

Determinant (5A)

Copyright (c) 2012 Young W. Lim.

Permission is granted to copy, distribute and/or modify this document under the terms of the GNU Free Documentation License, Version 1.2 or any later version published by the Free Software Foundation; with no Invariant Sections, no Front-Cover Texts, and no Back-Cover Texts. A copy of the license is included in the section entitled "GNU Free Documentation License".

Please send corrections (or suggestions) to youngwlim@hotmail.com.

This document was produced by using OpenOffice and Octave.

Determinant

Determinant of order 2

$$\begin{bmatrix} a_1 & a_2 \\ b_1 & b_2 \end{bmatrix} \quad \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} = a_1 b_2 - a_2 b_1$$

Determinant of order 3

$$\begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix} \quad \begin{bmatrix} + & - & + \\ - & + & - \\ + & - & + \end{bmatrix}$$

$$\begin{bmatrix} a_1 & & \\ b_2 & b_3 & \\ c_2 & c_3 & \end{bmatrix} \quad \begin{bmatrix} & a_2 & \\ b_1 & & b_3 \\ c_1 & & c_3 \end{bmatrix} \quad \begin{bmatrix} & & a_3 \\ & b_2 & \\ b_1 & b_2 & \\ c_1 & c_2 & \end{bmatrix}$$

Determinant

Determinant of order 3

$$\begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix} \quad \begin{bmatrix} + & - & + \\ - & + & - \\ + & - & + \end{bmatrix}$$

$$\begin{bmatrix} a_1 & & \\ b_2 & \cancel{b_3} & \\ c_2 & c_3 & \end{bmatrix} \quad \begin{bmatrix} & a_2 & \\ b_1 & \cancel{a_3} & b_3 \\ c_1 & c_3 & \end{bmatrix} \quad \begin{bmatrix} & & a_3 \\ b_1 & \cancel{b_2} & \\ c_1 & c_2 & \end{bmatrix}$$

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = + a_1 \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - a_2 \begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + a_3 \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}$$

Determinant – Rule of Sarrus

Determinant of order 3

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

Recursive Method

$$\begin{aligned} &= a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} \\ &= a_{11}(a_{22}a_{33} - a_{23}a_{32}) \\ &\quad - a_{12}(a_{21}a_{33} - a_{23}a_{31}) \\ &\quad + a_{13}(a_{21}a_{32} - a_{22}a_{31}) \end{aligned}$$

Determinant of order 3 only

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

Rule of Sarrus

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

Minor

The **minor** of entry a_{ij}

M_{ij}

The determinant of the submatrix
that remains after **deleting** i-th row and j-th column

Given matrix

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

A diagram illustrating the formation of a submatrix. A 3x3 matrix is shown with its first row and first column highlighted in red. A 2x2 submatrix is formed by removing these highlighted rows and columns, resulting in the matrix $\begin{bmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{bmatrix}$.

$$\begin{bmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{bmatrix}$$

Sub-matrix

$$\begin{bmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{bmatrix}$$

Minor

$$M_{11} = \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}$$

$n \times n$

$(n-1) \times (n-1)$

Minor

The **minor** of entry a_{ij}

M_{ij}

The determinant of the submatrix
that remains after **deleting** i-th row and j-th column

Given matrix

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

A diagram illustrating the formation of a submatrix. A 3x3 matrix is shown with its first row and first column highlighted in light blue. A large teal L-shaped bracket is positioned over the remaining elements, indicating the 2x2 submatrix formed by deleting the first row and first column.

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$n \times n$

Sub-matrix

$$\begin{bmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{bmatrix}$$

$(n-1) \times (n-1)$

Minor

$$M_{11} = \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}$$

Cofactor

The **cofactor** of entry a_{ij}

$$C_{ij} = (-1)^{i+j} M_{ij}$$

The **minor** of entry a_{ij}

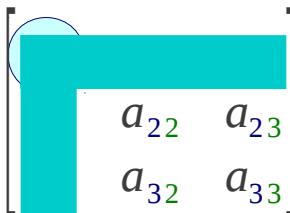
$$M_{ij}$$

The determinant of the submatrix
that remains after **deleting** i-th row and j-th column

Given matrix

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$n \times n$


$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

Sub-matrix

$$\begin{bmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{bmatrix}$$

$(n-1) \times (n-1)$

Minor

$$M_{11} = \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}$$

Cofactor

$$C_{11} = (-1)^{1+1} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}$$

$$\begin{bmatrix} +1 & -1 & +1 \\ -1 & +1 & -1 \\ +1 & -1 & +1 \end{bmatrix}$$

$$\begin{bmatrix} (-1)^{1+1} & (-1)^{1+2} & (-1)^{1+3} \\ (-1)^{2+1} & (-1)^{2+2} & (-1)^{2+3} \\ (-1)^{3+1} & (-1)^{3+2} & (-1)^{3+3} \end{bmatrix} = \begin{bmatrix} (-1)^2 & (-1)^3 & (-1)^4 \\ (-1)^3 & (-1)^4 & (-1)^5 \\ (-1)^4 & (-1)^5 & (-1)^6 \end{bmatrix}$$

Minor Example (1)

Given matrix

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

Sub-matrix

$$\begin{bmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{bmatrix}$$

Minor

$$M_{11} = \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}$$

$$C_{11} = +M_{11}$$

$$M_{12} = \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix}$$

$$C_{12} = -M_{12}$$

$$M_{13} = \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

$$C_{13} = +M_{13}$$

Minor Example (2)

Given matrix

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$\begin{bmatrix} & a_{12} & a_{13} \\ a_{21} & & \\ & a_{32} & a_{33} \end{bmatrix}$$

