

Matrices (3A)

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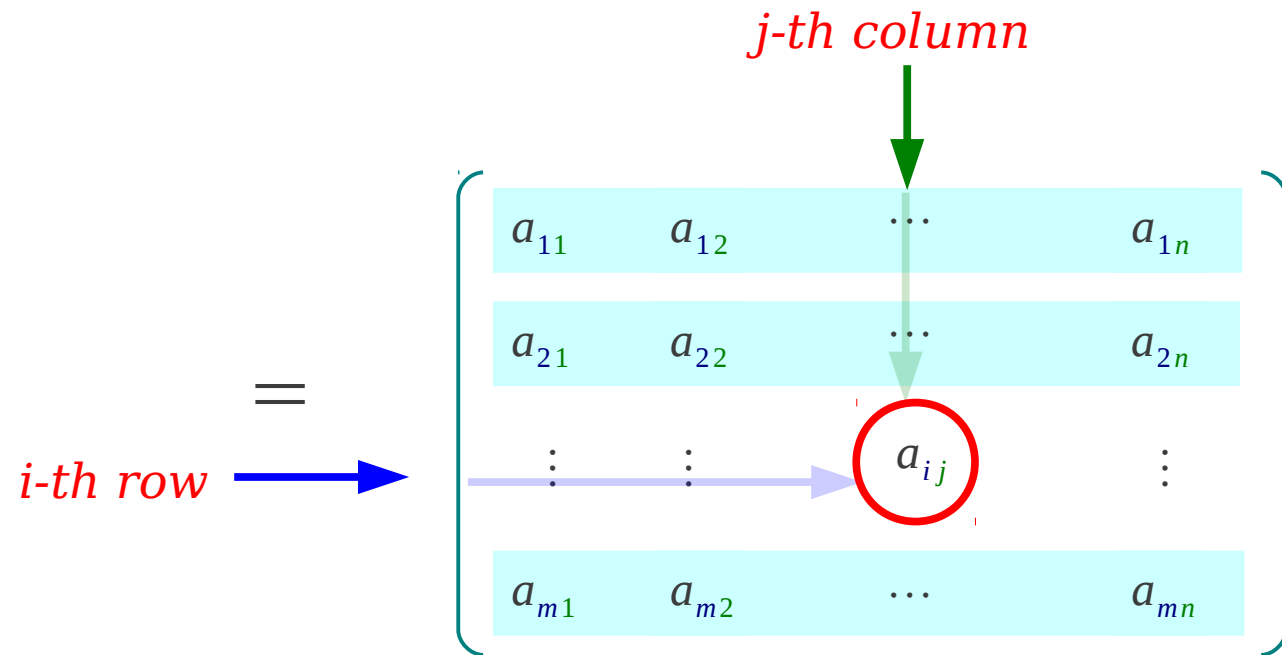
A General $m \times n$ Matrix

$$A = [a_{ij}]_{m \times n} = [a_{ij}]$$

$$= \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}$$

A Element of a Matrix

$$A = [a_{ij}]_{m \times n} = [a_{ij}]$$



$$(A)_{m \times n} = a_{ij}$$

Matrix Multiplication (1)

1st row →

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix}$$

2nd row →

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix}$$

m-th row →

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix}$$

$$\mathbf{A} = [a_{ij}]_{m \times n}$$

$$\mathbf{x} = [x_{ij}]_{n \times 1}$$

$$\mathbf{b} = [b_{ij}]_{m \times 1}$$

$$\mathbf{A} \quad \mathbf{x} = \mathbf{b}$$

$$(m) \times n \quad n \times (1) \quad (m \times 1)$$

Matrix Multiplication (2)

1st row →

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \\ \vdots & \vdots \\ b_{n1} & b_{n2} \end{pmatrix} = \begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \\ \vdots & \vdots \\ c_{m1} & c_{m2} \end{pmatrix}$$

↓ 1st column

$$A = [a_{ij}]_{m \times n}$$

$$B = [b_{ij}]_{n \times 2}$$

$$C = [c_{ij}]_{m \times 2}$$

2nd row →

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \\ \vdots & \vdots \\ b_{n1} & b_{n2} \end{pmatrix} = \begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \\ \vdots & \vdots \\ c_{m1} & c_{m2} \end{pmatrix}$$

