} \\ a_{12} & a_{22} & a_{32} \\ a_{13} & a_{23} & a_{33} \end{bmatrix} \qquad det(A) = det(A^T)$$

$$[a_{ij}] \qquad\qquad [a_{ji}]$$



Cofactor Expansion and Determinant

A $n \times n$ has zero row $\begin{bmatrix} & & \\ 0 & 0 & 0 \\ & & \end{bmatrix}$ or zero col $\begin{bmatrix} 0 & & \\ 0 & & \\ 0 & & \end{bmatrix}$ ➡ $det(A) = 0$

$$det(A) = a_{i1}C_{i1} + a_{i2}C_{i2} + a_{i3}C_{i3}$$ i-th row cofactor expansion

$$= a_{1j}C_{1j} + a_{2j}C_{2j} + a_{3j}C_{3j}$$ j-th column cofactor expansion

$$= 0$$

A $n \times n$ $\begin{bmatrix} & & \\ * & * & * \\ & & \end{bmatrix}$ i-th row

A^T $n \times n$ $\begin{bmatrix} * & & \\ * & & \\ * & & \end{bmatrix}$ i-th col

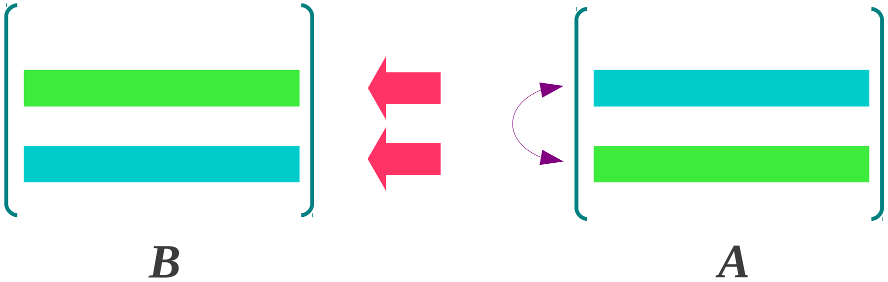
➡ $det(A^T) = det(A) = 0$

$$det(A) = a_{i1}C_{i1} + a_{i2}C_{i2} + a_{i3}C_{i3}$$ i-th row cofactor expansion of A

$$= a_{1i}C_{1i} + a_{2i}C_{2i} + a_{3i}C_{3i}$$ i-th column cofactor expansion of AT

Elementary Matrix and Determinant (1)

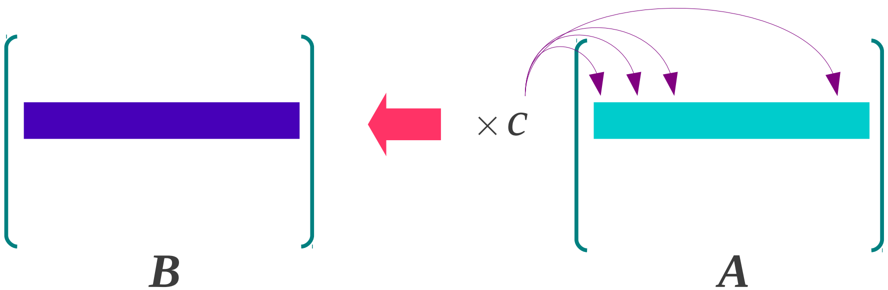
Interchange two rows



$$det(\boldsymbol{B}) = -\,det(\boldsymbol{A})$$

$$\begin{vmatrix} a_{21} & a_{22} & a_{23} \\ a_{11} & a_{12} & a_{13} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = -\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

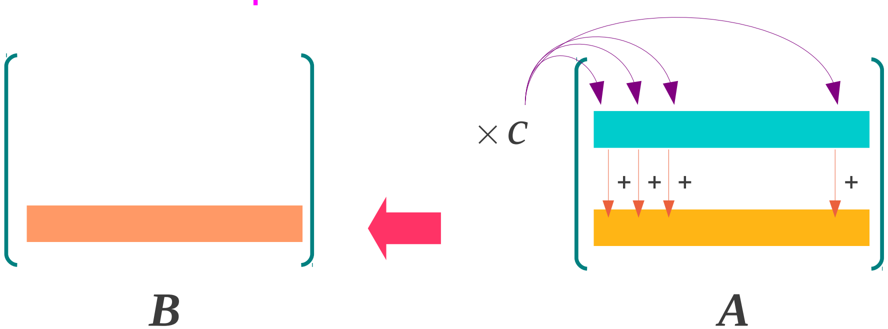
Multiply a row by a nonzero constant



$$det(\boldsymbol{B}) = c\; det(\boldsymbol{A})$$

$$\begin{vmatrix} c\,a_{11} & c\,a_{12} & c\,a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = c\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

