

CORDIC Background (4A)

•
•

Copyright (c) 2011-2012 Young W. Lim.

Permission is granted to copy, distribute and/or modify this document under the terms of the GNU Free Documentation License, Version 1.2 or any later version published by the Free Software Foundation; with no Invariant Sections, no Front-Cover Texts, and no Back-Cover Texts. A copy of the license is included in the section entitled "GNU Free Documentation License".

Please send corrections (or suggestions) to youngwlim@hotmail.com.

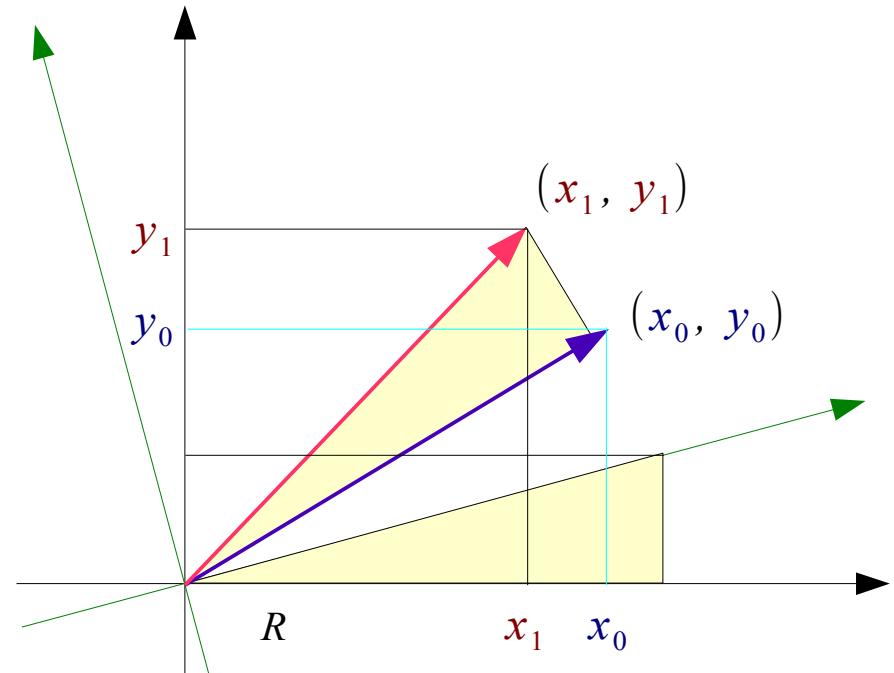
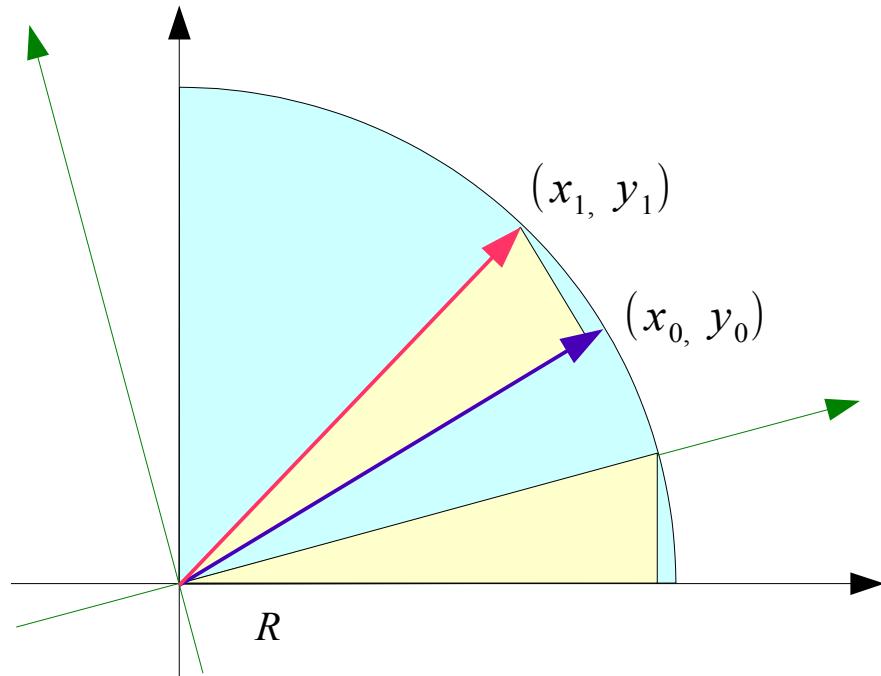
This document was produced by using OpenOffice and Octave.

CORDIC Background

J. P. Deschamps, G. A. Bioul, G.D. Sutter, Synthesis of Arithmetic Circuits

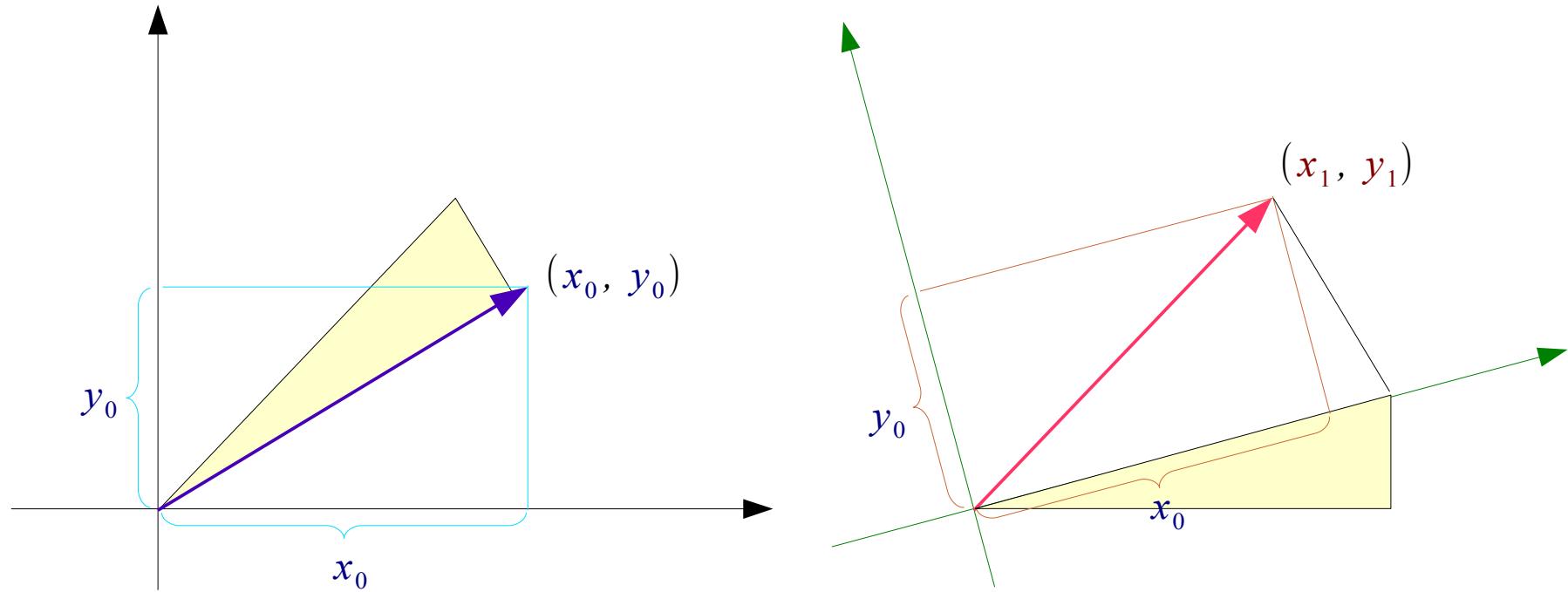
Vector Rotation (1)