Sub-matrix

$$\begin{bmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{bmatrix}$$

Minor

$$M_{21} = \begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix}$$

$$C_{21} = -M_{21}$$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$\begin{bmatrix} a_{11} & & a_{13} \\ & + & \\ a_{31} & & a_{33} \end{bmatrix}$$

$$\begin{bmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{bmatrix}$$

$$M_{22} = \begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix}$$

$$C_{22} = +M_{22}$$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$\begin{bmatrix} a_{11} & a_{12} & \\ & - & \\ a_{31} & a_{32} & \end{bmatrix}$$

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{bmatrix}$$

$$M_{23} = \begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix}$$

$$C_{23} = -M_{23}$$

Minor Example (2)

Given matrix

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

A diagram illustrating the formation of a 2x3 sub-matrix by removing the first column from the original 3x3 matrix. The sub-matrix is shown with a teal border around the elements $a_{12}, a_{13}, a_{22}, a_{23}$.

$$\begin{bmatrix} & a_{12} & a_{13} \\ & a_{22} & a_{23} \end{bmatrix}$$

Sub-matrix

$$\begin{bmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{bmatrix}$$

Minor / Cofactor

$$M_{31} = \begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix}$$

$$C_{31} = +M_{31}$$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

A diagram illustrating the formation of a 2x3 sub-matrix by removing the second column from the original 3x3 matrix. The sub-matrix is shown with a teal border around the elements $a_{11}, a_{13}, a_{21}, a_{23}$.

$$\begin{bmatrix} a_{11} & & a_{13} \\ a_{21} & & a_{23} \end{bmatrix}$$

$$\begin{bmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{bmatrix}$$

$$M_{32} = \begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix}$$

$$C_{32} = -M_{32}$$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

A diagram illustrating the formation of a 2x3 sub-matrix by removing the third column from the original 3x3 matrix. The sub-matrix is shown with a teal border around the elements $a_{11}, a_{12}, a_{21}, a_{22}$.

$$\begin{bmatrix} a_{11} & a_{12} & \\ a_{21} & a_{22} & \end{bmatrix}$$

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

$$M_{33} = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$$

$$C_{33} = +M_{33}$$

Determinant

The **determinant** of an $n \times n$ matrix \mathbf{A} $det(\mathbf{A})$

Cofactor expansion along the i-th row

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$\begin{aligned} det(\mathbf{A}) &= a_{11} C_{11} + a_{12} C_{12} + a_{13} C_{13} \\ &= a_{21} C_{21} + a_{22} C_{22} + a_{23} C_{23} \\ &= a_{31} C_{31} + a_{32} C_{32} + a_{33} C_{33} \end{aligned}$$

Cofactor expansion along the j-th column

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$\begin{aligned} det(\mathbf{A}) &= a_{11} C_{11} + a_{21} C_{21} + a_{31} C_{31} \\ &= a_{12} C_{12} + a_{22} C_{22} + a_{32} C_{32} \\ &= a_{13} C_{13} + a_{23} C_{23} + a_{33} C_{33} \end{aligned}$$

Adjoint

The **cofactor** of entry a_{ij} $C_{ij} = (-1)^{i+j} M_{ij}$

The **minor** of entry a_{ij} M_{ij}

The determinant of the submatrix
that remains after **deleting** i-th row and j-th column

Given matrix

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \quad \begin{array}{lll} a_{11} \Leftrightarrow C_{11} & a_{12} \Leftrightarrow C_{12} & a_{13} \Leftrightarrow C_{13} \\ a_{21} \Leftrightarrow C_{21} & a_{22} \Leftrightarrow C_{22} & a_{23} \Leftrightarrow C_{23} \\ a_{31} \Leftrightarrow C_{31} & a_{32} \Leftrightarrow C_{32} & a_{33} \Leftrightarrow C_{33} \end{array}$$

$n \times n$

Matrix of Cofactors

$$\begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix}$$

$n \times n$

transpose 

Adjoint

$$\begin{bmatrix} C_{11} & C_{21} & C_{31} \\ C_{12} & C_{22} & C_{32} \\ C_{13} & C_{23} & C_{33} \end{bmatrix}$$

$n \times n$

Inverse Matrix

The **cofactor** of entry a_{ij}

$$C_{ij} = (-1)^{i+j} M_{ij}$$

The **minor** of entry a_{ij}

$$M_{ij}$$

The determinant of the submatrix
that remains after **deleting** i-th row and j-th column

Given matrix

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$n \times n$

Matrix of Cofactors

$$\begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix}$$

$n \times n$

transpose 

Adjoint

$$\begin{bmatrix} C_{11} & C_{21} & C_{31} \\ C_{12} & C_{22} & C_{32} \\ C_{13} & C_{23} & C_{33} \end{bmatrix}$$

$n \times n$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}^{-1}$$

$$A^{-1} = \frac{1}{\det(A)} adj(A) = \frac{1}{\det(A)} \begin{bmatrix} C_{11} & C_{21} & C_{31} \\ C_{12} & C_{22} & C_{32} \\ C_{13} & C_{23} & C_{33} \end{bmatrix}$$