Add a multiple of one row to another

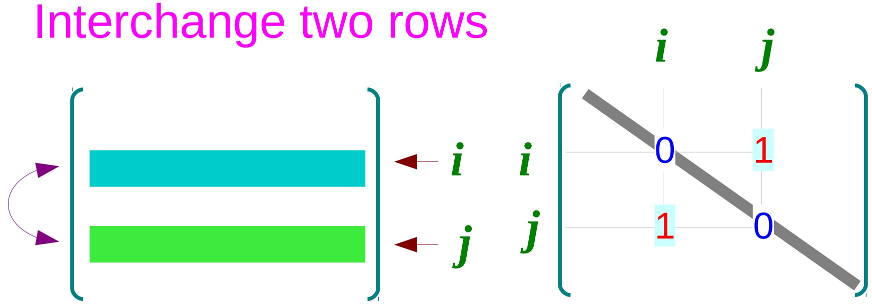


$$det(\boldsymbol{B}) = det(\boldsymbol{A})$$

$$\begin{vmatrix} a_{11}+c\,a_{21} & a_{12}+c\,a_{22} & a_{13}+c\,a_{23} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

Elementary Matrix and Determinant (2)

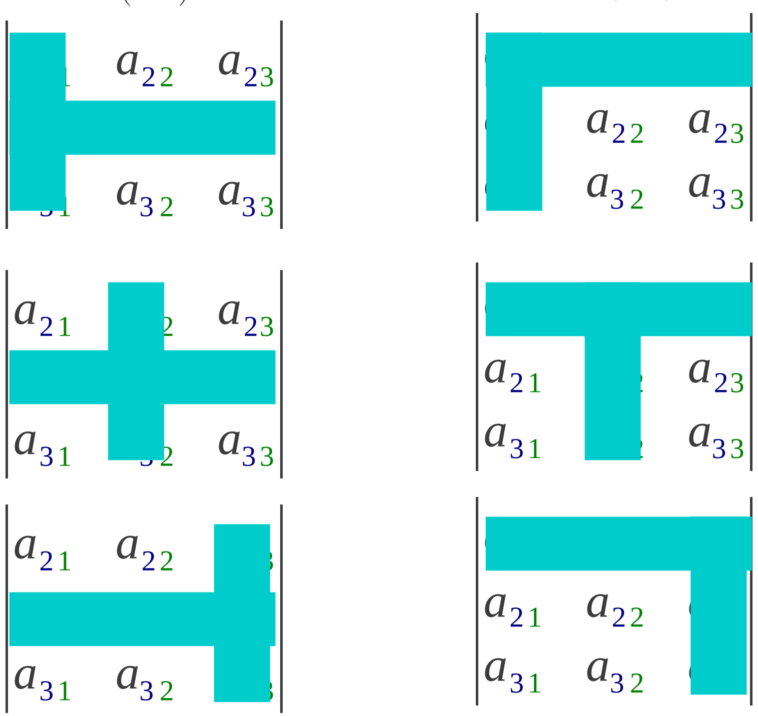
Interchange two rows



$$det(\boldsymbol{B}) = -\,det(\boldsymbol{A})$$

$$\begin{vmatrix} a_{21} & a_{22} & a_{23} \\ a_{11} & a_{12} & a_{13} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = - \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

$det(\boldsymbol{B})$ $det(\boldsymbol{A})$

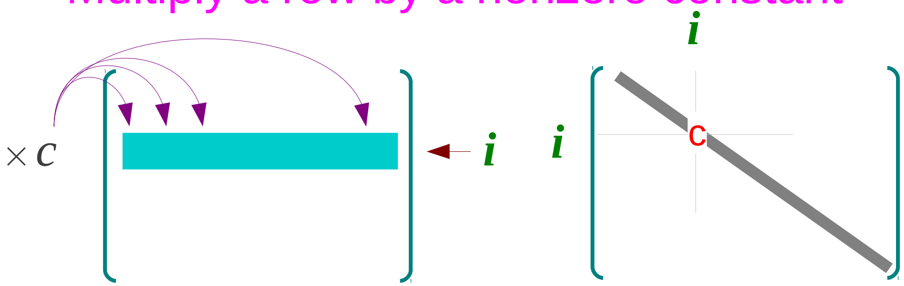


$$det(\boldsymbol{B}) = b_{21}C_{21} + b_{22}C_{22} + b_{23}C_{23}$$
$$= -a_{11}M_{21} + a_{12}M_{22} - a_{13}M_{23}$$

$$det(\boldsymbol{A}) = a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13}$$
$$= a_{11}M_{11} - a_{12}M_{12} + a_{13}M_{13}$$

Elementary Matrix and Determinant (3)