$$y_1 = x_0 \sin \alpha + y_0 \cos \alpha$$

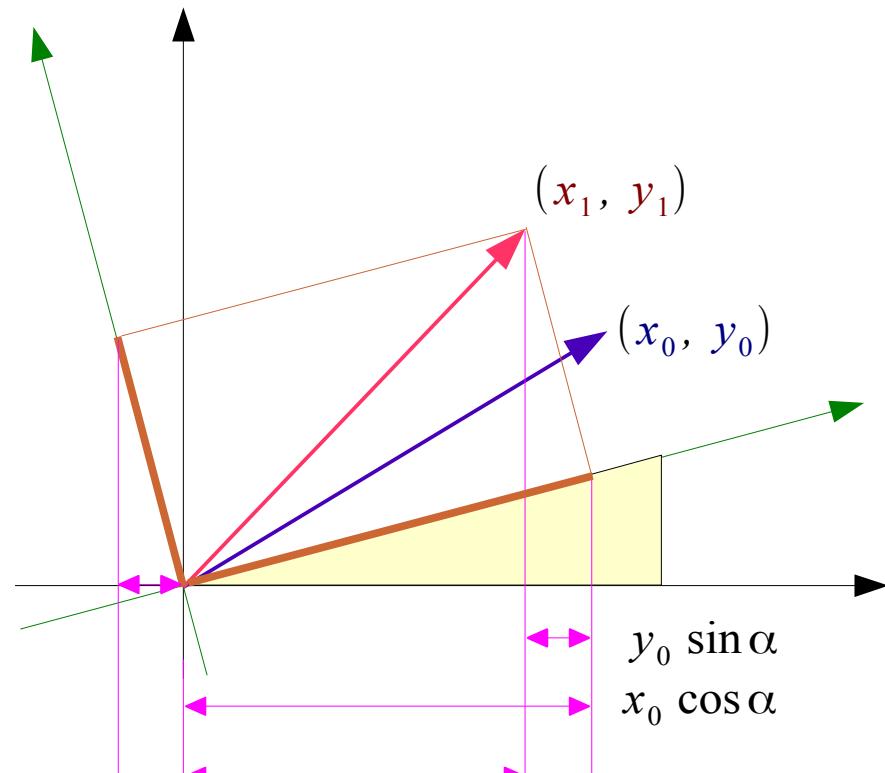


$$x_1 = x_0 \cos \alpha - y_0 \sin \alpha$$

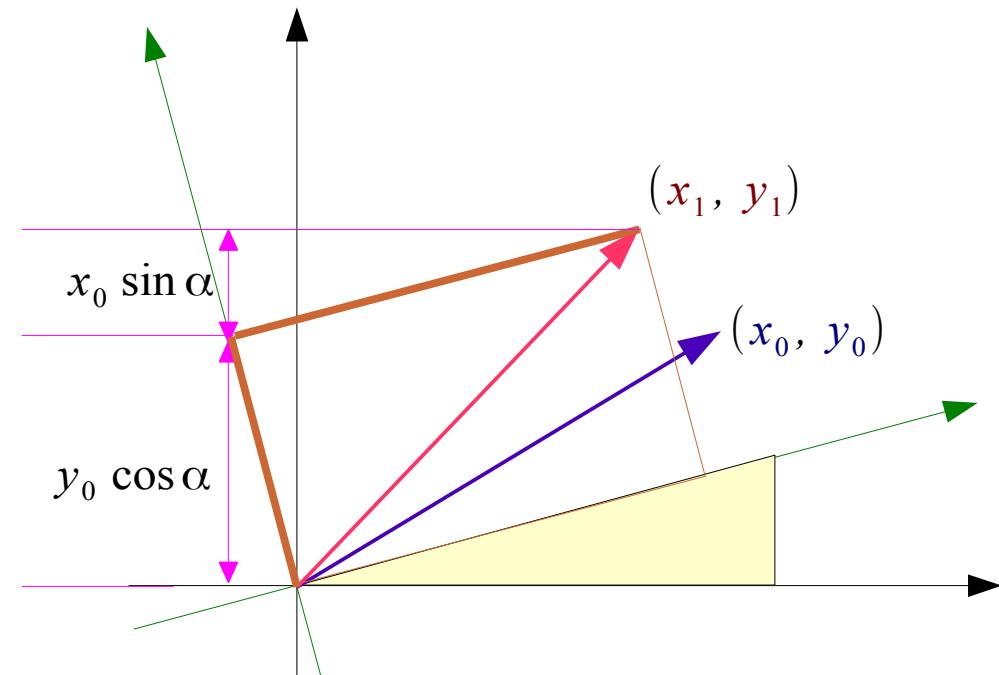
Vector Rotation (2)



Vector Rotation (3)



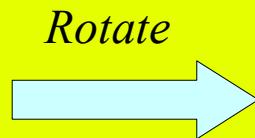
$$x_1 = x_0 \cos \alpha - y_0 \sin \alpha$$



$$y_1 = x_0 \sin \alpha + y_0 \cos \alpha$$

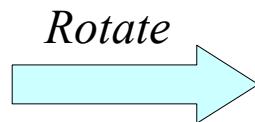
Iterative Rotation

Given Vector
Given Angle



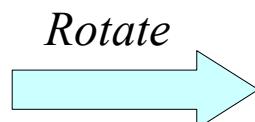
Computing
new coordinates

Given Unit Vector
Given Angle α



$$\begin{aligned}x &= \cos \alpha \\y &= \sin \alpha\end{aligned}$$

Given Vector (x_0, y_0)
Given Angle α



$$\begin{aligned}x_n &= x_0 \cos \alpha - y_0 \sin \alpha \\y_n &= x_0 \sin \alpha + y_0 \cos \alpha\end{aligned}$$

$\alpha_0 \rightarrow \dots \rightarrow \alpha_i \rightarrow \alpha_{i+1} \rightarrow \dots \rightarrow \alpha_n \rightarrow 0$

$$\begin{pmatrix} x_0 \\ y_0 \end{pmatrix} \rightarrow \dots \rightarrow \begin{pmatrix} x_i \\ y_i \end{pmatrix} \rightarrow \begin{pmatrix} x_{i+1} \\ y_{i+1} \end{pmatrix} \rightarrow \dots \rightarrow \begin{pmatrix} x_n \\ y_n \end{pmatrix}$$

CORDIC Rotation

$\alpha_0 \rightarrow \dots \alpha_i \rightarrow \alpha_{i+1} \rightarrow \dots \alpha_n \rightarrow 0$