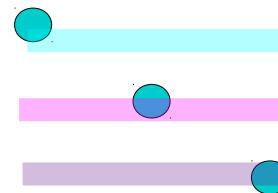
Matrix Transpose

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

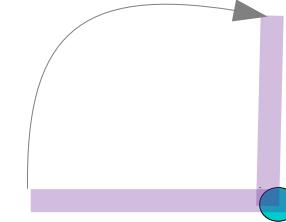
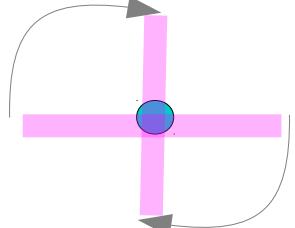
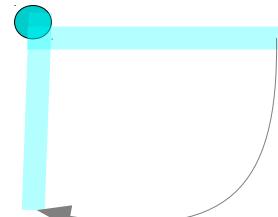
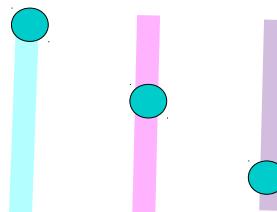
$$A^T = \begin{bmatrix} a_{11} & a_{21} & a_{31} \\ a_{12} & a_{22} & a_{32} \\ a_{13} & a_{23} & a_{33} \end{bmatrix}$$

$$\det(A) = \det(A^T)$$

$$[a_{ij}]$$



$$[a_{ji}]$$



Cofactor Expansion and Determinant

$A \quad n \times n$ zero row zero col
has $\begin{bmatrix} 0 & 0 & 0 \end{bmatrix}$ or $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ $\Rightarrow \det(A) = 0$

$$\begin{aligned}\det(A) &= a_{i_1} C_{i_1} + a_{i_2} C_{i_2} + a_{i_3} C_{i_3} && \text{i-th row cofactor expansion} \\ &= a_{1j} C_{1j} + a_{2j} C_{2j} + a_{3j} C_{3j} && \text{j-th column cofactor expansion} \\ &= 0\end{aligned}$$

$A \quad n \times n$ $A^T \quad n \times n$
 $\begin{bmatrix} * & * & * \end{bmatrix}$ i-th row $\begin{bmatrix} * \\ * \\ * \end{bmatrix}$ i-th col $\Rightarrow \det(A^T) = \det(A) = 0$

$$\begin{aligned}\det(A) &= a_{i_1} C_{i_1} + a_{i_2} C_{i_2} + a_{i_3} C_{i_3} && \text{i-th row cofactor expansion of } A \\ &= a_{1i} C_{1i} + a_{2i} C_{2i} + a_{3i} C_{3i} && \text{i-th column cofactor expansion of } A^T\end{aligned}$$

Elementary Matrix and Determinant (1)

Interchange two rows

The diagram shows two 3x1 matrices, A and B . Matrix A has rows colored green, cyan, and green. Matrix B has rows colored cyan, green, and cyan. Red arrows indicate the interchange of the first and second rows between A and B .

$$\begin{bmatrix} \text{green} \\ \text{cyan} \\ \text{green} \end{bmatrix} \leftrightarrow \begin{bmatrix} \text{cyan} \\ \text{green} \\ \text{green} \end{bmatrix}$$

B A

$$\det(B) = -\det(A)$$

$$\begin{vmatrix} a_{21} & a_{22} & a_{23} \\ a_{11} & a_{12} & a_{13} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = - \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

Multiply a row by a nonzero constant

The diagram shows two 3x1 matrices, A and B . Matrix A has a cyan row. Matrix B has a purple row. A red arrow labeled $\times c$ points from A to B . Purple arrows point from the cyan row of A to the purple row of B .

$$\begin{bmatrix} \text{purple} \end{bmatrix} \leftarrow \begin{bmatrix} \text{cyan} \end{bmatrix} \times c$$

B A

$$\det(B) = c \det(A)$$

$$\begin{vmatrix} c a_{11} & c a_{12} & c a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = c \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

Add a multiple of one row to another

The diagram shows two 3x1 matrices, A and B . Matrix A has a cyan row. Matrix B has an orange row. A red arrow labeled $\times c$ points from A to B . Purple arrows point from the cyan row of A to the orange row of B . Red arrows with '+' signs point from the cyan row of A to the orange row of B , indicating the addition of a multiple of one row to another.

$$\begin{bmatrix} \text{orange} \end{bmatrix} \leftarrow \begin{bmatrix} \text{cyan} \end{bmatrix} \times c$$

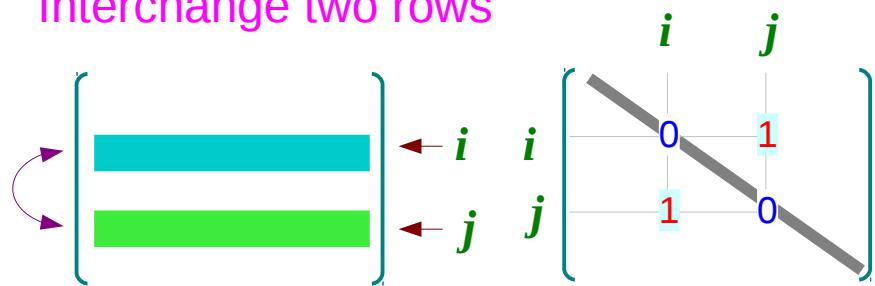
B A

$$\det(B) = \det(A)$$

$$\begin{vmatrix} a_{11} + c a_{21} & a_{12} + c a_{22} & a_{13} + c a_{23} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

Elementary Matrix and Determinant (2)

Interchange two rows



$$\det(\mathbf{B}) = -\det(\mathbf{A})$$

$$\begin{vmatrix} a_{21} & a_{22} & a_{23} \\ a_{11} & a_{12} & a_{13} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = - \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

$$\det(\mathbf{B})$$

$$\begin{vmatrix} & a_{22} & a_{23} \\ & a_{32} & a_{33} \\ a_{11} & & \end{vmatrix}$$

$$\det(\mathbf{A})$$

$$\begin{vmatrix} & a_{22} & a_{23} \\ a_{32} & & a_{33} \\ a_{11} & & \end{vmatrix}$$

$$\det(\mathbf{B})$$

$$\begin{aligned} \det(\mathbf{B}) &= b_{21} C_{21} + b_{22} C_{22} + b_{23} C_{23} \\ &= -a_{11} M_{21} + a_{12} M_{22} - a_{13} M_{23} \end{aligned}$$

$$\begin{vmatrix} a_{21} & & a_{23} \\ & a_{22} & \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