↓ 1st column

$$A \quad B = C$$

$$(m \times n) \quad (n \times 2) \quad (m \times 2)$$

Matrix Multiplication (3)

1st row →

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \begin{pmatrix} b_{11} \\ b_{12} \\ b_{21} \\ b_{22} \\ \vdots \\ \vdots \\ b_{n1} \\ b_{n2} \end{pmatrix} = \begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \\ \vdots & \vdots \\ c_{m1} & c_{m2} \end{pmatrix}$$

2nd column

$$\mathbf{A} = [a_{ij}]_{m \times n}$$

$$\mathbf{B} = [b_{ij}]_{n \times 2}$$

$$\mathbf{C} = [c_{ij}]_{m \times 2}$$

2nd row →

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \begin{pmatrix} b_{11} \\ b_{12} \\ b_{21} \\ b_{22} \\ \vdots \\ \vdots \\ b_{n1} \\ b_{n2} \end{pmatrix} = \begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \\ \vdots & \vdots \\ c_{m1} & c_{m2} \end{pmatrix}$$

2nd column

$$\mathbf{A} \mathbf{B} = \mathbf{C}$$

$$(m \times n) (n \times 2) = (m \times 2)$$

Matrix Multiplication (4)

m-th row →

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \\ \vdots & \vdots \\ b_{n1} & b_{n2} \end{pmatrix} = \begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \\ \vdots & \vdots \\ c_{m1} & c_{m2} \end{pmatrix}$$

1st column

$$A = [a_{ij}]_{m \times n}$$

$$B = [b_{ij}]_{n \times 2}$$

$$C = [c_{ij}]_{m \times 2}$$

m-th row →

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \\ \vdots & \vdots \\ b_{n1} & b_{n2} \end{pmatrix} = \begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \\ \vdots & \vdots \\ c_{m1} & c_{m2} \end{pmatrix}$$

2nd column

$$A \quad B = C$$

$$(m \times n) \quad (n \times 2) \quad (m \times 2)$$

Matrix Multiplication (4)

m-th row →

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \\ \vdots & \vdots \\ b_{n1} & b_{n2} \end{pmatrix} = \begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \\ \vdots & \vdots \\ c_{m1} & c_{m2} \end{pmatrix}$$

1st column

$$\mathbf{A} = [a_{ij}]_{m \times n}$$

$$\mathbf{B} = [b_{ij}]_{n \times 2}$$

$$\mathbf{C} = [c_{ij}]_{m \times 2}$$

m-th row →

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \\ \vdots & \vdots \\ b_{n1} & b_{n2} \end{pmatrix} = \begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \\ \vdots & \vdots \\ c_{m1} & c_{m2} \end{pmatrix}$$

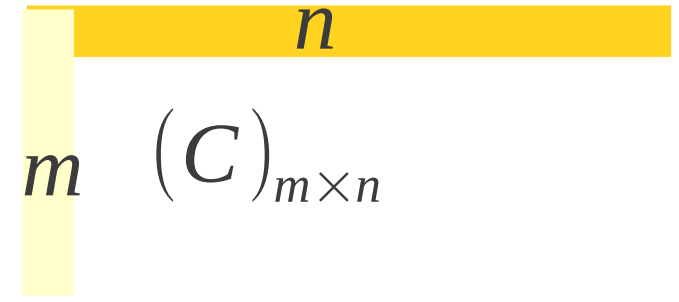
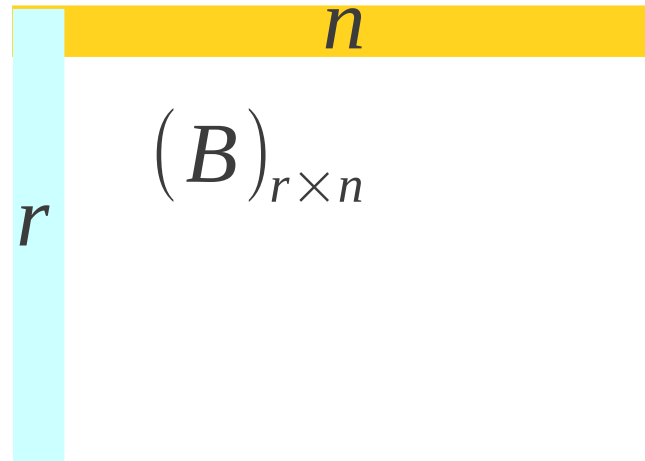
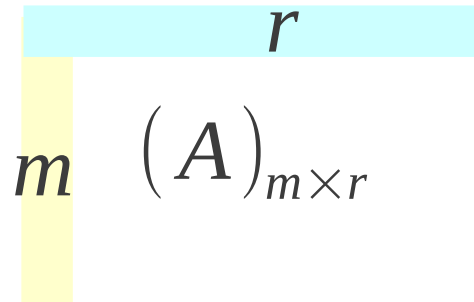
2nd column