$\begin{pmatrix} x_0 \\ y_0 \end{pmatrix} \rightarrow \dots \rightarrow \begin{pmatrix} x_i \\ y_i \end{pmatrix} \rightarrow \begin{pmatrix} x_{i+1} \\ y_{i+1} \end{pmatrix} \rightarrow \dots \rightarrow \begin{pmatrix} x_n \\ y_n \end{pmatrix}$

$$\begin{cases} x_{i+1} = x_i \cos \alpha_i - y_i \sin \alpha_i \\ y_{i+1} = x_i \sin \alpha_i + y_i \cos \alpha_i \end{cases}$$

$$\begin{cases} x_{i+1} = \cos \alpha_i (x_i - y_i \tan \alpha_i) \\ y_{i+1} = \cos \alpha_i (x_i \tan \alpha_i + y_i) \end{cases}$$

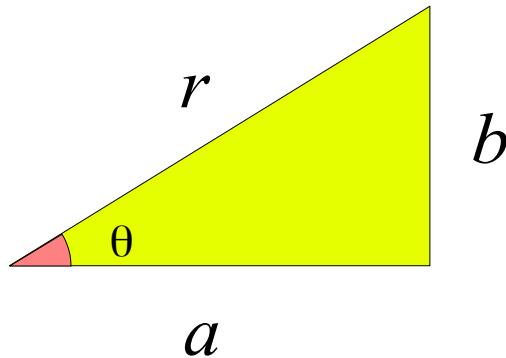
$$\begin{cases} x_{i+1} = \frac{1}{\sqrt{1 + \tan^2 \alpha_i}} (x_i - y_i \tan \alpha_i) \\ y_{i+1} = \frac{1}{\sqrt{1 + \tan^2 \alpha_i}} (x_i \tan \alpha_i + y_i) \end{cases}$$



Pseudo-rotation

$$\begin{cases} x'_{i+1} = (x_i - y_i \tan \alpha_i) \\ y'_{i+1} = (x_i \tan \alpha_i + y_i) \end{cases}$$