$$\begin{vmatrix} a_{21} & & a_{23} \\ a_{31} & & a_{33} \\ a_{11} & & \end{vmatrix}$$

$$\det(\mathbf{A})$$

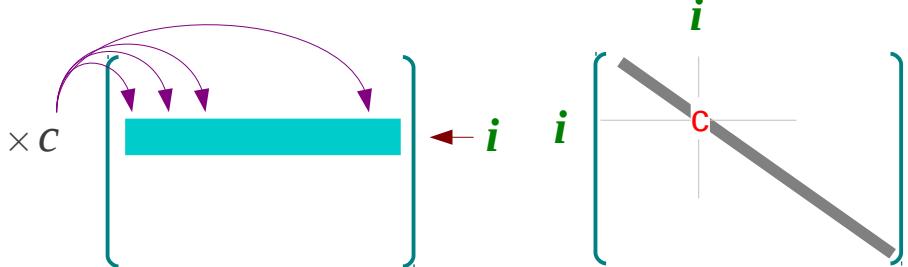
$$\begin{aligned} \det(\mathbf{A}) &= a_{11} C_{11} + a_{12} C_{12} + a_{13} C_{13} \\ &= a_{11} M_{11} - a_{12} M_{12} + a_{13} M_{13} \end{aligned}$$

$$\begin{vmatrix} a_{21} & a_{22} & \\ & a_{23} & \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

$$\begin{vmatrix} a_{21} & a_{22} & \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

Elementary Matrix and Determinant (3)

Multiply a row by a nonzero constant



$$\det(\mathbf{B}) = c \det(\mathbf{A})$$

$$\begin{vmatrix} c a_{11} & c a_{12} & c a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = c \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

$$\det(\mathbf{B})$$

$$\begin{vmatrix} c a_{11} & c a_{12} & c a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

$$\det(\mathbf{B}) = b_{11} C_{11} + b_{12} C_{12} + b_{13} C_{13}$$

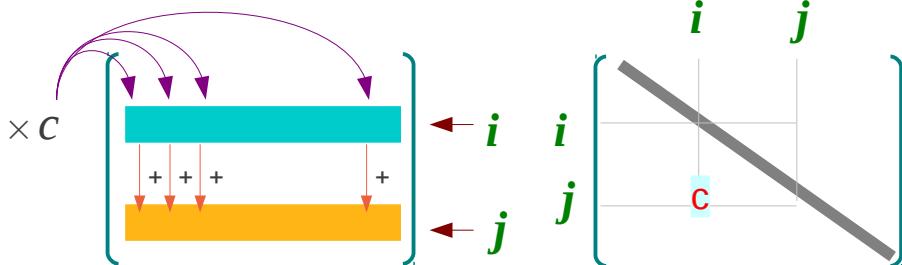
$$= c \cdot a_{11} C_{11} + c \cdot a_{12} C_{12} + c \cdot a_{13} C_{13}$$

$$= c (a_{11} C_{11} + a_{12} C_{12} + a_{13} C_{13})$$

$$= c \cdot \det(\mathbf{A})$$

Elementary Matrix and Determinant (4)

Add a multiple of one row to another



$$\det(\mathbf{B}) = \det(\mathbf{A})$$

$$\begin{vmatrix} a_{11} + c a_{21} & a_{12} + c a_{22} & a_{13} + c a_{23} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

$$\det(\mathbf{B})$$

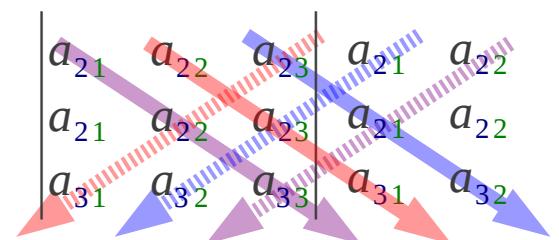
$$\begin{vmatrix} a_{11} + c a_{21} & a_{12} + c a_{22} & a_{13} + c a_{23} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

$$\det(\mathbf{B}) = b_{11} C_{11} + b_{12} C_{12} + b_{13} C_{13}$$

$$\begin{aligned} &= (a_{11} + c a_{21}) C_{11} + (a_{12} + c a_{22}) C_{12} + (a_{13} + c a_{23}) C_{13} \\ &= (a_{11} C_{11} + a_{12} C_{12} + a_{13} C_{13}) \xrightarrow{\textcolor{red}{\rightarrow}} \det(\mathbf{A}) \\ &\quad + c(a_{21} C_{11} + a_{22} C_{12} + a_{23} C_{13}) \xrightarrow{\textcolor{red}{\rightarrow}} 0 \end{aligned}$$

$$\begin{vmatrix} a_{21} & a_{22} & a_{23} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} \begin{vmatrix} a_{21} & a_{22} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

$$(a_{21} C_{11} + a_{22} C_{12} + a_{23} C_{13}) = 0$$



Determinant of Diagonal Matrix

Lower Triangular Matrix

$$\begin{vmatrix} a_{11} & 0 & 0 \\ a_{21} & a_{22} & 0 \\ a_{31} & a_{32} & a_{33} \end{vmatrix} \quad \begin{matrix} a_{11} & 0 \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{matrix} \quad \det(A) = a_{11} a_{22} a_{33}$$

Upper Triangular Matrix

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & 0 & a_{33} \end{vmatrix} \quad \begin{matrix} a_{11} & a_{12} \\ 0 & a_{22} \\ 0 & 0 \end{matrix} \quad \det(A) = a_{11} a_{22} a_{33}$$

Diagonal Triangular Matrix

$$\begin{vmatrix} a_{11} & 0 & 0 \\ 0 & a_{22} & 0 \\ 0 & 0 & a_{33} \end{vmatrix} \quad \begin{matrix} a_{11} & a_{12} \\ 0 & a_{22} \\ 0 & 0 \end{matrix} \quad \det(A) = a_{11} a_{22} a_{33}$$