Multiply a row by a nonzero constant



$$\det(\boldsymbol{B}) = c\,\det(\boldsymbol{A})$$

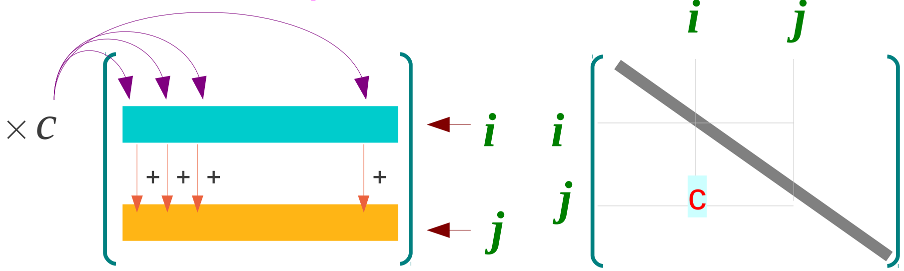
$$\begin{vmatrix} c\,a_{11} & c\,a_{12} & c\,a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = c \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

$$\det(\boldsymbol{B}) = \begin{vmatrix} c\,a_{11} & c\,a_{12} & c\,a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

$$\begin{aligned} \det(\boldsymbol{B}) &= b_{11}C_{11} + b_{12}C_{12} + b_{13}C_{13} \\ &= c\cdot a_{11}C_{11} + c\cdot a_{12}C_{12} + c\cdot a_{13}C_{13} \\ &= c\,(a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13}) \\ &= c\cdot\det(\boldsymbol{A}) \end{aligned}$$

Elementary Matrix and Determinant (4)

Add a multiple of one row to another



$$det(\boldsymbol{B}) = det(\boldsymbol{A})$$

$$\begin{vmatrix} a_{11}+c\,a_{21} & a_{12}+c\,a_{22} & a_{13}+c\,a_{23} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

$$det(\boldsymbol{B})$$

$$\begin{vmatrix} a_{11}+c\,a_{21} & a_{12}+c\,a_{22} & a_{13}+c\,a_{23} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

$$det(\boldsymbol{B}) = b_{11}C_{11} + b_{12}C_{12} + b_{13}C_{13}$$

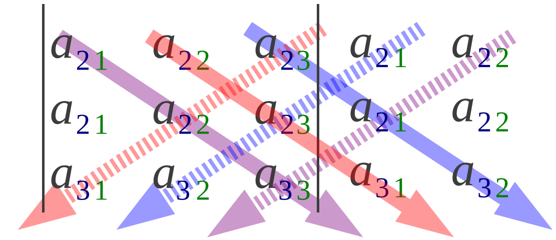
$$= (a_{11}+c\,a_{21})C_{11} + (a_{12}+c\,a_{22})C_{12} + (a_{13}+c\,a_{23})C_{13}$$

$$= (a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13}) \Rightarrow det(\boldsymbol{A})$$

$$+ \; c(a_{21}C_{11} + a_{22}C_{12} + a_{23}C_{13}) \Rightarrow 0$$

$$\left|\begin{matrix} a_{21} & a_{22} & a_{23} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{matrix}\right| \begin{matrix} a_{21} & a_{22} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{matrix}$$

$$(a_{21}C_{11} + a_{22}C_{12} + a_{23}C_{13}) = 0$$



Determinant of Diagonal Matrix

Lower Triangular Matrix

$$\begin{vmatrix} a_{11} & 0 & 0 \\ a_{21} & a_{22} & 0 \\ a_{31} & a_{32} & a_{33} \end{vmatrix} \begin{matrix} a_{11} & 0 \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{matrix}$$

$$det(\boldsymbol{A}) = a_{11} a_{22} a_{33}$$

Upper Triangular Matrix

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & 0 & a_{33} \end{vmatrix} \begin{matrix} a_{11} & a_{12} \\ 0 & a_{22} \\ 0 & 0 \end{matrix}$$

$$det(\boldsymbol{A}) = a_{11} a_{22} a_{33}$$

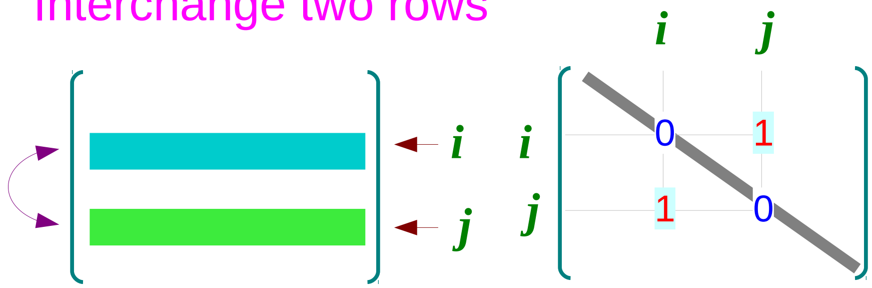
Diagonal Triangular Matrix

$$\begin{vmatrix} a_{11} & 0 & 0 \\ 0 & a_{22} & 0 \\ 0 & 0 & a_{33} \end{vmatrix} \begin{matrix} a_{11} & a_{12} \\ 0 & a_{22} \\ 0 & 0 \end{matrix}$$

