DFT (4B)

- CTFS and DFT
- CTFT and DFT
- DTFT and DFT
- CTFT and DFT
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CTFS with Complex Coefficients

$$x(t) = a_0 + \sum_{k=1}^{\infty} \left(a_k \cos(k\omega_0 t) + b_k \sin(k\omega_0 t) \right)$$

$$x(t) = \sum_{k=0}^{\infty} \left(A_k\, e^{+jk\omega_0 t} + B_k\, e^{-jk\omega_0 t} \right)$$

$$a_0 = \frac{1}{T} \int_0^T x(t)\, dt$$

$$a_k = \frac{2}{T} \int_0^T x(t) \cos(k\omega_0 t)\, dt$$

$$b_k = \frac{2}{T} \int_0^T x(t) \sin(k\omega_0 t)\, dt$$

$$k = 1,\ 2,\ ...$$

$$A_k = \frac{1}{T} \int_0^T x(t)\, e^{-jk\omega_0 t}\, dt$$

$$k = 0,\ 1,\ 2,\ ...$$

$$B_k = \frac{1}{T} \int_0^T x(t)\, e^{+jk\omega_0 t}\, dt$$

$$k = 1,\ 2,\ ...$$

$$x(t) = A_0 + \sum_{k=1}^{\infty} \left(A_k\, e^{jk\omega_0 t} + B_k\, e^{-jk\omega_0 t} \right)$$

$$x(t) = \sum_{k=-\infty}^{+\infty} C_k\, e^{+jk\omega_0 t}$$

$$A_0 = a_0$$

$$A_k = \frac{1}{2}\left(a_k - j b_k \right)$$

$$B_k = \frac{1}{2}\left(a_k + j b_k \right)$$

$$k = 1,\ 2,\ ...$$

$$C_k = \frac{1}{T} \int_0^T x(t)\, e^{-jk\omega_0 t}\, dt$$

$$k = -2,\ -1,\ 0,\ +1,\ +2,\ ...$$

$$C_k = \begin{cases} A_0 = a_0 & (k = 0) \\[2mm] A_k = \frac{1}{2}(a_k - jb_k) & (k > 0) \\[2mm] B_k = \frac{1}{2}(a_k + jb_k) & (k < 0) \end{cases}$$

CTFS and DTFS

$$x(t) = \sum_{k=-\infty}^{+\infty} C_k \, e^{+jk\omega_0 t} \qquad \textbf{CTFS}$$

$$C_k = \frac{1}{T} \int_0^T x(t) \, e^{-jk\omega_0 t} \, dt$$

$$k = -2, -1, 0, +1, +2, \ldots$$

$$x(t) \approx \sum_{k=-M}^{+M} C_k \, e^{+jk\omega_0 t} \qquad N = 2M + 1$$

$$jk\omega_0 t \to k\left(\frac{2\pi}{T}\right) n\left(\frac{T}{N}\right) = \left(\frac{2\pi}{T}\right) nk$$
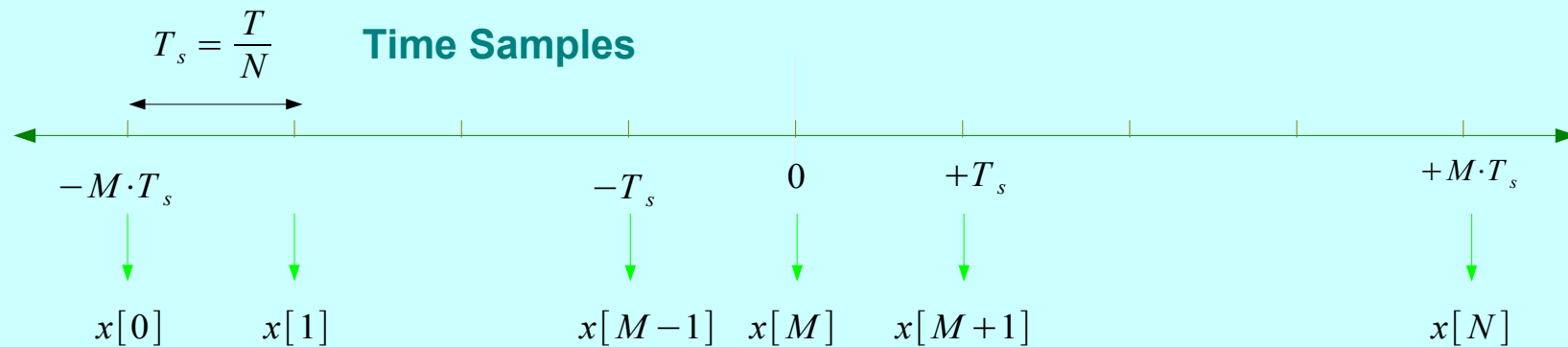
$$x[n] = \sum_{k=-M}^{+M} \gamma_k \, e^{+j\left(\frac{2\pi}{N}\right)nk}$$