Determinant of an Elementary Matrix

Interchange two rows

A diagram illustrating the interchange of two rows in a 2x2 matrix. On the left, a teal bracket labeled i and a green bracket labeled j enclose two horizontal bars: a teal bar above a green bar. A curved arrow above the teal bar points to the right, indicating the swap. To the right, a 2x2 matrix is shown with columns labeled i and j . The top-left entry is 0, the top-right is 1, the bottom-left is 1, and the bottom-right is 0. A thick grey diagonal line connects the top-left and bottom-right entries.

$$\det(E_k) = -1$$

Multiply a row by a nonzero constant

A diagram illustrating multiplying a row by a constant c . On the left, a teal bracket labeled i encloses a teal bar. A curved arrow above the bar points to the right, with three purple arrows pointing from the bar to the arrow. To the right, a 2x2 matrix is shown with columns labeled i and j . The top-left entry is c , and the other entries are 0. A thick grey diagonal line connects the top-left and bottom-right entries.

$$\det(E_k) = c$$

Add a multiple of one row to another

A diagram illustrating adding a multiple of one row to another. On the left, a teal bracket labeled i encloses a teal bar, and a green bracket labeled j encloses an orange bar. A curved arrow above the teal bar points to the right, with three purple arrows pointing from the teal bar to the arrow. Below the teal bar, four red arrows point down to the orange bar, each labeled with a '+'. To the right, a 2x2 matrix is shown with columns labeled i and j . The top-left entry is 0, the top-right is 1, the bottom-left is 1, and the bottom-right is c . A thick grey diagonal line connects the top-left and bottom-right entries.

$$\det(E_k) = 1$$

Properties of Determinants

$$\det(kA) = k^n \det(A)$$

$$\det(A+B) \neq \det(A)+\det(B)$$

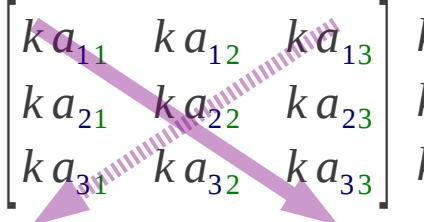
$$\det(AB) = \det(A)\det(B)$$

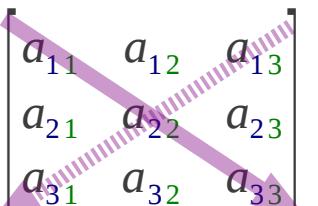
Proof of $\det(kA) = k^n \det(A)$

$$\det(kA) = k^n \det(A)$$

$$\det(A+B) \neq \det(A)+\det(B)$$

$$\det(AB) = \det(A)\det(B)$$

$$\begin{bmatrix} ka_{11} & ka_{12} & ka_{13} \\ ka_{21} & ka_{22} & ka_{23} \\ ka_{31} & ka_{32} & ka_{33} \end{bmatrix} \begin{array}{ll} ka_{11} & ka_{12} \\ ka_{21} & ka_{22} \\ ka_{31} & ka_{32} \end{array}$$


$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{array}{ll} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{array}$$


Proof of $\det(A+B) \neq \det(A) + \det(B)$

$$\det(kA) = k^n \det(A)$$

$$\det(A+B) \neq \det(A) + \det(B)$$

$$\det(AB) = \det(A)\det(B)$$

$$\begin{bmatrix} a_{11}+b_{11} & a_{12}+b_{12} & a_{13}+b_{13} \\ a_{21}+b_{21} & a_{22}+b_{22} & a_{23}+b_{23} \\ a_{31}+b_{31} & a_{32}+b_{32} & a_{33}+b_{33} \end{bmatrix} \begin{array}{l} a_{11}+b_{11} \quad a_{12}+b_{12} \\ a_{21}+b_{21} \quad a_{22}+b_{22} \\ a_{31}+b_{31} \quad a_{32}+b_{32} \end{array}$$

$$A \quad n \times n \quad B \quad n \times n$$

$$\begin{bmatrix} \$ \$ \$ \\ \$ \$ \$ \end{bmatrix} + \begin{bmatrix} \# \# \# \\ \# \# \# \end{bmatrix}$$

$$C = A + B$$



$$\det(C) = \det(A) + \det(B)$$

$$\begin{bmatrix} a_{11}+b_{11} & a_{12}+b_{12} & a_{13}+b_{13} \\ 2a_{21} & 2a_{22} & 2a_{23} \\ 2a_{31} & 2a_{32} & 2a_{33} \end{bmatrix} \begin{array}{l} a_{11}+b_{11} \quad a_{12}+b_{12} \\ 2a_{21} \quad 2a_{22} \\ 2a_{31} \quad 2a_{32} \end{array}$$

Proof of $\det(AB) = \det(A) \det(B)$ (1)

E_k (Elementary Matrices)