$\cos \theta$



$$r = \sqrt{a^2 + b^2}$$

$$\cos \theta = \frac{a}{r} = \frac{a}{\sqrt{a^2 + b^2}}$$

$$\cos \theta = \frac{a}{r}$$

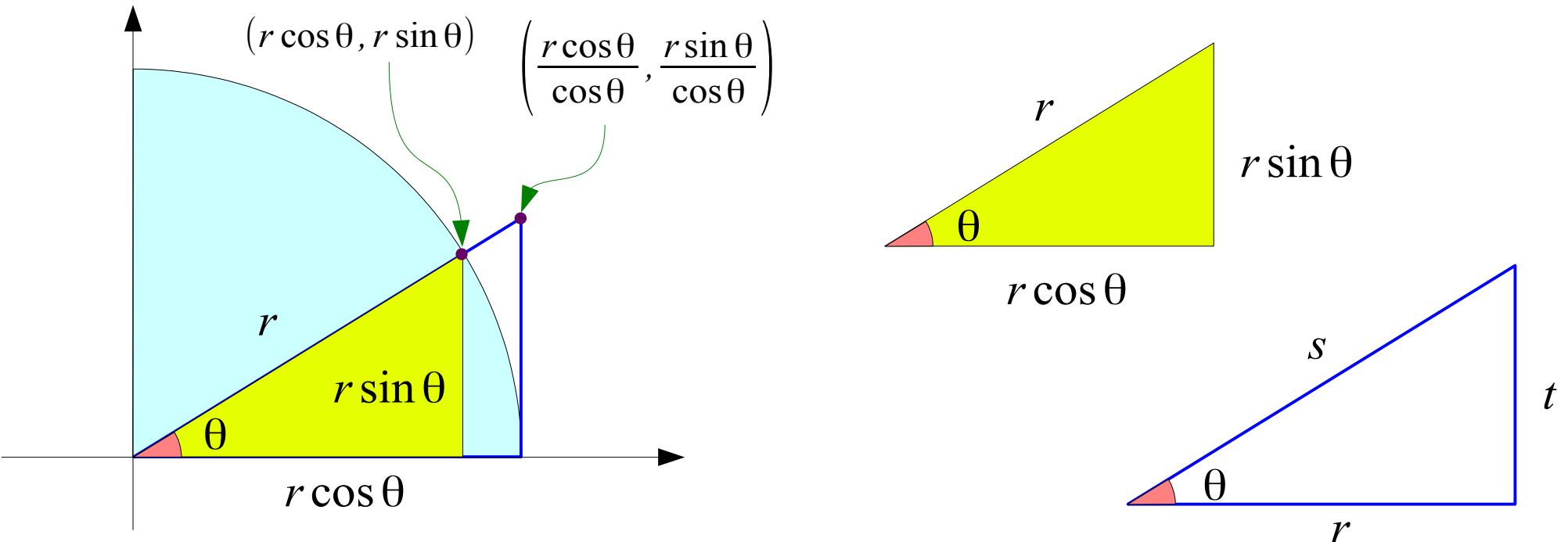
$$= \frac{1}{\sqrt{1 + (b/a)^2}}$$

$$\sin \theta = \frac{b}{r}$$

$$= \frac{1}{\sqrt{1 + \tan^2 \theta}}$$

$$\tan \theta = \frac{b}{a}$$

Pseudo-rotation – factor of $1/\cos \theta$



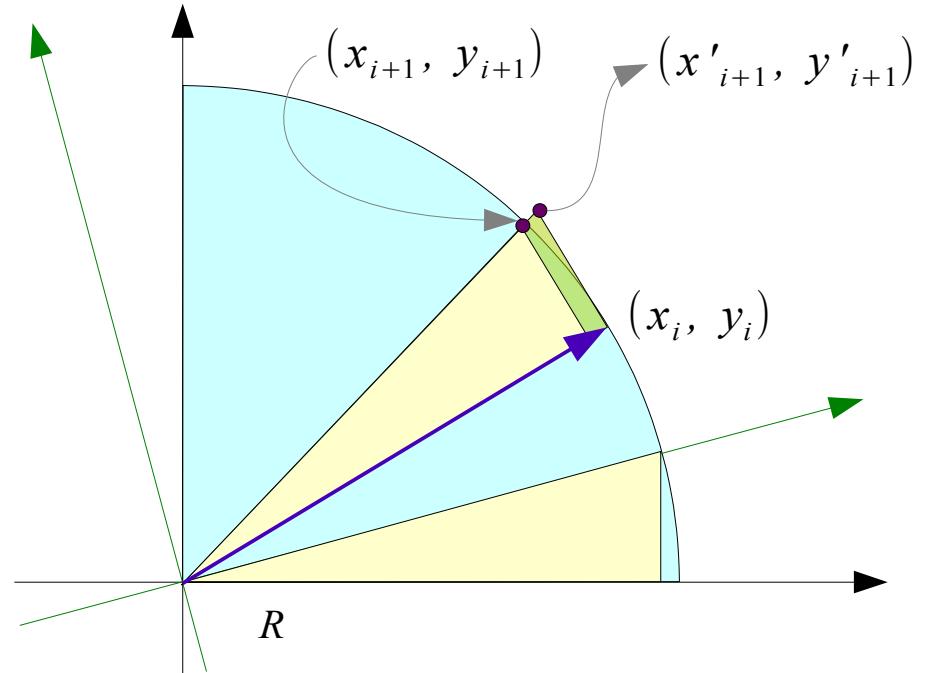
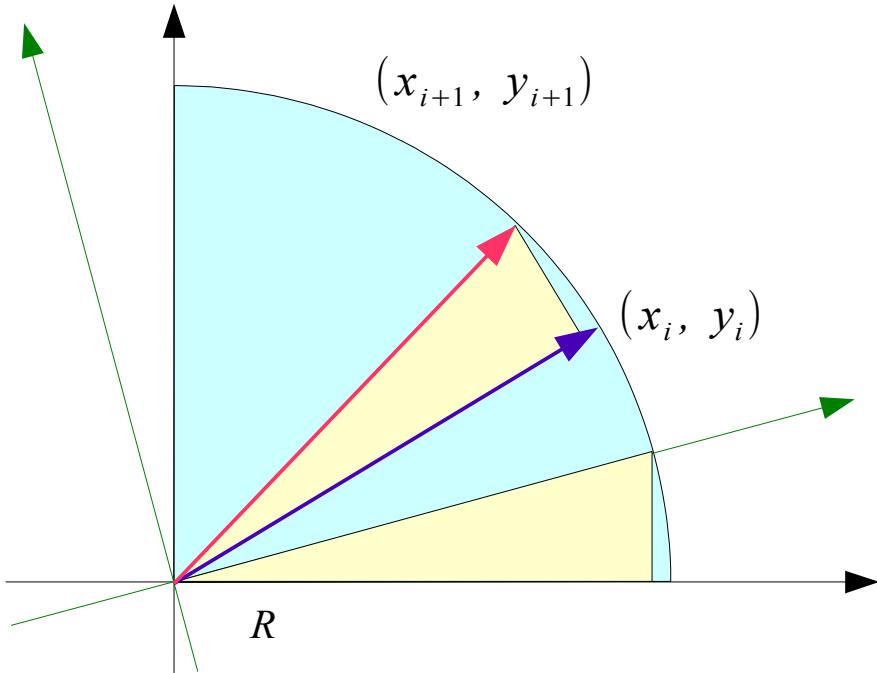
$$r : r \cos \theta = s : r$$

$$r \cos \theta : r \sin \theta = r : t$$

$$s = \frac{r}{\cos \theta}$$

$$s = \frac{r \sin \theta}{\cos \theta}$$

Pseudo-rotation (1)

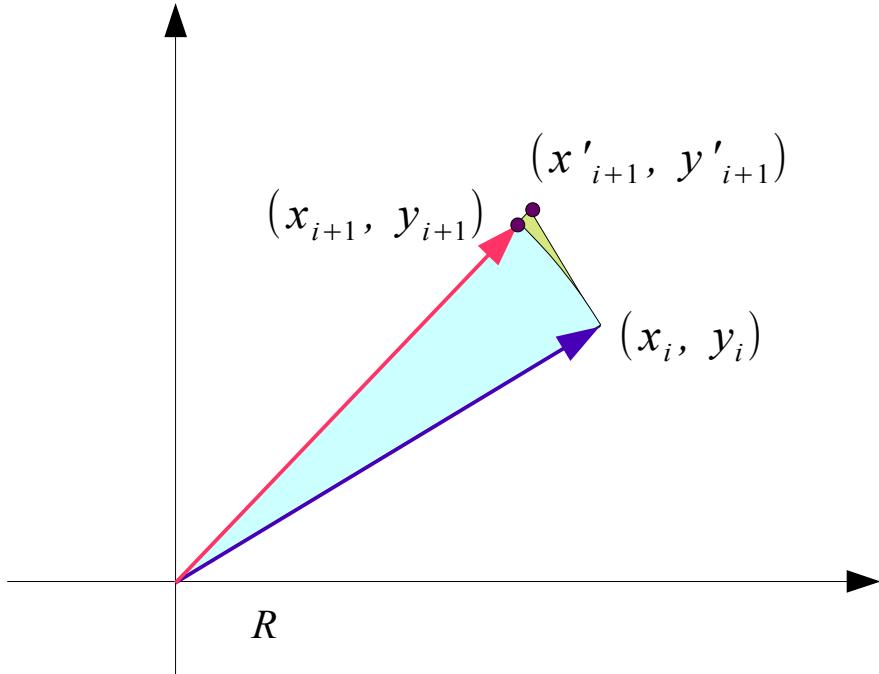


$$\begin{cases} x_{i+1} = \cos \alpha_i (x_i - y_i \tan \alpha_i) \\ y_{i+1} = \cos \alpha_i (x_i \tan \alpha_i + y_i) \end{cases}$$



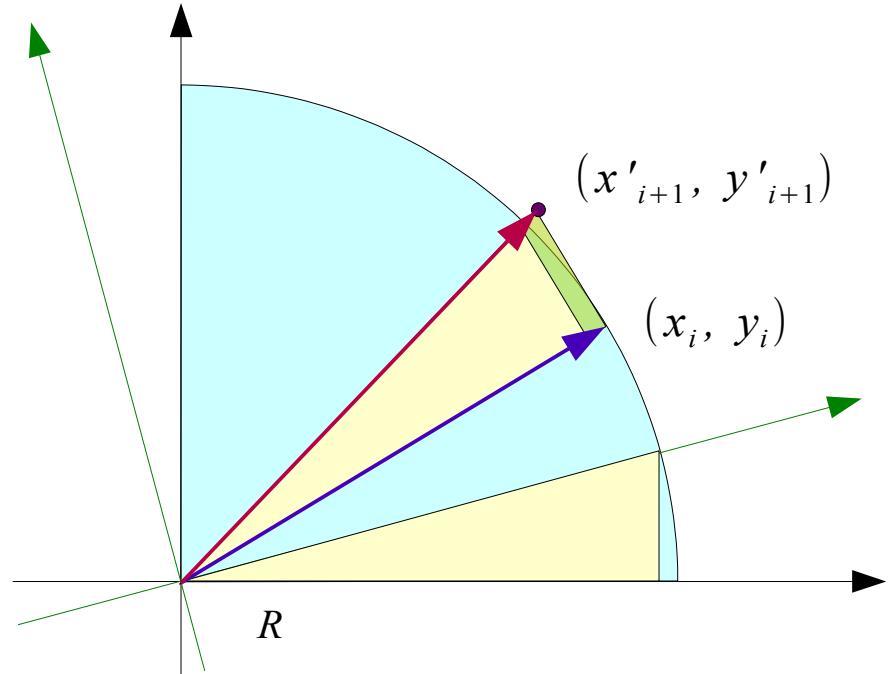
$$\begin{cases} x'_{i+1} = (x_i - y_i \tan \alpha_i) \\ y'_{i+1} = (x_i \tan \alpha_i + y_i) \end{cases}$$