$$n = 0, 1, 2, \ldots, N-1,$$

$$\gamma_k = \frac{1}{N} \sum_{n=0}^{N-1} x[n] \, e^{-j\left(\frac{2\pi}{N}\right)nk\omega_0 t}$$

$$k = -M, \ldots, 0, \ldots, +M$$

$$x_{FS}(t) = \sum_{k=-\infty}^{+\infty} \gamma_k \, e^{+jk\omega_0 t} \qquad \textbf{DTFS}$$

$$T_s = \frac{T}{N} \qquad \textbf{Time Samples}$$

$-M \cdot T_s \qquad \qquad -T_s \qquad 0 \qquad +T_s \qquad \qquad +M \cdot T_s$

$x[0] \qquad x[1] \qquad x[M-1] \quad x[M] \quad x[M+1] \qquad x[N]$

$$0 \le n < L \qquad\qquad T_s \to 0$$

CTFT and DFT

Continuous Time Fourier Transform

$$X(j\omega) = \int_{-\infty}^{+\infty} x(t)\, e^{-j\omega t}\, dt \quad \Longleftrightarrow \quad x(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} X(j\omega)\, e^{+j\omega t}\, d\omega$$
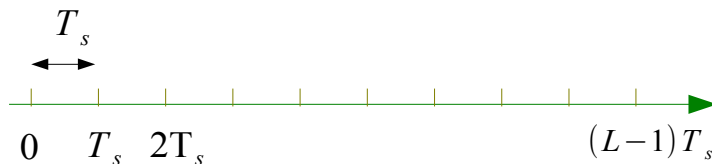
Discrete Fourier Transform

$$X[k] = \sum_{n=0}^{N-1} x[n]\, e^{-j(2\pi/N)kn} \quad \Longleftrightarrow \quad x[n] = \frac{1}{N} \sum_{k=0}^{N-1} X[k]\, e^{+j(2\pi/N)kn}$$

From CTFT to DFT (1)

Continuous Time Fourier <u>Transform</u>

$$X(j\omega) = \int_{-\infty}^{+\infty} x(t)\, e^{-j\omega t}\, dt \quad \Longleftrightarrow \quad x(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} X(j\omega)\, e^{+j\omega t}\, d\omega$$
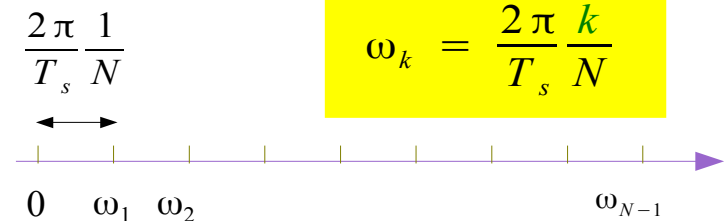
Time Samples

T_s

$$0 \quad T_s \quad 2T_s \qquad\qquad (L-1)T_s$$

$$t \to nT_s \qquad dt \to T_s \qquad \int \to \sum$$

$$0 \le n < L \qquad\qquad T_s \to 0$$

Frequency Samples

$$\frac{2\pi}{T_s}\frac{1}{N} \qquad\qquad \omega_k = \frac{2\pi}{T_s}\frac{k}{N}$$

$$0 \quad \omega_1 \quad \omega_2 \qquad\qquad \omega_{N-1}$$

$$\omega \to \omega_k \qquad d\omega \to \frac{2\pi}{T_s}\frac{1}{N} \qquad \int \to \sum$$

$$0 \le k < N \qquad\qquad 0 \le \omega_k < \frac{2\pi}{T_s}$$

Continuous Time Fourier Transform

$$X(j\omega) = \int_{-\infty}^{+\infty} x(t)\, e^{-j\omega t}\, dt \quad \Longleftrightarrow \quad x(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} X(j\omega)\, e^{+j\omega t}\, d\omega$$