$$det(\boldsymbol{A}) = a_{11} a_{22} a_{33}$$

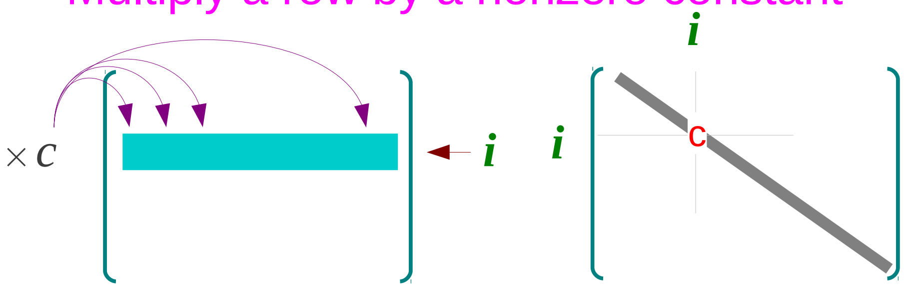
Determinant of an Elementary Matrix

Interchange two rows



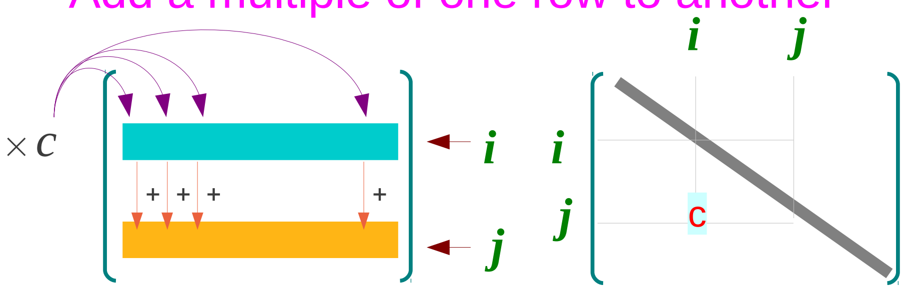
$$det(\boldsymbol{E}_k) = -1$$

Multiply a row by a nonzero constant



$$det(\boldsymbol{E}_k) = c$$

Add a multiple of one row to another



$$det(\boldsymbol{E}_k) = 1$$

Properties of Determinants

$$det(k\,\mathbf{A}) \;=\; k^n\,det(\mathbf{A})$$

$$det(\mathbf{A}+\mathbf{B}) \;\neq\; det(\mathbf{A})+det(\mathbf{B})$$

$$det(\mathbf{A}\,\mathbf{B}) \;=\; det(\mathbf{A})\,det(\mathbf{B})$$

Proof of det(kA) = k^n det(A)

$$det(k\boldsymbol{A}) = k^n det(\boldsymbol{A})$$

$$det(\boldsymbol{A}+\boldsymbol{B}) \neq det(\boldsymbol{A})+det(\boldsymbol{B})$$

$$det(\boldsymbol{A}\boldsymbol{B}) = det(\boldsymbol{A})\,det(\boldsymbol{B})$$

$$\begin{bmatrix} k a_{11} & k a_{12} & k a_{13} \\ k a_{21} & k a_{22} & k a_{23} \\ k a_{31} & k a_{32} & k a_{33} \end{bmatrix} \begin{matrix} k a_{11} & k a_{12} \\ k a_{21} & k a_{22} \\ k a_{31} & k a_{32} \end{matrix}$$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{matrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{matrix}$$

Proof of det(A+B) ≠ det(A) + det(B)

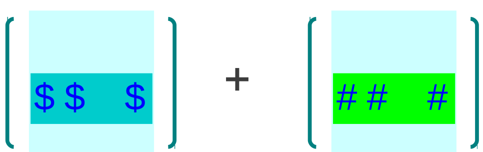
$$det(k\,\boldsymbol{A}) = k^n\,det(\boldsymbol{A})$$

$$det(\boldsymbol{A}+\boldsymbol{B}) \neq det(\boldsymbol{A})+det(\boldsymbol{B})$$

$$det(\boldsymbol{A}\,\boldsymbol{B}) = det(\boldsymbol{A})\,det(\boldsymbol{B})$$

$$\begin{bmatrix} a_{11}+b_{11} & a_{12}+b_{12} & a_{13}+b_{13} \\ a_{21}+b_{21} & a_{22}+b_{22} & a_{23}+b_{23} \\ a_{31}+b_{31} & a_{32}+b_{32} & a_{33}+b_{33} \end{bmatrix} \begin{matrix} a_{11}+b_{11} & a_{12}+b_{12} \\ a_{21}+b_{21} & a_{22}+b_{22} \\ a_{31}+b_{31} & a_{32}+b_{32} \end{matrix}$$