Pseudo-rotation (2)



$$x'_{i+1} = x_{i+1} / \cos \alpha_i$$

$$y'_{i+1} = y_{i+1} / \cos \alpha_i$$



$$x'_{i+1} > x_{i+1}$$

$$y'_{i+1} > y_{i+1}$$

CORDIC Iteration Equations (1)

$$\alpha_0 \rightarrow \dots \rightarrow \alpha_i \rightarrow \alpha_{i+1} \rightarrow \dots \rightarrow \alpha_n \rightarrow 0$$

$$\begin{pmatrix} x_0 \\ y_0 \end{pmatrix} \rightarrow \dots \rightarrow \begin{pmatrix} x_i \\ y_i \end{pmatrix} \rightarrow \begin{pmatrix} x_{i+1} \\ y_{i+1} \end{pmatrix} \rightarrow \dots \rightarrow \begin{pmatrix} x_n \\ y_n \end{pmatrix}$$

$$\begin{aligned} x_{i+1} &= x_i \cos \alpha_i - y_i \sin \alpha_i & = \cos \alpha_i (x_i - y_i \tan \alpha_i) & = (1/\sqrt{1 + \tan^2 \alpha_i}) (x_i - y_i \tan \alpha_i) \\ y_{i+1} &= x_i \sin \alpha_i + y_i \cos \alpha_i & = \cos \alpha_i (x_i \tan \alpha_i + y_i) & = (1/\sqrt{1 + \tan^2 \alpha_i}) (x_i \tan \alpha_i + y_i) \end{aligned}$$

Pseudo-rotation

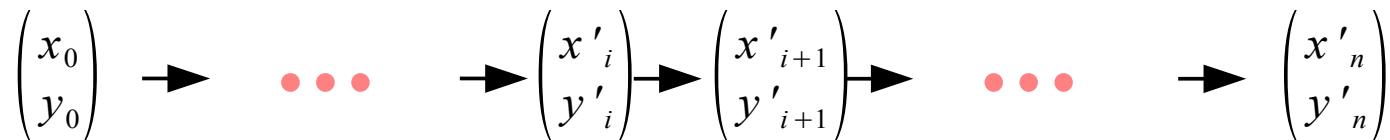
$$\begin{pmatrix} x_0 \\ y_0 \end{pmatrix} \rightarrow \dots \rightarrow \begin{pmatrix} x'_i \\ y'_i \end{pmatrix} \rightarrow \begin{pmatrix} x'_{i+1} \\ y'_{i+1} \end{pmatrix} \rightarrow \dots \rightarrow \begin{pmatrix} x'_n \\ y'_n \end{pmatrix}$$

$$\begin{aligned} x'_{i+1} &= (x'_i - y'_i \tan \alpha_i) & = (x'_i \cos \alpha_i - y'_i \sin \alpha_i) \sqrt{1 + \tan^2 \alpha_i} \\ y'_{i+1} &= (x'_i \tan \alpha_i + y'_i) & = (x'_i \sin \alpha_i + y'_i \cos \alpha_i) \sqrt{1 + \tan^2 \alpha_i} \end{aligned}$$

$$\begin{aligned} x'_n &= \{x_0 \cos(\sum \alpha_i) - y_0 \sin(\sum \alpha_i)\} \cdot \prod \{\sqrt{1 + \tan^2 \alpha_i}\} & \alpha_n &= \alpha - \sum \alpha_i \\ y'_n &= \{x_0 \sin(\sum \alpha_i) + y_0 \cos(\sum \alpha_i)\} \cdot \prod \{\sqrt{1 + \tan^2 \alpha_i}\} \end{aligned}$$

CORDIC Iteration Equations (2)

Pseudo-rotation



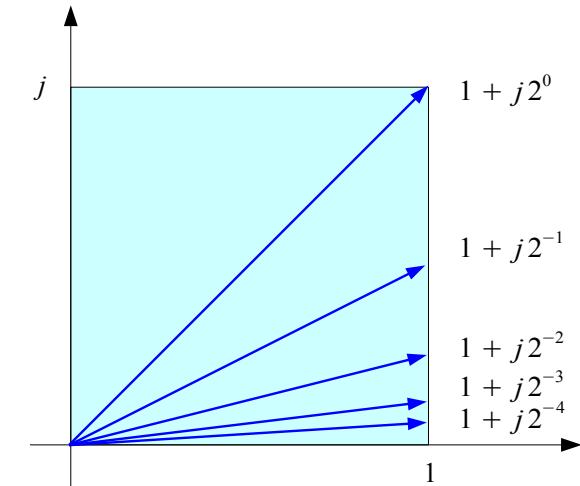
$$\begin{aligned} x'_{i+1} &= (x'_i - y'_i \tan \alpha_i) &= (x'_i \cos \alpha_i - y'_i \sin \alpha_i) \sqrt{1 + \tan^2 \alpha_i} \\ y'_{i+1} &= (x'_i \tan \alpha_i + y'_i) &= (x'_i \sin \alpha_i + y'_i \cos \alpha_i) \sqrt{1 + \tan^2 \alpha_i} \end{aligned}$$

$$\begin{aligned} x'_n &= \{x_0 \cos(\sum \alpha_i) - y_0 \sin(\sum \alpha_i)\} \cdot \prod \{\sqrt{1 + \tan^2 \alpha_i}\} & \alpha_n &= \alpha - \sum \alpha_i \\ y'_n &= \{x_0 \sin(\sum \alpha_i) + y_0 \cos(\sum \alpha_i)\} \cdot \prod \{\sqrt{1 + \tan^2 \alpha_i}\} \end{aligned}$$

Choose α_i such that $\tan \alpha_i = \begin{cases} +2^{-i} \\ -2^{-i} \end{cases}$

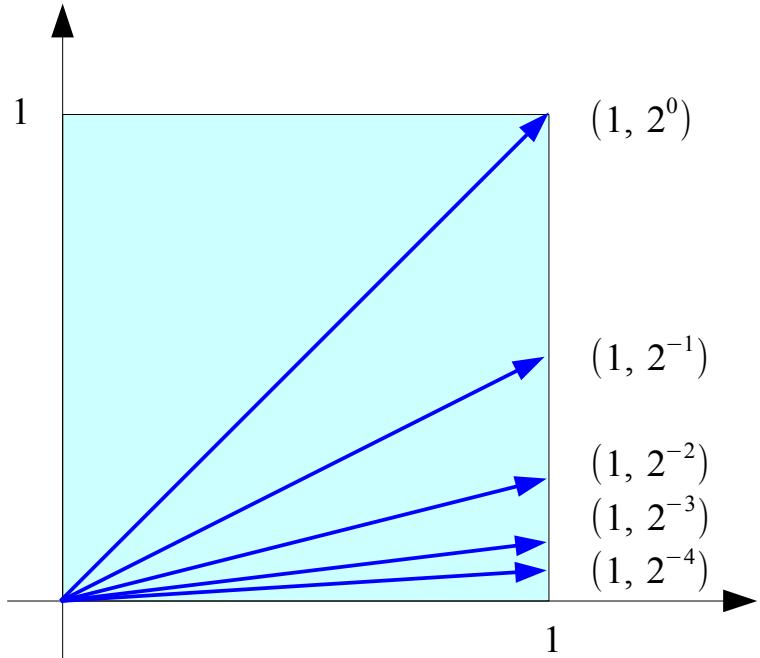
$$\tan \alpha_i = \sigma_i 2^{-i} \quad \sigma_i \in \{+1, -1\}$$

$$\begin{aligned} x'_{i+1} &= (x'_i - y'_i \sigma_i 2^{-i}) \\ y'_{i+1} &= (x'_i \sigma_i 2^{-i} + y'_i) \\ \alpha_{i+1} &= \alpha_i - \tan(\sigma_i 2^{-i}) \end{aligned}$$