$$E_k \begin{bmatrix} i & j \\ i & j \end{bmatrix}$$

$$E_k \begin{bmatrix} i & j \\ i & j \end{bmatrix}$$

$$E_k \begin{bmatrix} i & j \\ i & j \end{bmatrix}$$



$$\det(E_k B) = \det(E_k) \det(B)$$

$A \quad n \times n$: invertible

$$AA^{-1} = A^{-1}A = I$$



$$\det(A) \neq 0$$

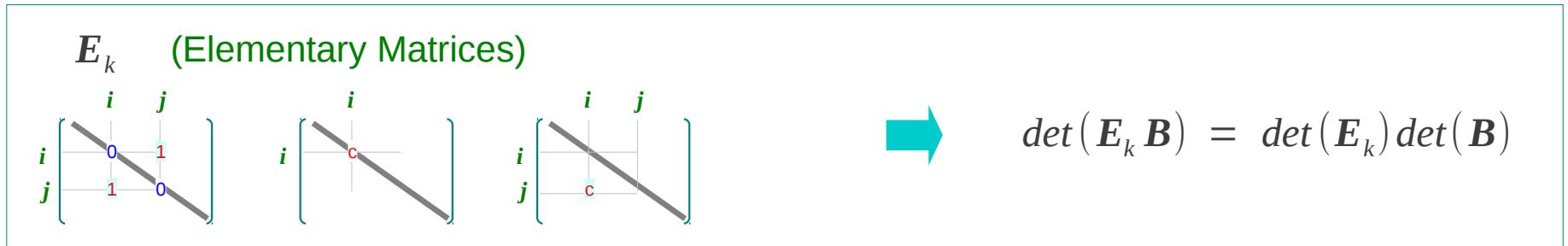
$A \quad n \times n$: invertible

$$AA^{-1} = A^{-1}A = I$$



$$\det(A^{-1}) = \frac{1}{\det(A)}$$

Proof of $\det(AB) = \det(A) \det(B)$ (2)



$$E_k = \begin{bmatrix} i & j \\ i & j \end{bmatrix}$$

$$E_k = \begin{bmatrix} i & j \\ i & j \end{bmatrix}$$

$$E_k = \begin{bmatrix} i & j \\ i & j \end{bmatrix}$$

$$\det(E_k B) = -\det(B)$$

$$\det(E_k) = -1$$

$$\det(E_k B) = \det(E_k) \det(B)$$

$$\det(E_k B) = c \cdot \det(B)$$

$$\det(E_k) = c$$

$$\det(E_k B) = \det(E_k) \det(B)$$

$$\det(E_k B) = \det(B)$$

$$\det(E_k) = 1$$

$$\det(E_k B) = \det(E_k) \det(B)$$

Proof of $\det(AB) = \det(A) \det(B)$ (3)

A $n \times n$: invertible

$$AA^{-1} = A^{-1}A = I$$



$$\det(A) \neq 0$$

$$E_r \cdots E_2 E_1 A = R$$

Reduced Row Echelon Form

$$\underbrace{\det(E_r) \cdots \det(E_2) \det(E_1)}_{\text{non-zero}} \det(A) = \det(R)$$

A $n \times n$: invertible



$$R = I \quad \det(R) = 1 (\neq 0)$$

$$\det(A) \neq 0$$



$$\det(R) \neq 0$$

No zero row $R = I$

A $n \times n$: invertible

Proof of $\det(AB) = \det(A) \det(B)$ (4)

$$\det(AB) = \det(A) + \det(B)$$

A $n \times n$: not invertible \rightarrow AB $n \times n$: not invertible

$$\det(A) = 0$$

$$\det(AB) = 0$$

A $n \times n$: invertible \rightarrow $A = E_r \cdots E_2 E_1$
 $AB = E_r \cdots E_2 E_1 B$

$$\det(AB) = \det(E_r) \cdots \det(E_2) \det(E_1) \det(B)$$

$$\det(AB) = \boxed{\det(E_r \cdots E_2 E_1) \det(B)}$$

$$\det(AB) = \boxed{\det(A) \det(B)}$$

Proof of $\det(AB) = \det(A) \det(B)$ (5)

A $n \times n$: invertible

$$AA^{-1} = A^{-1}A = I$$



$$\det(A^{-1}) = \frac{1}{\det(A)}$$

$$AA^{-1} = I$$

$$\det(AA^{-1}) = \det(I)$$

$$\det(A)\det(A^{-1}) = 1$$

Computing $A \ adj(A) - \text{diagonal elements}$

Given matrix

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$n \times n$

Matrix of Cofactors

$$\begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix}$$

$n \times n$

Adjoint

$$\begin{bmatrix} C_{11} & C_{21} & C_{31} \\ C_{12} & C_{22} & C_{32} \\ C_{13} & C_{23} & C_{33} \end{bmatrix}$$

$n \times n$

transpose 

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$\begin{bmatrix} C_{11} & C_{21} & C_{31} \\ C_{12} & C_{22} & C_{32} \\ C_{13} & C_{23} & C_{33} \end{bmatrix}$$

$$det(\mathbf{A}) = a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13}$$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$\begin{bmatrix} C_{11} & C_{21} & C_{31} \\ C_{12} & C_{22} & C_{32} \\ C_{13} & C_{23} & C_{33} \end{bmatrix}$$

$$det(\mathbf{A}) = a_{21}C_{21} + a_{22}C_{22} + a_{23}C_{23}$$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$\begin{bmatrix} C_{11} & C_{21} & C_{31} \\ C_{12} & C_{22} & C_{32} \\ C_{13} & C_{23} & C_{33} \end{bmatrix}$$

$$det(\mathbf{A}) = a_{31}C_{31} + a_{32}C_{32} + a_{33}C_{33}$$

Computing $A \ adj(A) - \text{off-diagonal elements}$ (1)

Given matrix

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$n \times n$

Matrix of Cofactors

$$\begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix}$$

$n \times n$

Adjoint

$$\begin{bmatrix} C_{11} & C_{21} & C_{31} \\ C_{12} & C_{22} & C_{32} \\ C_{13} & C_{23} & C_{33} \end{bmatrix}$$

$n \times n$

transpose 

\mathbf{A}

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$adj(\mathbf{A})$

$$\begin{bmatrix} C_{11} & C_{21} & C_{31} \\ C_{12} & C_{22} & C_{32} \\ C_{13} & C_{23} & C_{33} \end{bmatrix}$$

$$det(\mathbf{A}) = a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13}$$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$\begin{bmatrix} C_{11} & C_{21} & C_{31} \\ C_{12} & C_{22} & C_{32} \\ C_{13} & C_{23} & C_{33} \end{bmatrix}$$

$$0 = a_{21}C_{11} + a_{22}C_{12} + a_{23}C_{13}$$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$\begin{bmatrix} C_{11} & C_{21} & C_{31} \\ C_{12} & C_{22} & C_{32} \\ C_{13} & C_{23} & C_{33} \end{bmatrix}$$

$$0 = a_{31}C_{11} + a_{32}C_{12} + a_{33}C_{13}$$

Computing $A \ adj(A) - \text{off-diagonal elements}$ (2)