$$\mathbf{A} \mathbf{B} = \mathbf{C}$$

$$(m \times n) (n \times 2) = (m \times 2)$$

Multiplication of Matrices (1)

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1r} \\ a_{21} & a_{22} & \cdots & a_{2r} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mr} \end{pmatrix} \begin{pmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2n} \\ \vdots & \vdots & & \vdots \\ b_{r1} & b_{r2} & \cdots & b_{rn} \end{pmatrix} = \begin{pmatrix} c_{11} & c_{12} & \cdots & c_{1n} \\ c_{21} & c_{22} & \cdots & c_{2n} \\ \vdots & \vdots & \boxed{c_{ii}} & \vdots \\ c_{r1} & c_{r2} & \cdots & c_{rn} \end{pmatrix}$$



Multiplication of Matrices (2)

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1r} \\ a_{21} & a_{22} & \cdots & a_{2r} \\ \vdots & \vdots & \ddots & \vdots \\ a_{i1} & a_{i2} & \cdots & a_{ir} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mr} \end{pmatrix} \begin{pmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ b_{r1} & b_{r2} & \cdots & b_{rn} \end{pmatrix} = \begin{pmatrix} c_{11} & c_{12} & \cdots & c_{1n} \\ c_{21} & c_{22} & \cdots & c_{2n} \\ \vdots & \vdots & \boxed{c_{ij}} & \vdots \\ c_{r1} & c_{r2} & \cdots & c_{rn} \end{pmatrix}$$

$$(AB)_{ij} = c_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + a_{i3}b_{3j} \cdots + a_{ir}b_{rj}$$

$$= \sum_{k=1}^r a_{ik}b_{kj}$$

Partitioned Matrix

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \\ \vdots & \vdots \\ b_{n1} & b_{n2} \end{pmatrix} = \begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \\ \vdots & \vdots \\ c_{m1} & c_{m2} \end{pmatrix}$$

$$\mathbf{A} = [a_{ij}]_{m \times n}$$

$$\mathbf{B} = [b_{ij}]_{n \times 2}$$

$$\mathbf{C} = [c_{ij}]_{m \times 2}$$

$$\begin{matrix} \mathbf{B} \\ n \times 2 \end{matrix} = \begin{bmatrix} \mathbf{b}_1 & \mathbf{b}_2 \end{bmatrix} \begin{matrix} n \times 1 & n \times 1 \end{matrix}$$

$$\begin{matrix} \mathbf{A} \mathbf{B} \\ m \times 2 \end{matrix} = \begin{bmatrix} \mathbf{A} \mathbf{b}_1 & \mathbf{A} \mathbf{b}_2 \end{bmatrix} \begin{matrix} m \times 1 & m \times 1 \end{matrix}$$

$$\begin{matrix} m \times n & n \times 2 & m \times n & n \times 1 & m \times n & n \times 1 \end{matrix}$$