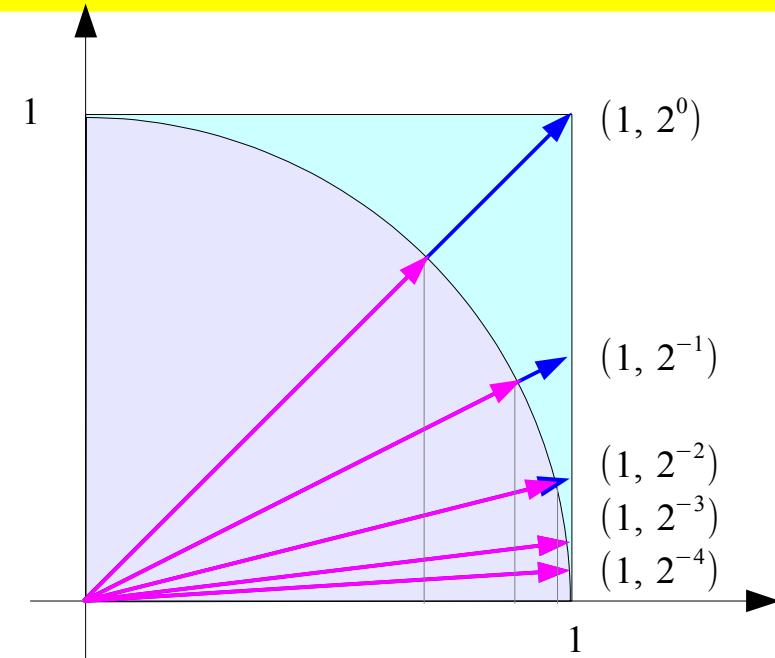
CORDIC Iteration Equations (2)

Choose α_i such that $\tan \alpha_i = \begin{cases} +2^{-i} \\ -2^{-i} \end{cases}$

$$\tan \alpha_i = \sigma_i 2^{-i} \quad \sigma_i \in \{+1, -1\}$$


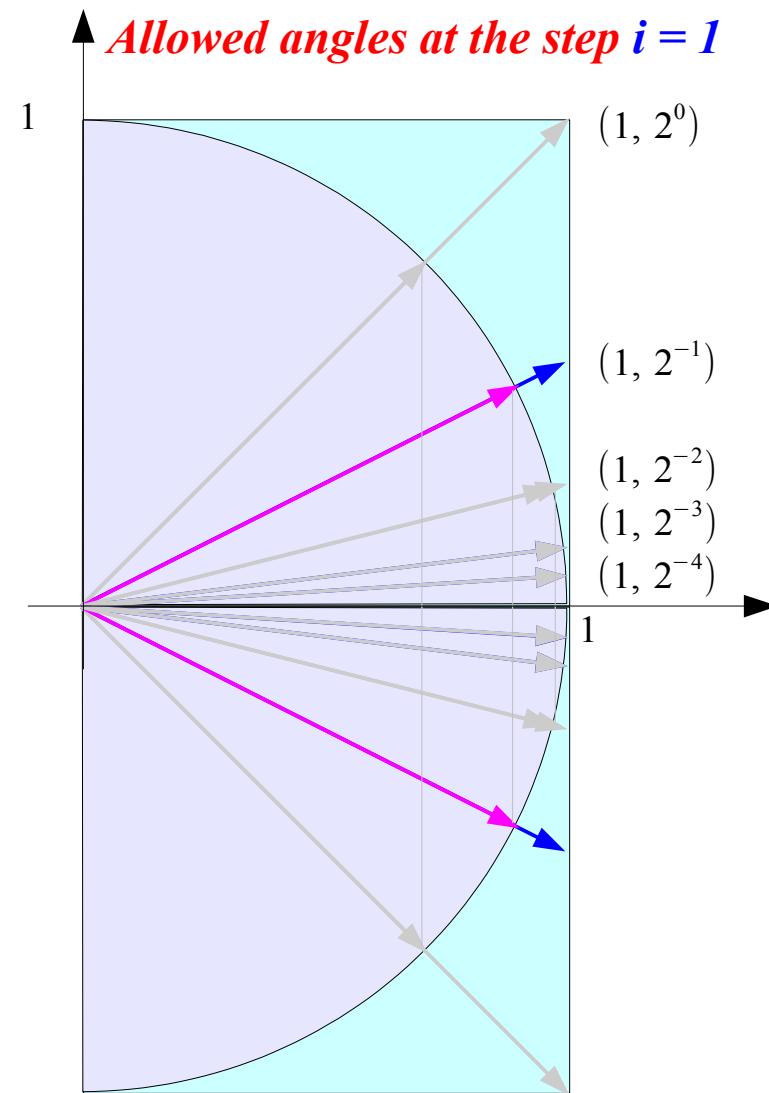
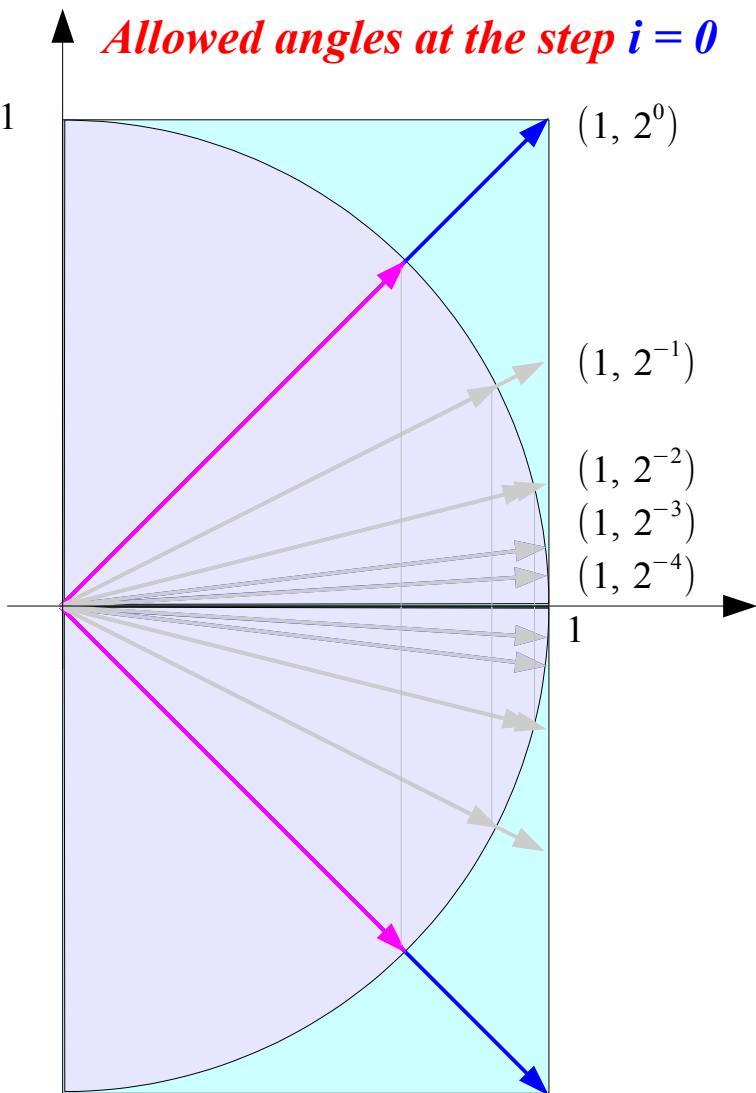
$$\tan \alpha_i = \pm 2^{-i} \quad \cos \alpha_i = \frac{+1}{\sqrt{1 + 2^{-2i}}} \\ \sin \alpha_i = \frac{\pm 2^{-i}}{\sqrt{1 + 2^{-2i}}}$$

$$x'_{i+1} = (x'_i - y'_i \sigma_i 2^{-i}) \\ y'_{i+1} = (x'_i \sigma_i 2^{-i} + y'_i) \\ \alpha_{i+1} = \alpha_i - \tan(\sigma_i 2^{-i})$$