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \quad \begin{bmatrix} C_{11} \\ C_{12} \\ C_{13} \end{bmatrix}$$



$$\begin{bmatrix} a_{21} & a_{22} & a_{23} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \quad \begin{bmatrix} C'_{11} \\ C'_{12} \\ C'_{13} \end{bmatrix}$$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$\begin{bmatrix} & a_{22} & a_{23} \\ a_{32} & a_{33} \end{bmatrix}$$

$$\begin{bmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{bmatrix}$$

$$M_{11} = \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}$$

$$C_{11} = +M_{11}$$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$\begin{bmatrix} & & a_{23} \\ a_{21} & & a_{23} \\ a_{31} & & a_{33} \end{bmatrix}$$

$$\begin{bmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{bmatrix}$$

$$M_{12} = \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix}$$

$$C_{12} = -M_{12}$$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$\begin{bmatrix} & & a_{13} \\ a_{21} & a_{22} & \\ a_{31} & a_{32} & \end{bmatrix}$$

$$\begin{bmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix}$$

$$M_{13} = \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

$$C_{13} = +M_{13}$$

The redundant row

- The same cofactors
- Determinant = 0

$$a_{21} C'_{11} + a_{22} C'_{12} + a_{23} C'_{13} \\ = a_{21} C_{11} + a_{22} C_{12} + a_{23} C_{13} = 0$$

$$\begin{array}{|ccc|cc|} \hline a_{21} & a_{22} & a_{23} & a_{21} & a_{22} \\ \hline a_{21} & a_{22} & a_{23} & a_{21} & a_{22} \\ \hline a_{31} & a_{32} & a_{33} & a_{31} & a_{32} \\ \hline \end{array}$$

$$(a_{21} C_{11} + a_{22} C_{12} + a_{23} C_{13}) = 0$$

$$\begin{array}{|ccc|cc|} \hline a_{21} & a_{22} & a_{23} & a_{21} & a_{22} \\ \hline a_{21} & a_{22} & a_{23} & a_{21} & a_{22} \\ \hline a_{31} & a_{32} & a_{33} & a_{31} & a_{32} \\ \hline \end{array}$$

Result of $\mathbf{A} \ adj(\mathbf{A})$

Given matrix

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$n \times n$

Matrix of Cofactors

$$\begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix}$$

$n \times n$

transpose 

Adjoint

$$\begin{bmatrix} C_{11} & C_{21} & C_{31} \\ C_{12} & C_{22} & C_{32} \\ C_{13} & C_{23} & C_{33} \end{bmatrix}$$

$n \times n$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$\begin{bmatrix} C_{11} & C_{21} & C_{31} \\ C_{12} & C_{22} & C_{32} \\ C_{13} & C_{23} & C_{33} \end{bmatrix}$$

$$= \begin{bmatrix} \det(\mathbf{A}) & 0 & 0 \\ 0 & \det(\mathbf{A}) & 0 \\ 0 & 0 & \det(\mathbf{A}) \end{bmatrix}$$

$$= \det(\mathbf{A}) \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

matrix  value 

$$\mathbf{A} adj(\mathbf{A}) = \det(\mathbf{A}) \mathbf{I}$$

$$\mathbf{A} \left[\frac{adj(\mathbf{A})}{\det(\mathbf{A})} \right] = \mathbf{I}$$

$$\mathbf{A} [\mathbf{A}^{-1}] = \mathbf{I}$$

Linear Equations

$$(\text{Eq 1}) \rightarrow a_{11} x_1 + a_{12} x_2 + \cdots + a_{1n} x_n = b_1$$

$$(\text{Eq 2}) \rightarrow a_{21} x_1 + a_{22} x_2 + \cdots + a_{2n} x_n = b_2$$

 \vdots \vdots \vdots \vdots

$$(\text{Eq 3}) \rightarrow a_{n1} x_1 + a_{n2} x_2 + \cdots + a_{nn} x_n = b_n$$

$$\left[\begin{array}{cccc} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{array} \right] \left[\begin{array}{c} x_1 \\ x_2 \\ \vdots \\ x_n \end{array} \right] = \left[\begin{array}{c} b_1 \\ b_2 \\ \vdots \\ b_n \end{array} \right]$$

$$\mathbf{A} \mathbf{x} = \mathbf{b}$$

Cramer's Rule (1)

$$\begin{matrix} \mathbf{A} \\ \left[\begin{array}{cccc} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{array} \right] \end{matrix} \begin{matrix} \mathbf{x} \\ \left[\begin{array}{c} x_1 \\ x_2 \\ \vdots \\ x_n \end{array} \right] \end{matrix} = \begin{matrix} \mathbf{b} \\ \left[\begin{array}{c} b_1 \\ b_2 \\ \vdots \\ b_n \end{array} \right] \end{matrix}$$

$$\begin{matrix} \mathbf{A}_1 \\ \left[\begin{array}{ccccc} b_1 & a_{12} & \cdots & a_{1n} & | \\ b_2 & a_{22} & \cdots & a_{2n} & | \\ \vdots & \vdots & & \vdots & | \\ b_n & a_{n2} & \cdots & a_{nn} & | \end{array} \right] \end{matrix} \quad \begin{matrix} \mathbf{A}_2 \\ \left[\begin{array}{ccccc} a_{11} & b_1 & \cdots & a_{1n} & | \\ a_{21} & b_2 & \cdots & a_{2n} & | \\ \vdots & \vdots & & \vdots & | \\ a_{n1} & b_n & \cdots & a_{nn} & | \end{array} \right] \end{matrix} \quad \begin{matrix} \mathbf{A}_n \\ \left[\begin{array}{ccccc} a_{11} & a_{12} & \cdots & b_1 & | \\ a_{21} & a_{22} & \cdots & b_2 & | \\ \vdots & \vdots & & \vdots & | \\ a_{n1} & a_{n2} & \cdots & b_n & | \end{array} \right] \end{matrix}$$