Partitioned Matrix

$$\begin{array}{l}
 \mathbf{a}_1 \rightarrow \\
 \mathbf{a}_2 \rightarrow \\
 \mathbf{a}_m \rightarrow
 \end{array}
 \begin{pmatrix}
 a_{11} & a_{12} & \cdots & a_{1n} \\
 a_{21} & a_{22} & \cdots & a_{2n} \\
 \vdots & \vdots & & \vdots \\
 a_{m1} & a_{m2} & \cdots & a_{mn}
 \end{pmatrix}
 \begin{pmatrix}
 b_{11} & b_{12} \\
 b_{21} & b_{22} \\
 \vdots & \vdots \\
 b_{n1} & b_{n2}
 \end{pmatrix}
 =
 \begin{pmatrix}
 c_{11} & c_{12} \\
 c_{21} & c_{22} \\
 \vdots & \vdots \\
 c_{m1} & c_{m2}
 \end{pmatrix}$$

$$\mathbf{A} = [a_{ij}]_{m \times n}$$

$$\mathbf{B} = [b_{ij}]_{n \times 2}$$

$$\mathbf{C} = [c_{ij}]_{m \times 2}$$

$$\begin{array}{l}
 \mathbf{A} \\
 m \times n
 \end{array}
 =
 \begin{bmatrix}
 \mathbf{a}_1 \\
 \mathbf{a}_2 \\
 \vdots \\
 \mathbf{a}_m
 \end{bmatrix}
 \begin{array}{l}
 1 \times n \\
 1 \times n \\
 \\
 1 \times n
 \end{array}
 \quad
 \begin{array}{l}
 \mathbf{A} \mathbf{B} \\
 m \times 2 \\
 m \times n \quad n \times 2
 \end{array}
 =
 \begin{bmatrix}
 \mathbf{a}_1 \\
 \mathbf{a}_2 \\
 \vdots \\
 \mathbf{a}_m
 \end{bmatrix}
 \begin{array}{l}
 1 \times 2 \quad 1 \times n \quad n \times 2 \\
 1 \times 2 \quad 1 \times n \quad n \times 2 \\
 \\
 1 \times 2 \quad 1 \times n \quad n \times 2
 \end{array}$$

Determinant

Determinant of order 2

$$\begin{bmatrix} a_1 & a_2 \\ b_1 & b_2 \end{bmatrix} \quad \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} = a_1 b_2 - a_2 b_1$$

Determinant of order 3

$$\begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix} \quad \begin{bmatrix} + & - & + \\ - & + & - \\ + & - & + \end{bmatrix}$$

$$\begin{bmatrix} a_1 & & \\ & b_2 & b_3 \\ & c_2 & c_3 \end{bmatrix} \quad \begin{bmatrix} & a_2 & \\ b_1 & & b_3 \\ c_1 & & c_3 \end{bmatrix} \quad \begin{bmatrix} & & a_3 \\ b_1 & b_2 & \\ c_1 & c_2 & \end{bmatrix}$$

Determinant

Determinant of order 3

$$\begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix} \quad \begin{bmatrix} + & - & + \\ - & + & - \\ + & - & + \end{bmatrix}$$

$$\begin{bmatrix} a_1 & & \\ b_2 & b_3 & \\ c_2 & c_3 & \end{bmatrix} \quad \begin{bmatrix} & a_2 & \\ b_1 & & b_3 \\ c_1 & & c_3 \end{bmatrix} \quad \begin{bmatrix} & & a_3 \\ b_1 & b_2 & \\ c_1 & c_2 & \end{bmatrix}$$

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = + a_1 \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - a_2 \begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + a_3 \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}$$

Minor

The **minor** of entry a_{ij} M_{ij}

The determinant of the submatrix
that remains after deleting i -th row and j -th column

$$= (a_2 b_3 - a_3 b_2)\mathbf{i} - (a_1 b_3 - a_3 b_1)\mathbf{j} + (a_1 b_2 - a_2 b_1)\mathbf{k}$$

Minor

The **minor** of entry a_{ij}

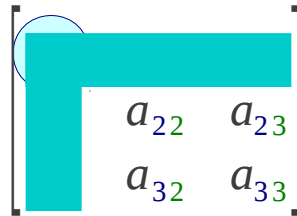
$$M_{ij}$$

The determinant of the submatrix
that remains after **deleting** i -th row and j -th column