$\boldsymbol{A}$ $n \times n$ $\quad$ $\boldsymbol{B}$ $n \times n$

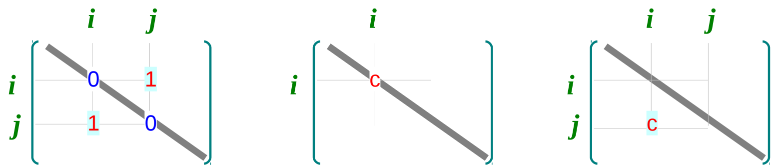


$$\boldsymbol{C} = \boldsymbol{A} + \boldsymbol{B} \Rightarrow det(\boldsymbol{C}) = det(\boldsymbol{A}) + det(\boldsymbol{B})$$

$$\begin{bmatrix} a_{11}+b_{11} & a_{12}+b_{12} & a_{13}+b_{13} \\ 2a_{21} & 2a_{22} & 2a_{23} \\ 2a_{31} & 2a_{32} & 2a_{33} \end{bmatrix} \begin{matrix} a_{11}+b_{11} & a_{12}+b_{12} \\ 2a_{21} & 2a_{22} \\ 2a_{31} & 2a_{32} \end{matrix}$$

Proof of det(AB) = det(A) det(B)   (1)

E_k (Elementary Matrices)



$$det(E_k B) = det(E_k) det(B)$$

A $n \times n$: invertible

$A A^{-1} = A^{-1} A = I$

$$det(A) \neq 0$$

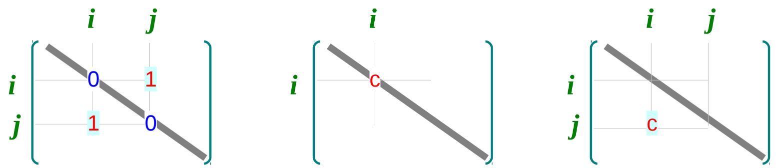
A $n \times n$: invertible

$A A^{-1} = A^{-1} A = I$

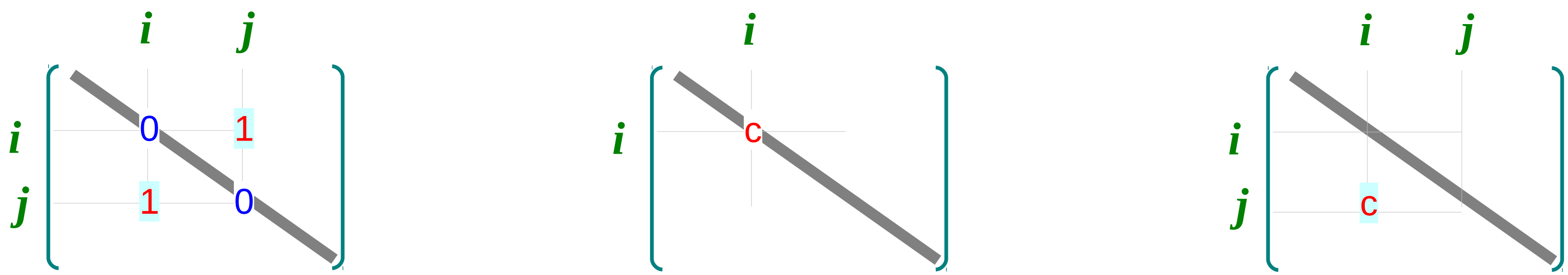
$$det(A^{-1}) = \frac{1}{det(A)}$$

Proof of det(AB) = det(A) det(B)   (2)

E_k (Elementary Matrices)



$$det(E_k B) = det(E_k)\, det(B)$$



$$det(E_k B) = -det(B)$$
$$det(E_k) = -1$$
$$det(E_k B) = det(E_k)\, det(B)$$

$$det(E_k B) = c \cdot det(B)$$
$$det(E_k) = c$$
$$det(E_k B) = det(E_k)\, det(B)$$

$$det(E_k B) = det(B)$$
$$det(E_k) = 1$$
$$det(E_k B) = det(E_k)\, det(B)$$

Proof of det(AB) = det(A) det(B)   (3)

$\boldsymbol{A}$ $n \times n$: invertible

$\boldsymbol{A}\boldsymbol{A}^{-1} = \boldsymbol{A}^{-1}\boldsymbol{A} = \boldsymbol{I}$

⟺

$det(\boldsymbol{A}) \neq 0$

$\boldsymbol{E}_r \cdots \boldsymbol{E}_2\boldsymbol{E}_1\boldsymbol{A} = \boldsymbol{R}$ Reduced Row Echelon Form

$det(\boldsymbol{E}_r) \cdots det(\boldsymbol{E}_2)\, det(\boldsymbol{E}_1)\, det(\boldsymbol{A}) = det(\boldsymbol{R})$

non-zero

$\boldsymbol{A}$ $n \times n$: invertible   ⇒ $\boldsymbol{R} = I$ $det(\boldsymbol{R}) = 1 (\neq 0)$

$det(\boldsymbol{A}) \neq 0$ ⇒ $det(\boldsymbol{R}) \neq 0$

No zero row $\boldsymbol{R} = I$

$\boldsymbol{A}$ $n \times n$: invertible

Proof of det(AB) = det(A) det(B) (4)

$$det(\boldsymbol{A}\,\boldsymbol{B}) = det(\boldsymbol{A}) + det(\boldsymbol{B})$$