$$x_1 = \frac{\det(\mathbf{A}_1)}{\det(\mathbf{A})}$$

$$x_2 = \frac{\det(\mathbf{A}_2)}{\det(\mathbf{A})}$$

$$x_n = \frac{\det(\mathbf{A}_n)}{\det(\mathbf{A})}$$

Cramer's Rule (2)

$$\mathbf{A}_1 \left[\begin{array}{cccc} b_1 & a_{12} & \cdots & a_{1n} \\ b_2 & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ b_n & a_{n2} & \cdots & a_{nn} \end{array} \right]$$

$$\det(\mathbf{A}_1) = b_1 C_{11} + b_2 C_{21} + \cdots + b_n C_{n1}$$

$$x_1 = \frac{\det(\mathbf{A}_1)}{\det(\mathbf{A})}$$

$$\mathbf{A}_2 \left[\begin{array}{cccc} a_{11} & b_1 & \cdots & a_{1n} \\ a_{21} & b_2 & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & b_n & \cdots & a_{nn} \end{array} \right]$$

$$\det(\mathbf{A}_2) = b_1 C_{12} + b_2 C_{22} + \cdots + b_n C_{n2}$$

$$x_2 = \frac{\det(\mathbf{A}_2)}{\det(\mathbf{A})}$$

$$\mathbf{A}_n \left[\begin{array}{cccc} a_{11} & a_{12} & \cdots & b_1 \\ a_{21} & a_{22} & \cdots & b_2 \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & b_n \end{array} \right]$$

$$\det(\mathbf{A}_n) = b_1 C_{1n} + b_2 C_{2n} + \cdots + b_n C_{nn}$$

$$x_n = \frac{\det(\mathbf{A}_n)}{\det(\mathbf{A})}$$

Cramer's Rule (3)

$$\mathbf{A}\mathbf{x} = \mathbf{b} \quad \mathbf{x} = \mathbf{A}^{-1}\mathbf{b} = \frac{\text{adj}(\mathbf{A})}{\det(\mathbf{A})}\mathbf{b} = \begin{pmatrix} C_{11} & C_{21} & \cdots & C_{n1} \\ C_{12} & C_{22} & \cdots & C_{n2} \\ \vdots & \vdots & & \vdots \\ C_{1n} & C_{2n} & \cdots & C_{nn} \end{pmatrix} \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}$$

$$\mathbf{x} = \frac{1}{\det(\mathbf{A})} \begin{pmatrix} b_1 C_{11} + b_2 C_{21} + \cdots + b_n C_{n1} \\ b_1 C_{12} + b_2 C_{22} + \cdots + b_n C_{n2} \\ \vdots & \vdots & \vdots \\ b_1 C_{1n} + b_2 C_{2n} + \cdots + b_n C_{nn} \end{pmatrix}$$

$$\begin{aligned} x_1 &= \frac{\det(\mathbf{A}_1)}{\det(\mathbf{A})} \\ x_2 &= \frac{\det(\mathbf{A}_2)}{\det(\mathbf{A})} \\ x_n &= \frac{\det(\mathbf{A}_n)}{\det(\mathbf{A})} \end{aligned}$$

Equivalent Statements

A : invertible

$$\begin{bmatrix} A \end{bmatrix} \begin{bmatrix} A^{-1} \end{bmatrix} = \begin{bmatrix} A^{-1} \end{bmatrix} \begin{bmatrix} A \end{bmatrix} = \begin{bmatrix} I_n \end{bmatrix}$$
$$\begin{bmatrix} \text{[cyan square]} \end{bmatrix} \begin{bmatrix} \text{[green square]} \end{bmatrix} = \begin{bmatrix} \text{[green square]} \end{bmatrix} \begin{bmatrix} \text{[cyan square]} \end{bmatrix} = \begin{bmatrix} \text{[blue 1, red 0 diagonal]} \\ \text{[blue 0, red 1 diagonal]} \end{bmatrix}$$

$$Ax = 0$$

only the trivial solution

$$\begin{bmatrix} A \end{bmatrix} \begin{bmatrix} x \end{bmatrix} = \begin{bmatrix} 0 \end{bmatrix}$$
$$\begin{bmatrix} \text{[cyan square]} \end{bmatrix} \begin{bmatrix} \text{[orange bar]} \end{bmatrix} = \begin{bmatrix} \text{[blue 0, blue 0, blue 0]} \end{bmatrix}$$

A the RREF is I_n
(Reduced Row Echelon Form)

$$\begin{bmatrix} A \end{bmatrix} \xrightarrow{\text{Elem Row Op}} \begin{bmatrix} I_n \end{bmatrix}$$
$$\begin{bmatrix} \text{[cyan square]} \end{bmatrix} \xrightarrow{\text{[yellow arrow]}} \begin{bmatrix} \text{[blue 1, red 0 diagonal]} \\ \text{[blue 0, red 1 diagonal]} \end{bmatrix}$$

A can be written as a product of E_k
(Elementary Matrices)

$$\begin{bmatrix} i & j \end{bmatrix} \quad \begin{bmatrix} i & j \end{bmatrix} \quad \begin{bmatrix} i & j \end{bmatrix}$$

References

- [1] <http://en.wikipedia.org/>
- [2] Anton & Busby, "Contemporary Linear Algebra"
- [3] Anton & Rorres, "Elementary Linear Algebra"