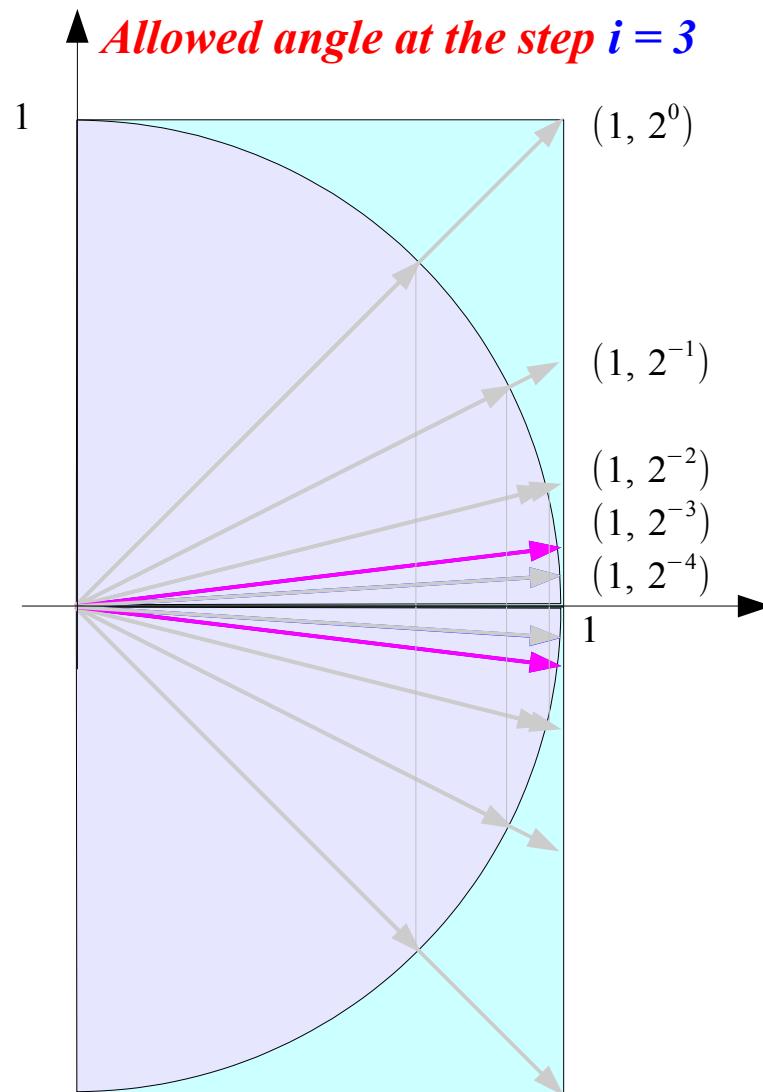
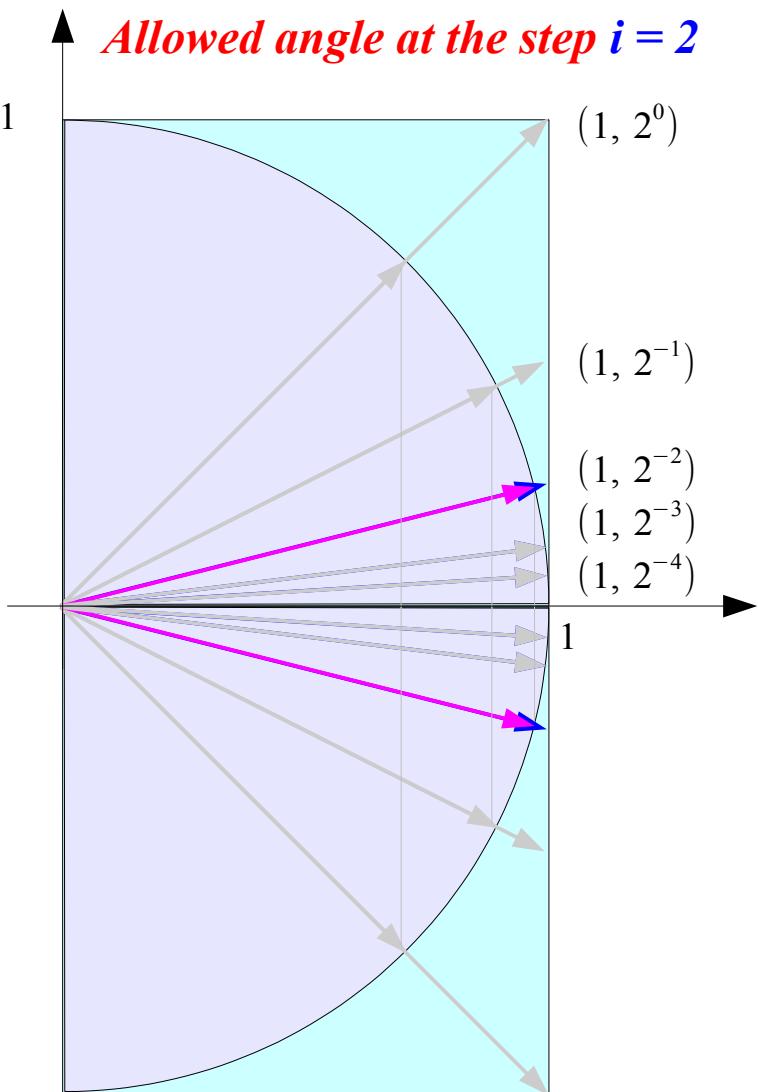


$$\begin{pmatrix} +\cos \alpha_i & -\sin \alpha_i \\ +\sin \alpha_i & +\cos \alpha_i \end{pmatrix} = \frac{1}{\sqrt{1 + 2^{-2i}}} \begin{pmatrix} +1 & \mp 2^{-i} \\ \pm 2^{-i} & +1 \end{pmatrix}$$

CORDIC Iteration Equations (3)



CORDIC Iteration Equations (4)



CORDIC Iteration Equations (2)