$\boldsymbol{A}$ $n \times n$: not invertible ⇒ $\boldsymbol{AB}$ $n \times n$: not invertible

$det(\boldsymbol{A}) = 0$ $det(\boldsymbol{AB}) = 0$

$\boldsymbol{A}$ $n \times n$: invertible ⇒

$$\boldsymbol{A} = \boldsymbol{E}_r \cdots \boldsymbol{E}_2 \boldsymbol{E}_1$$

$$\boldsymbol{AB} = \boldsymbol{E}_r \cdots \boldsymbol{E}_2 \boldsymbol{E}_1 \boldsymbol{B}$$

$$det(\boldsymbol{AB}) = det(\boldsymbol{E}_r) \cdots det(\boldsymbol{E}_2) det(\boldsymbol{E}_1) det(\boldsymbol{B})$$

$$det(\boldsymbol{AB}) = det(\boldsymbol{E}_r \cdots \boldsymbol{E}_2 \boldsymbol{E}_1) det(\boldsymbol{B})$$

$$det(\boldsymbol{AB}) = det(\boldsymbol{A}) det(\boldsymbol{B})$$

Proof of det(AB) = det(A) det(B)   (5)

$\boldsymbol{A}$ $n \times n$: invertible

$\boldsymbol{A}\boldsymbol{A}^{-1} = \boldsymbol{A}^{-1}\boldsymbol{A} = \boldsymbol{I}$

$\Rightarrow$ $det(\boldsymbol{A}^{-1}) = \dfrac{1}{det(\boldsymbol{A})}$

$$\boldsymbol{A}\boldsymbol{A}^{-1} = \boldsymbol{I}$$

$$det(\boldsymbol{A}\boldsymbol{A}^{-1}) = det(\boldsymbol{I})$$

$$det(\boldsymbol{A})\,det(\boldsymbol{A}^{-1}) = 1$$

Computing A adj(A) – diagonal elements

Given matrix

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$n \times n$

Matrix of Cofactors

$$\begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix}$$

$n \times n$

transpose →

Adjoint

$$\begin{bmatrix} C_{11} & C_{21} & C_{31} \\ C_{12} & C_{22} & C_{32} \\ C_{13} & C_{23} & C_{33} \end{bmatrix}$$

$n \times n$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} C_{11} & C_{21} & C_{31} \\ C_{12} & C_{22} & C_{32} \\ C_{13} & C_{23} & C_{33} \end{bmatrix} \quad det(\boldsymbol{A}) = a_{11} C_{11} + a_{12} C_{12} + a_{13} C_{13}$$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} C_{11} & C_{21} & C_{31} \\ C_{12} & C_{22} & C_{32} \\ C_{13} & C_{23} & C_{33} \end{bmatrix} \quad det(\boldsymbol{A}) = a_{21} C_{21} + a_{22} C_{22} + a_{23} C_{23}$$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} C_{11} & C_{21} & C_{31} \\ C_{12} & C_{22} & C_{32} \\ C_{13} & C_{23} & C_{33} \end{bmatrix} \quad det(\boldsymbol{A}) = a_{31} C_{31} + a_{32} C_{32} + a_{33} C_{33}$$

Computing A adj(A) – off-diagonal elements (1)

Given matrix

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$n \times n$

Matrix of Cofactors

$$\begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix}$$

$n \times n$

transpose

Adjoint

$$\begin{bmatrix} C_{11} & C_{21} & C_{31} \\ C_{12} & C_{22} & C_{32} \\ C_{13} & C_{23} & C_{33} \end{bmatrix}$$

$n \times n$

$\boldsymbol{A}$ $\quad adj(\boldsymbol{A})$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} C_{11} & C_{21} & C_{31} \\ C_{12} & C_{22} & C_{32} \\ C_{13} & C_{23} & C_{33} \end{bmatrix} \qquad det(\boldsymbol{A}) = a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13}$$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} C_{11} & C_{21} & C_{31} \\ C_{12} & C_{22} & C_{32} \\ C_{13} & C_{23} & C_{33} \end{bmatrix} \qquad 0 = a_{21}C_{11} + a_{22}C_{12} + a_{23}C_{13}$$

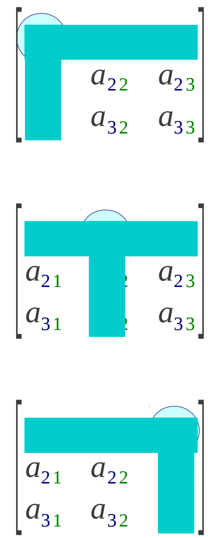
$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} C_{11} & C_{21} & C_{31} \\ C_{12} & C_{22} & C_{32} \\ C_{13} & C_{23} & C_{33} \end{bmatrix} \qquad 0 = a_{31}C_{11} + a_{32}C_{12} + a_{33}C_{13}$$