$$t \rightarrow nT_s \qquad dt \rightarrow T_s \qquad \int \rightarrow \sum \qquad 0 \le n < L$$

$$\hat{X}(j\omega) = \sum_{n=-\infty}^{+\infty} x(nT_s)\, e^{-j\omega nT_s} \cdot T_s \quad \Longleftrightarrow \quad x(nT_s) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \hat{X}(j\omega)\, e^{+j\omega nT_s}\, d\omega$$

$$\omega \rightarrow \omega_k \qquad d\omega \rightarrow \frac{2\pi}{T_s} \frac{1}{N} \qquad \int \rightarrow \sum \qquad 0 \le k < N \qquad \boxed{\omega_k = \frac{2\pi}{T_s} \frac{k}{N}}$$

$$\hat{X}(j\omega_k) = T_s \sum_{n=0}^{L-1} x[n]\, e^{-j\omega_k nT_s} \quad \Longleftrightarrow \quad x[n] = \frac{1}{2\pi} \sum_{k=0}^{N-1} \hat{X}(j\omega_k)\, e^{+j\omega_k nT_s}\, \frac{2\pi}{T_s} \frac{1}{N}$$

From CTFT to DFT (3)

Continuous Time Fourier Transform

$$X(j\omega) = \int_{-\infty}^{+\infty} x(t)\, e^{-j\omega t}\, dt \quad \longleftrightarrow \quad x(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} X(j\omega)\, e^{+j\omega t}\, d\omega$$

Time Samples

$$t \rightarrow nT_s \quad dt \rightarrow T_s \quad \int \rightarrow \sum$$

Frequency Samples

$$\omega \rightarrow \omega_k \quad d\omega \rightarrow \frac{2\pi}{T_s}\frac{1}{N} \quad \int \rightarrow \sum$$

$$\hat{X}(j\omega_k) = T_s \sum_{n=0}^{L-1} x[n]\, e^{-j\omega_k n T_s} \quad \longleftrightarrow \quad x[n] = \frac{1}{2\pi} \sum_{k=0}^{N-1} \hat{X}(j\omega_k)\, e^{+j\omega_k n T_s}\, \frac{2\pi}{T_s}\frac{1}{N}$$

$$\omega_k T_s \rightarrow \frac{2\pi}{N} k \qquad \Rightarrow \qquad \omega_k n T_s \rightarrow \frac{2\pi}{N} k n \qquad\qquad \omega_k = \frac{2\pi}{T_s}\frac{k}{N}$$

$$\frac{1}{T_s}\hat{X}(j\omega_k) = \sum_{n=0}^{L-1} x[n]\, e^{-j\left(\frac{2\pi}{N}\right)kn} \quad \longleftrightarrow \quad x[n] = \frac{1}{N} \sum_{k=0}^{N-1} \frac{1}{T_s}\hat{X}(j\omega_k)\, e^{+j\left(\frac{2\pi}{N}\right)kn}$$

From CTFT to DFT (4)

Continuous Time Fourier Transform

$$X(j\omega) = \int_{-\infty}^{+\infty} x(t)\, e^{-j\omega t}\, dt \quad\longleftrightarrow\quad x(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} X(j\omega)\, e^{+j\omega t}\, d\omega$$

Time Samples

$$t \to nT_s \qquad dt \to T_s \qquad \int \to \sum$$

Frequency Samples

$$\omega \to \omega_k \qquad d\omega \to \frac{2\pi}{T_s}\frac{1}{N} \qquad \int \to \sum$$

$$\omega_k T_s \to \frac{2\pi}{N}k \qquad\Longrightarrow\qquad \omega_k nT_s \to \frac{2\pi}{N}kn \qquad\qquad \omega_k = \frac{2\pi}{T_s}\frac{k}{N}$$

$$\frac{1}{T_s}\hat{X}(j\omega_k) = \sum_{n=0}^{L-1} x[n]\, e^{-j\left(\frac{2\pi}{N}\right)kn} \quad\longleftrightarrow\quad x[n] = \frac{1}{N}\sum_{k=0}^{N-1} \frac{1}{T_s}\hat{X}(j\omega_k)\, e^{+j\left(\frac{2\pi}{N}\right)kn}$$

From CTFT to DFT (5)