Given matrix

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$n \times n$$


$$\begin{bmatrix} & & \\ & a_{22} & a_{23} \\ & a_{32} & a_{33} \end{bmatrix}$$

Sub-matrix

$$\begin{bmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{bmatrix}$$

$$(n-1) \times (n-1)$$

Minor

$$M_{11} = \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}$$

Cofactor

The **cofactor** of entry a_{ij}

$$C_{ij} = (-1)^{i+j} M_{ij}$$

The **minor** of entry a_{ij}

$$M_{ij}$$

The determinant of the submatrix that remains after **deleting** i -th row and j -th column

Given matrix

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$n \times n$

$$\begin{bmatrix} & & \\ & a_{22} & a_{23} \\ & a_{32} & a_{33} \end{bmatrix}$$

Sub-matrix

$$\begin{bmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{bmatrix}$$

$(n-1) \times (n-1)$

Minor

$$M_{11} = \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}$$

Cofactor

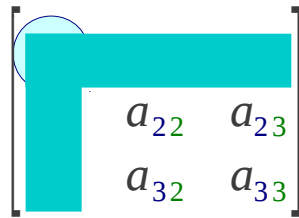
$$C_{11} = (-1)^{1+1} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}$$

$$\begin{bmatrix} +1 & -1 & +1 \\ -1 & +1 & -1 \\ +1 & -1 & +1 \end{bmatrix} \quad \begin{bmatrix} (-1)^{1+1} & (-1)^{1+2} & (-1)^{1+3} \\ (-1)^{2+1} & (-1)^{2+2} & (-1)^{2+3} \\ (-1)^{3+1} & (-1)^{3+2} & (-1)^{3+3} \end{bmatrix} = \begin{bmatrix} (-1)^2 & (-1)^3 & (-1)^4 \\ (-1)^3 & (-1)^4 & (-1)^5 \\ (-1)^4 & (-1)^5 & (-1)^6 \end{bmatrix}$$

Minor Example (1)

Given matrix

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$


$$\begin{bmatrix} & & & \\ & a_{22} & a_{23} & \\ & a_{32} & a_{33} & \end{bmatrix}$$

Sub-matrix

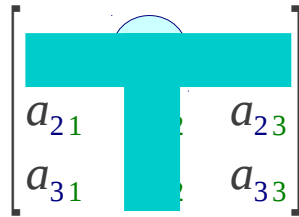
$$\begin{bmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{bmatrix}$$

Minor

$$M_{11} = \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}$$

$$C_{11} = +M_{11}$$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

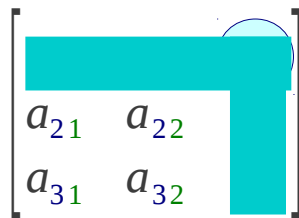

$$\begin{bmatrix} & & & \\ a_{21} & & a_{23} & \\ a_{31} & & a_{33} & \end{bmatrix}$$

$$\begin{bmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{bmatrix}$$

$$M_{12} = \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix}$$

$$C_{12} = -M_{12}$$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$


$$\begin{bmatrix} & & & \\ a_{21} & a_{22} & & \\ a_{31} & a_{32} & & \end{bmatrix}$$

$$\begin{bmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix}$$

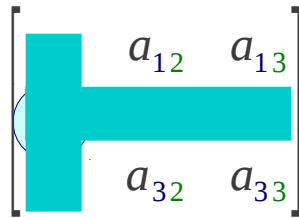
$$M_{13} = \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

$$C_{13} = +M_{13}$$

Minor Example (2)

Given matrix

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$


$$\begin{bmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{bmatrix}$$

Sub-matrix

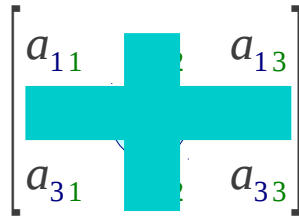
$$\begin{bmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{bmatrix}$$