Computing A adj(A) – off-diagonal elements (2)

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \quad \begin{bmatrix} C_{11} \\ C_{12} \\ C_{13} \end{bmatrix}$$

$$\begin{bmatrix} a_{21} & a_{22} & a_{23} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \quad \begin{bmatrix} C'_{11} \\ C'_{12} \\ C'_{13} \end{bmatrix}$$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$



$$\begin{bmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{bmatrix} \quad M_{11} = \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} \quad C_{11} = +M_{11}$$

$$\begin{bmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{bmatrix} \quad M_{12} = \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} \quad C_{12} = -M_{12}$$

$$\begin{bmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix} \quad M_{13} = \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} \quad C_{13} = +M_{13}$$

The redundant row

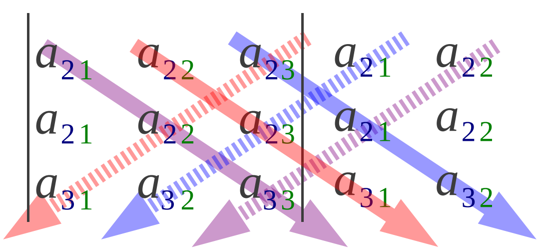
- The same cofactors
- Determinant = 0

$$a_{21} C'_{11} + a_{22} C'_{12} + a_{23} C'_{13}$$

$$= a_{21} C_{11} + a_{22} C_{12} + a_{23} C_{13} = 0$$

$$\begin{vmatrix} a_{21} & a_{22} & a_{23} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} \begin{matrix} a_{21} & a_{22} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{matrix}$$

$$(a_{21} C_{11} + a_{22} C_{12} + a_{23} C_{13}) = 0$$



Result of A adj(A)

Given matrix

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$n \times n$

Matrix of Cofactors

$$\begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix}$$

$n \times n$

transpose

Adjoint

$$\begin{bmatrix} C_{11} & C_{21} & C_{31} \\ C_{12} & C_{22} & C_{32} \\ C_{13} & C_{23} & C_{33} \end{bmatrix}$$

$n \times n$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} C_{11} & C_{21} & C_{31} \\ C_{12} & C_{22} & C_{32} \\ C_{13} & C_{23} & C_{33} \end{bmatrix} = \begin{bmatrix} det(\boldsymbol{A}) & 0 & 0 \\ 0 & det(\boldsymbol{A}) & 0 \\ 0 & 0 & det(\boldsymbol{A}) \end{bmatrix} = det(\boldsymbol{A}) \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

matrix value

$$\boldsymbol{A}\, adj(\boldsymbol{A}) = det(\boldsymbol{A}) \boldsymbol{I}$$

$$\boldsymbol{A} \left[\frac{adj(\boldsymbol{A})}{det(\boldsymbol{A})} \right] = \boldsymbol{I}$$

$$\boldsymbol{A} [\boldsymbol{A}^{-1}] = \boldsymbol{I}$$

Linear Equations

(Eq 1) $a_{11} x_1 + a_{12} x_2 + \cdots + a_{1n} x_n = b_1$

(Eq 2) $a_{21} x_1 + a_{22} x_2 + \cdots + a_{2n} x_n = b_2$

$\vdots$

(Eq 3) $a_{n1} x_1 + a_{n2} x_2 + \cdots + a_{nn} x_n = b_n$

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}$$

$$\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}$$

Cramer's Rule (1)

$$\overbrace{\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}}^{\boldsymbol{A}} \overbrace{\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}}^{\boldsymbol{x}} \begin{matrix} = \\ = \\ \\ = \end{matrix} \overbrace{\begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}}^{\boldsymbol{b}}$$

$$\boldsymbol{A}_1 = \begin{pmatrix} b_1 & a_{12} & \cdots & a_{1n} \\ b_2 & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ b_n & a_{n2} & \cdots & a_{nn} \end{pmatrix} \quad \boldsymbol{A}_2 = \begin{pmatrix} a_{11} & b_1 & \cdots & a_{1n} \\ a_{21} & b_2 & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & b_n & \cdots & a_{nn} \end{pmatrix} \quad \boldsymbol{A}_n = \begin{pmatrix} a_{11} & a_{12} & \cdots & b_1 \\ a_{21} & a_{22} & \cdots & b_2 \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & b_n \end{pmatrix}$$

$$x_1 = \frac{det(\boldsymbol{A}_1)}{det(\boldsymbol{A})} \qquad x_2 = \frac{det(\boldsymbol{A}_2)}{det(\boldsymbol{A})} \qquad x_n = \frac{det(\boldsymbol{A}_n)}{det(\boldsymbol{A})}$$

