

Spectra (1A)

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Single-Sided Spectrum

$$x(t) = a_0 + \sum_{n=1}^{\infty} (a_n \cos(n\omega_0 t) + b_n \sin(n\omega_0 t))$$

$$a_0 = \frac{1}{T} \int_0^T x(t) dt$$

$$a_n = \frac{2}{T} \int_0^T x(t) \cos(n\omega_0 t) dt$$

$$b_n = \frac{2}{T} \int_0^T x(t) \sin(n\omega_0 t) dt$$

$$n = 1, 2, \dots$$

$$x(t) = g_0 + \sum_{n=1}^{\infty} g_n \cos(n\omega_0 t + \phi_n)$$

$$g_0 = a_0$$

$$g_n = \sqrt{a_n^2 + b_n^2}$$

$$\phi_n = \tan^{-1} \left(-\frac{b_n}{a_n} \right)$$

$$n = 1, 2, \dots$$

$$\cos(\alpha + \beta) = \cos(\alpha) \cos(\beta) - \sin(\alpha) \sin(\beta)$$

$$g_n \cos(n\omega_0 t + \phi_n) = g_n \cos(\phi_n) \cos(n\omega_0 t) - g_n \sin(\phi_n) \sin(n\omega_0 t)$$

Two-Sided Spectrum

$$x(t) = a_0 + \sum_{n=1}^{\infty} (a_n \cos(n\omega_0 t) + b_n \sin(n\omega_0 t))$$

$$a_0 = \frac{1}{T} \int_0^T x(t) dt$$

$$a_n = \frac{2}{T} \int_0^T x(t) \cos(n\omega_0 t) dt$$

$$b_n = \frac{2}{T} \int_0^T x(t) \sin(n\omega_0 t) dt$$

$$n = 1, 2, \dots$$

$$x(t) = A_0 + \sum_{n=1}^{\infty} (A_n e^{jn\omega_0 t} + B_n e^{-jn\omega_0 t})$$

$$A_0 = a_0$$

$$A_n = \frac{1}{2} (a_n - j b_n)$$

$$B_n = \frac{1}{2} (a_n + j b_n)$$

$$x(t) = \sum_{n=-\infty}^{+\infty} C_n e^{jn\omega_0 t}$$

$$C_n = \frac{1}{T} \int_0^T x(t) e^{-jn\omega_0 t} dt$$

$$n = -2, -1, 0, +1, +2, \dots$$

$$C_n = \begin{cases} A_0 & (n = 0) \\ A_n & (n > 0) \\ B_n & (n < 0) \end{cases}$$

$$|C_n| = \frac{A_n}{2} \quad (n \neq 0)$$

$$\text{Arg}(C_n) = \begin{cases} +\phi_n = \tan^{-1} \left(-\frac{b_n}{a_n} \right) & (n > 0) \\ -\phi_n = \tan^{-1} \left(+\frac{b_n}{a_n} \right) & (n < 0) \end{cases}$$

Periodogram

$$x(t) = a_0 + \sum_{n=1}^{\infty} (a_n \cos(n\omega_0 t) + b_n \sin(n\omega_0 t))$$

$$a_0 = \frac{1}{T} \int_0^T x(t) dt$$

$$a_n = \frac{2}{T} \int_0^T x(t) \cos(n\omega_0 t) dt$$

$$b_n = \frac{2}{T} \int_0^T x(t) \sin(n\omega_0 t) dt$$

$$n = 1, 2, \dots$$

$$a_n = \frac{2}{T} \int_0^T x(t) \cos(n\omega_0 t) dt$$

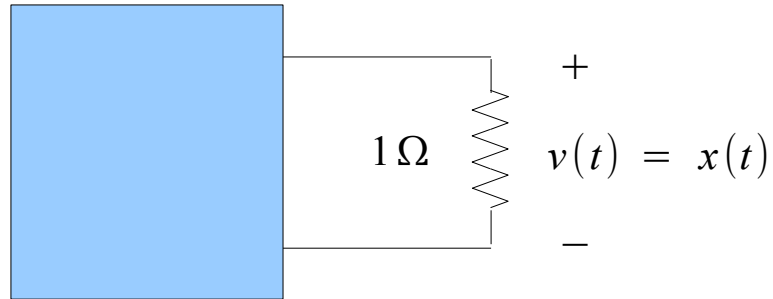
Positive Frequency Only

$$f = \frac{n}{T} \quad n = 1, 2, \dots$$

Single-sided Spectrum

$$a_n^2 + b_n^2$$

Constant Periodic Signal



instantaneous power $x^2(t)$

average power $\frac{1}{T} \int_0^T x^2(t) dt$

T : period

$v(t) = x(t)$ Continuous Periodic

➔ CTFS (Fourier Series)