Minor

$$M_{21} = \begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix}$$

$$C_{21} = -M_{21}$$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

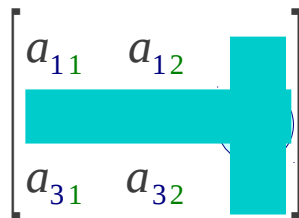

$$\begin{bmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{bmatrix}$$

$$\begin{bmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{bmatrix}$$

$$M_{22} = \begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix}$$

$$C_{22} = +M_{22}$$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$


$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

$$M_{23} = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$$

$$C_{23} = -M_{23}$$

Minor Example (2)

Given matrix

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$\begin{bmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{bmatrix}$$

Sub-matrix

$$\begin{bmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{bmatrix}$$

Minor / Cofactor

$$M_{31} = \begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix}$$

$$C_{31} = +M_{31}$$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$\begin{bmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{bmatrix}$$

$$\begin{bmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{bmatrix}$$

$$M_{32} = \begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix}$$

$$C_{32} = -M_{32}$$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

$$M_{33} = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$$

$$C_{33} = +M_{33}$$

Determinant

The **determinant** of an $n \times n$ matrix \mathbf{A} $\det(\mathbf{A})$

Cofactor expansion along the i -th row

Cofactor expansion along the j -th column

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$\begin{aligned} \det(\mathbf{A}) &= a_{11}M_{11} + a_{12}M_{12} + a_{13}M_{13} \\ &= a_{21}M_{21} + a_{22}M_{22} + a_{23}M_{23} \\ &= a_{31}M_{31} + a_{32}M_{32} + a_{33}M_{33} \end{aligned}$$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$\begin{aligned} \det(\mathbf{A}) &= a_{11}M_{11} + a_{21}M_{21} + a_{31}M_{31} \\ &= a_{12}M_{12} + a_{22}M_{22} + a_{32}M_{32} \\ &= a_{13}M_{13} + a_{23}M_{23} + a_{33}M_{33} \end{aligned}$$

Adjoint

The **cofactor** of entry a_{ij}

$$C_{ij} = (-1)^{i+j} M_{ij}$$

The **minor** of entry a_{ij}

$$M_{ij}$$

The determinant of the submatrix
that remains after **deleting** i -th row and j -th column

Given matrix

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$n \times n$

$$a_{11} \Leftrightarrow C_{11}$$

$$a_{12} \Leftrightarrow C_{12}$$

$$a_{13} \Leftrightarrow C_{13}$$

$$a_{21} \Leftrightarrow C_{21}$$

$$a_{22} \Leftrightarrow C_{22}$$

$$a_{23} \Leftrightarrow C_{23}$$

$$a_{31} \Leftrightarrow C_{31}$$

$$a_{32} \Leftrightarrow C_{32}$$

$$a_{33} \Leftrightarrow C_{33}$$

Matrix of Cofactors

$$\begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix}$$

$n \times n$

transpose



Adjoint

$$\begin{bmatrix} C_{11} & C_{21} & C_{31} \\ C_{12} & C_{22} & C_{32} \\ C_{13} & C_{23} & C_{33} \end{bmatrix}$$

$n \times n$

Inverse Matrix

The **cofactor** of entry a_{ij} $C_{ij} = (-1)^{i+j} M_{ij}$

The **minor** of entry a_{ij} M_{ij}

The determinant of the submatrix
that remains after **deleting** i -th row and j -th column

Given matrix

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$n \times n$

Matrix of Cofactors

$$\begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix}$$

$n \times n$

transpose



Adjoint

$$\begin{bmatrix} C_{11} & C_{21} & C_{31} \\ C_{12} & C_{22} & C_{32} \\ C_{13} & C_{23} & C_{33} \end{bmatrix}$$

$n \times n$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}^{-1}$$

$$\mathbf{A}^{-1} = \frac{1}{\det(\mathbf{A})} \text{adj}(\mathbf{A}) = \frac{1}{\det(\mathbf{A})} \begin{bmatrix} C_{11} & C_{21} & C_{31} \\ C_{12} & C_{22} & C_{32} \\ C_{13} & C_{23} & C_{33} \end{bmatrix}$$

Adjoint Matrix

References

- [1] <http://en.wikipedia.org/>
- [2] Anton & Busby, "Contemporary Linear Algebra"
- [3] Anton & Rorres, "Elementary Linear Algebra"