Cramer's Rule (2)

$$A_1 \quad \begin{pmatrix} b_1 & a_{12} & \cdots & a_{1n} \\ b_2 & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ b_n & a_{n2} & \cdots & a_{nn} \end{pmatrix}$$

$$det(A_1) = b_1 C_{11} + b_2 C_{21} + \cdots + b_n C_{n1}$$

$$x_1 = \frac{det(A_1)}{det(A)}$$

$$A_2 \quad \begin{pmatrix} a_{11} & b_1 & \cdots & a_{1n} \\ a_{21} & b_2 & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & b_n & \cdots & a_{nn} \end{pmatrix}$$

$$det(A_2) = b_1 C_{12} + b_2 C_{22} + \cdots + b_n C_{n2}$$

$$x_2 = \frac{det(A_2)}{det(A)}$$

$$A_n \quad \begin{pmatrix} a_{11} & a_{12} & \cdots & b_1 \\ a_{21} & a_{22} & \cdots & b_2 \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & b_n \end{pmatrix}$$

$$det(A_n) = b_1 C_{1n} + b_2 C_{2n} + \cdots + b_n C_{nn}$$

$$x_n = \frac{det(A_n)}{det(A)}$$

Cramer's Rule (3)

$$\mathbf{A}\mathbf{x} = \mathbf{b}$$

$$\mathbf{x} = \mathbf{A}^{-1}\mathbf{b} = \frac{adj(\mathbf{A})}{det(\mathbf{A})}\mathbf{b} = \begin{pmatrix} C_{11} & C_{21} & \cdots & C_{n1} \\ C_{12} & C_{22} & \cdots & C_{n2} \\ \vdots & \vdots & & \vdots \\ C_{1n} & C_{2n} & \cdots & C_{nn} \end{pmatrix} \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}$$

$$\mathbf{x} = \frac{1}{det(\mathbf{A})} \begin{pmatrix} b_1 C_{11} + b_2 C_{21} + \cdots + b_n C_{n1} \\ b_1 C_{12} + b_2 C_{22} + \cdots + b_n C_{n2} \\ \vdots \\ b_1 C_{1n} + b_2 C_{2n} + \cdots + b_n C_{nn} \end{pmatrix}$$

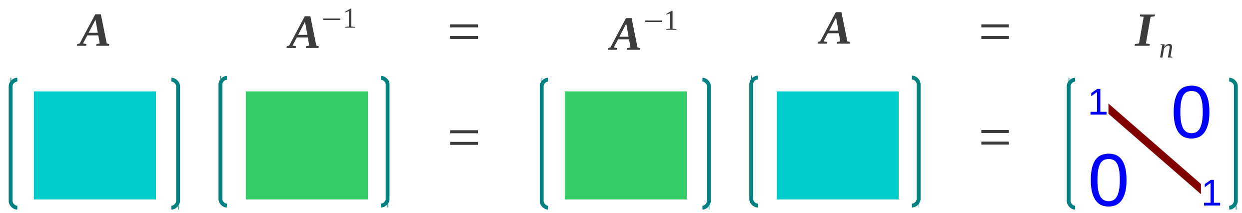
$$x_1 = \frac{det(\mathbf{A}_1)}{det(\mathbf{A})}$$

$$x_2 = \frac{det(\mathbf{A}_2)}{det(\mathbf{A})}$$

$$x_n = \frac{det(\mathbf{A}_n)}{det(\mathbf{A})}$$

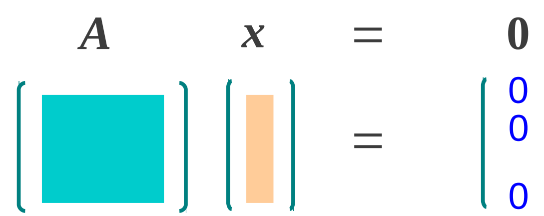
Equivalent Statements

A : invertible



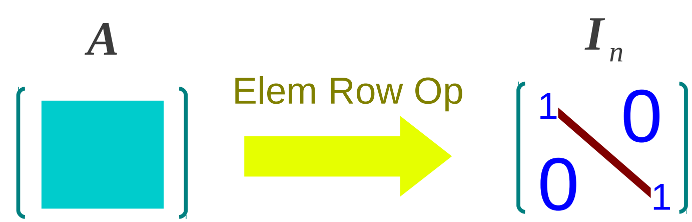
$Ax = 0$

only the trivial solution



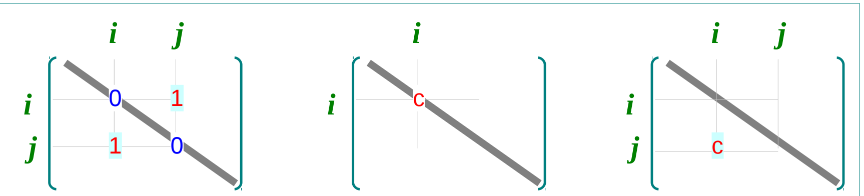
A the RREF is I_n

(Reduced Row Echelon Form)



A can be written as a product of E_k

(Elementary Matrices)



References

[1] http://en.wikipedia.org/
[2] Anton & Busby, “Contemporary Linear Algebra”
[3] Anton & Rorres, “Elementary Linear Algebra”