Parseval's Theorem

$$x(t) = \sum_{n=-\infty}^{+\infty} C_n e^{+jn\omega_0 t}$$

$$C_n = \frac{1}{T} \int_0^T x(t) e^{-jn\omega_0 t} dt$$

$$n = -2, -1, 0, +1, +2, \dots$$

$$\frac{1}{T} \int_0^T x^2(t) dt = \sum_{n=-\infty}^{+\infty} |C_n|^2$$

average power

sum of
power spectrum C_n

CTFS and CTFT

Continuous Time Fourier Series

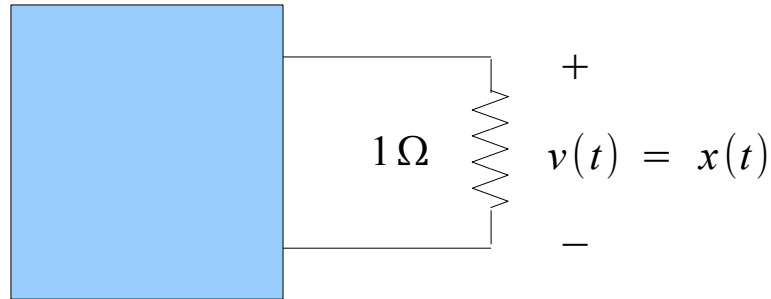
$$C_n = \frac{1}{T} \int_0^T x(t) e^{-jn\omega_0 t} dt \quad \longleftrightarrow \quad x(t) = \sum_{n=-\infty}^{\infty} C_n e^{jn\omega_0 t}$$

Continuous Time Fourier Transform

$$X(j\omega) = \int_{-\infty}^{+\infty} x(t) e^{-j\omega t} dt \quad \longleftrightarrow \quad x(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} X(j\omega) e^{+j\omega t} d\omega$$

$$X(f) = \int_{-\infty}^{+\infty} x(t) e^{-j2\pi f t} dt \quad \longleftrightarrow \quad x(t) = \int_{-\infty}^{+\infty} X(f) e^{+j2\pi f t} df$$

Constant Aperiodic Signal



instantaneous power

$$x^2(t)$$

total energy

$$\int_0^T x^2(t) dt$$

T : period

$$v(t) = x(t)$$

Continuous Aperiodic



CTFT (Fourier Integral)

Parseval's Theorem

$$x(t) = \int_{-\infty}^{+\infty} X(f) e^{+j2\pi f t} df$$

$$X(f) = \int_{-\infty}^{+\infty} x(t) e^{-j2\pi f t} dt$$

$$\int_0^T x^2(t) dt = \int_{-\infty}^{\infty} |X(f)|^2 df$$

total energy

integral of

energy spectral

density $|X(f)|^2$

Periodogram

A truncated sample function

$$\begin{aligned}x_T(t) &= x(t) \quad -T/2 < t < +T/2 \\ &= 0 \quad \textit{otherwise}\end{aligned}$$

Fourier Transform

$$\begin{aligned}X_T(\omega) &= \int_{-\infty}^{+\infty} x_T(t) e^{-j\omega t} dt & x_T(t) &= \frac{1}{2\pi} \int_{-\infty}^{+\infty} X_T(\omega) e^{+j\omega t} d\omega \\ X_T(f) &= \int_{-\infty}^{+\infty} x_T(t) e^{-j2\pi f t} dt & x_T(t) &= \int_{-\infty}^{+\infty} X_T(f) e^{+j2\pi f t} df\end{aligned}$$

Parseval's Theorem

$$\int_{-\infty}^{+\infty} |x_T(t)|^2 dt = \int_{-\infty}^{+\infty} |X_T(f)|^2 df$$

Average Power

$$\frac{1}{T} \int_{-\infty}^{+\infty} x_T^2(t) dt = \int_{-\infty}^{+\infty} \frac{|X_T(f)|^2}{T} df$$

Average Power

A truncated sample function

$$\begin{aligned}x_T(t) &= x(t) \quad -T/2 < t < +T/2 \\ &= 0 \quad \textit{otherwise}\end{aligned}$$