Choose α_i such that $\tan \alpha_i = \begin{cases} +2^{-i} \\ -2^{-i} \end{cases}$

$$\tan \alpha_i = \sigma_i 2^{-i} \quad \sigma_i \in \{+1, -1\}$$

$$\begin{aligned} x'_{i+1} &= (x'_i - y'_i \sigma_i 2^{-i}) \\ y'_{i+1} &= (x'_i \sigma_i 2^{-i} + y'_i) \\ \alpha_{i+1} &= \alpha_i - \tan(\sigma_i 2^{-i}) \end{aligned}$$

$$\begin{pmatrix} +\cos \alpha & -\sin \alpha \\ +\sin \alpha & +\cos \alpha \end{pmatrix}$$

$$\begin{aligned} &= \frac{1}{\sqrt{1 + 2^{-2 \cdot 0}}} \begin{pmatrix} +1 & \mp 2^{-0} \\ \pm 2^{-0} & +1 \end{pmatrix} \cdot \frac{1}{\sqrt{1 + 2^{-2 \cdot 1}}} \begin{pmatrix} +1 & \mp 2^{-1} \\ \pm 2^{-1} & +1 \end{pmatrix} \cdots \frac{1}{\sqrt{1 + 2^{-2 \cdot n}}} \begin{pmatrix} +1 & \mp 2^{-n} \\ \pm 2^{-n} & +1 \end{pmatrix} \\ &= \frac{1}{\sqrt{1 + 2^{-2 \cdot 0}}} \cdot \frac{1}{\sqrt{1 + 2^{-2 \cdot 1}}} \cdots \frac{1}{\sqrt{1 + 2^{-2 \cdot n}}} \begin{pmatrix} +1 & \mp 2^{-0} \\ \pm 2^{-0} & +1 \end{pmatrix} \begin{pmatrix} +1 & \mp 2^{-1} \\ \pm 2^{-1} & +1 \end{pmatrix} \cdots \begin{pmatrix} +1 & \mp 2^{-n} \\ \pm 2^{-n} & +1 \end{pmatrix} \end{aligned}$$

→ $1 / K = 0.607$

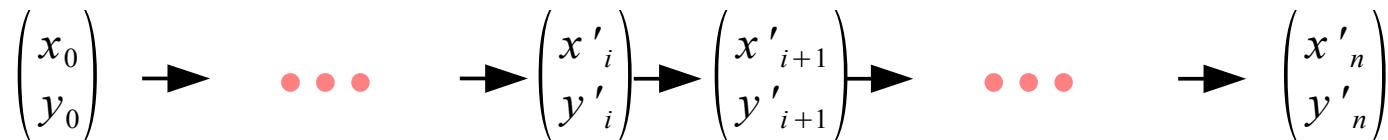
→ $\begin{pmatrix} +\cos(\sum \alpha_i) & -\sin(\sum \alpha_i) \\ +\sin(\sum \alpha_i) & +\cos(\sum \alpha_i) \end{pmatrix}$

$$\begin{pmatrix} +\cos(\sum \alpha_i) & -\sin(\sum \alpha_i) \\ +\sin(\sum \alpha_i) & +\cos(\sum \alpha_i) \end{pmatrix} = K \cdot \begin{pmatrix} +\cos \alpha & -\sin \alpha \\ +\sin \alpha & +\cos \alpha \end{pmatrix}$$

$$K = \prod \{\sqrt{1 + \tan^2 \alpha_i}\} = 1.647$$

CORDIC Iteration Equations (5)

Pseudo-rotation



$$\begin{aligned}x'_{i+1} &= (x'_i - y'_i \tan \alpha_i) &= (x'_i \cos \alpha_i - y'_i \sin \alpha_i) \sqrt{1 + \tan^2 \alpha_i} \\y'_{i+1} &= (x'_i \tan \alpha_i + y'_i) &= (x'_i \sin \alpha_i + y'_i \cos \alpha_i) \sqrt{1 + \tan^2 \alpha_i}\end{aligned}$$

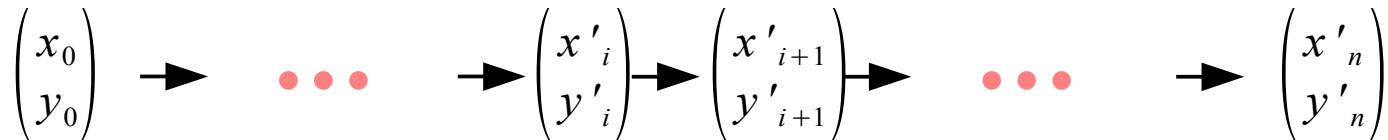
$$\begin{aligned}x'_n &= \{x_0 \cos(\sum \alpha_i) - y_0 \sin(\sum \alpha_i)\} \cdot \prod \{\sqrt{1 + \tan^2 \alpha_i}\} & \alpha_n &= \alpha - \sum \alpha_i \\y'_n &= \{x_0 \sin(\sum \alpha_i) + y_0 \cos(\sum \alpha_i)\} \cdot \prod \{\sqrt{1 + \tan^2 \alpha_i}\}\end{aligned}$$

Choose α_i such that $\tan \alpha_i = \sigma_i 2^{-i}$ $\sigma_i \in \{+1, -1\}$

$$\begin{aligned}x'_{i+1} &= (x'_i - y'_i \sigma_i 2^{-i}) & x'_n &= K(x_0 \cos \alpha - y_0 \sin \alpha) \\y'_{i+1} &= (x'_i \sigma_i 2^{-i} + y'_i) & y'_n &= K(x_0 \sin \alpha + y_0 \cos \alpha) \\ \alpha_{i+1} &= \alpha_i - \tan^{-1}(\sigma_i 2^{-i}) & \alpha_n &= \alpha - \sum \tan^{-1}(\sigma_i 2^{-i}) \\ K &= \prod \{\sqrt{1 + \tan^2 \alpha_i}\}\end{aligned}$$

CORDIC Iteration Equations (4)

Pseudo-rotation



Choose α_i such that $\tan \alpha_i = \sigma_i 2^{-i}$ $\sigma_i \in \{+1, -1\}$

$$x'_{i+1} = (x'_i - y'_i \sigma_i 2^{-i})$$

$$y'_{i+1} = (x'_i \sigma_i 2^{-i} + y'_i)$$

$$\alpha_{i+1} = \alpha_i - \tan^{-1}(\sigma_i 2^{-i})$$

$$x'_n = K(x_0 \cos \alpha - y_0 \sin \alpha)$$

$$y'_n = K(x_0 \sin \alpha + y_0 \cos \alpha)$$

$$\alpha_n = \alpha - \sum \tan^{-1}(\sigma_i 2^{-i})$$

pre-compute $K = \prod \{\sqrt{1 + \tan^2 \alpha_i}\} = 1.647$

set $x_0 = 1 / K = 0.607$

$$y_0 = 0$$

then $x'_n = K(1/K \cos \alpha - 0 \cdot \sin \alpha) = \cos \alpha$

$$y'_n = K(1/K \sin \alpha + 0 \cdot \cos \alpha) = \sin \alpha$$

References

- [1] <http://en.wikipedia.org/>
- [2] CORDIC FAQ, www.dspguru.com
- [3] R. Andraka, A survey of CORDIC algorithms for FPGA based computers
- [4] J. S. Walther, A Unified Algorithm for Elementary Functions
- [5] J. P. Deschamps, G. A. Bioul, G.D. Sutter, Synthesis of Arithmetic Circuits