Fourier Transform

$$X_T(\omega) = \int_{-\infty}^{+\infty} x_T(t) e^{-j\omega t} dt$$

$$x_T(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} X_T(\omega) e^{+j\omega t} d\omega$$

$$X_T(f) = \int_{-\infty}^{+\infty} x_T(t) e^{-j2\pi ft} dt$$

$$x_T(t) = \int_{-\infty}^{+\infty} X_T(f) e^{+j2\pi ft} df$$

Parseval's Theorem

$$\int_{-\infty}^{+\infty} |x_T(t)|^2 dt = \int_{-\infty}^{+\infty} |X_T(f)|^2 df$$

Average Power

$$\frac{1}{T} \int_{-\infty}^{+\infty} x_T^2(t) dt = \int_{-\infty}^{+\infty} \frac{|X_T(f)|^2}{T} df$$

Raw Power Spectral Density

Average Power

$$\frac{1}{T} \int_{-\infty}^{+\infty} x_T^2(t) dt = \int_{-\infty}^{+\infty} \frac{|X_T(f)|^2}{T} df \Rightarrow \int_{-\infty}^{+\infty} \hat{S}_{xx}(f) df$$

Raw Power Spectral Density

$$\frac{|X_T(f)|^2}{T} = \hat{S}_{xx}(f)$$

$$\lim_{T \rightarrow \infty} \frac{1}{T} \int_{-\infty}^{+\infty} x_T^2(t) dt = \lim_{T \rightarrow \infty} \int_{-\infty}^{+\infty} \frac{|X_T(f)|^2}{T} df$$

$$E \left[\lim_{T \rightarrow \infty} \frac{1}{T} \int_{-\infty}^{+\infty} x_T^2(t) dt \right] = E \left[\lim_{T \rightarrow \infty} \int_{-\infty}^{+\infty} \frac{|X_T(f)|^2}{T} df \right]$$

Power Spectral Density

Average Power

$$\frac{1}{T} \int_{-\infty}^{+\infty} x_T^2(t) dt = \int_{-\infty}^{+\infty} \frac{|X_T(f)|^2}{T} df \Rightarrow \int_{-\infty}^{+\infty} \hat{S}_{xx}(f) df$$

Raw Power Spectral Density

$$\frac{|X_T(f)|^2}{T} = \hat{S}_{xx}(f)$$

$$E \left[\lim_{T \rightarrow \infty} \frac{1}{T} \int_{-\infty}^{+\infty} x_T^2(t) dt \right] = E \left[\lim_{T \rightarrow \infty} \int_{-\infty}^{+\infty} \frac{|X_T(f)|^2}{T} df \right]$$

$$\text{Var}(x(t)) = \sigma_x^2 = \lim_{T \rightarrow \infty} \int_{-\infty}^{+\infty} \frac{E[|X_T(f)|^2]}{T} df \Rightarrow \int_{-\infty}^{+\infty} \hat{S}_{xx}(f) df$$

Power Spectral Density

$$\lim_{T \rightarrow \infty} \frac{E[|X_T(f)|^2]}{T} = S_{xx}(f)$$

Power and Power Density Spectra

Average Power

$$E[x^2(t)] = \frac{1}{T} \int_{-\infty}^{+\infty} x_T^2(t) dt$$

Average Power of N sample of x(t)

$$\frac{1}{N} \sum_{n=0}^{N-1} x_n^2 \approx \frac{1}{N \Delta t} \int_0^{N \Delta t} x^2(t) dt = \frac{1}{N^2} \sum_{m=0}^{N-1} |X_m|^2$$

$$\sum_{n=0}^{N-1} x_n^2 = \frac{1}{N} \sum_{m=0}^{N-1} |X_m|^2 \quad m = 0, 1, \dots, N-1$$

Periodogram:

$$P_{xx}(m) = \frac{1}{N} |X_m|^2$$
$$m = 0, 1, \dots, N-1$$

Average Power

$$\sum_{n=0}^{N-1} x_n^2 = \frac{1}{N} \sum_{m=0}^{N-1} P_{xx}(m)$$

Total Energy

$$T \sum_{n=0}^{N-1} x_n^2 = T \sum_{m=0}^{N-1} P_{xx}(m)$$
$$\approx \int_0^{N \Delta t} x^2(t) dt$$

References

- [1] <http://en.wikipedia.org/>
- [2] J.H. McClellan, et al., Signal Processing First, Pearson Prentice Hall, 2003
- [3] K.H. Shin, J.K. Hammond, Fundamentals of Signal Processing for Sound and Vibration Engineers, Wiley, 2008