Continuous Time Fourier Transform

$$X(j\omega) = \int_{-\infty}^{+\infty} x(t)\, e^{-j\omega t}\, dt \quad \Longleftrightarrow \quad x(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} X(j\omega)\, e^{+j\omega t}\, d\omega$$

$$\frac{1}{T_s}\hat{X}(j\omega_k) = \sum_{n=0}^{L-1} x[n]\, e^{-j\left(\frac{2\pi}{N}\right)kn} \quad \Longleftrightarrow \quad x[n] = \frac{1}{N}\sum_{k=0}^{N-1}\frac{1}{T_s}\hat{X}(j\omega_k)\, e^{+j\left(\frac{2\pi}{N}\right)kn}$$

Discrete Fourier Transform

$$L = N$$

$$X[k] = \sum_{n=0}^{N-1} x[n]\, e^{-j(2\pi/N)kn} \quad \Longleftrightarrow \quad x[n] = \frac{1}{N}\sum_{k=0}^{N-1} X[k]\, e^{+j(2\pi/N)kn}$$

CTFT of a Sampled Signal

Continuous Time Fourier <u>Transform</u>

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} X(j\omega)\, e^{+j\omega t}\, d\omega \qquad \Longleftrightarrow \qquad X(j\omega) = \int_{-\infty}^{+\infty} x(t)\, e^{-j\omega t}\, dt$$

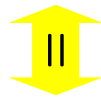$$x_s(t) = x_c(t) \sum_{n=-\infty}^{+\infty} \delta(t - nT_s)$$

$$x_s(t) = \sum_{n=-\infty}^{+\infty} x_c(nT_s)\delta(t - nT_s)$$

CTFT

$$X_s(j\omega) = \sum_{n=-\infty}^{+\infty} x_c(nT_s)\, e^{-j\omega nT_s}$$

$$X_s(j\omega) = \sum_{n=-\infty}^{+\infty} x[n]\, e^{-j\omega nT_s}$$

CTFS $\parallel$

DTFT $\parallel$

$$x_s(t) = x_c(t) \sum_{k=-\infty}^{+\infty} \frac{1}{T_s}\, e^{jk\omega_s t}$$

CTFT

$$X_s(j\omega) = \frac{1}{T_s} \sum_{k=-\infty}^{+\infty} X_c\big(j(\omega - k\omega_s)\big)$$

$$\omega_s = \frac{2\pi}{T_s}$$

$$\omega_s = \frac{2\pi}{T_s}$$

DTFT and CTFT

Continuous Time Fourier Transform   CTFT

$$X(j\omega) = \int_{-\infty}^{+\infty} x(t)\, e^{-j\omega t}\, dt \qquad \Longleftrightarrow \qquad x(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} X(j\omega)\, e^{+j\omega t}\, d\omega$$

Discrete Time Fourier Transform   DTFT

$$X(e^{j\hat{\omega}}) = \sum_{n=-\infty}^{+\infty} x[n]\, e^{-j\hat{\omega} n} \qquad \Longleftrightarrow \qquad x[n] = \frac{1}{2\pi} \int_{-\pi}^{+\pi} X(e^{j\hat{\omega}})\, e^{+j\hat{\omega} n}$$

$$X_s(j\omega) = \sum_{n=-\infty}^{+\infty} x_c(nT_s)\, e^{-j\omega n T_s}$$

$$X_s(j\omega) = \sum_{n=-\infty}^{+\infty} x[n]\, e^{-j\omega n T_s}$$

$$\Longleftrightarrow$$
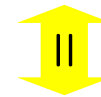
$$X_s(j\omega) = \frac{1}{T_s} \sum_{k=-\infty}^{+\infty} X_c(j(\omega - k\omega_s))$$

$$\omega_s = \frac{2\pi}{T_s}$$

DTFT and CTFT

$$X_s(j\omega) = \sum_{n=-\infty}^{+\infty} x_c(nT_s) e^{-j\omega nT_s}$$

$$X_s(j\omega) = \sum_{n=-\infty}^{+\infty} x[n] e^{-j\omega nT_s}$$

$\Longleftarrow$

$$X_s(j\omega) = \frac{1}{T_s} \sum_{k=-\infty}^{+\infty} X_c(j(\omega - k\omega_s))$$

$$\omega_s = \frac{2\pi}{T_s}$$

CTFT of a sampled signal

$$X(e^{j\hat{\omega}})\bigg|_{\hat{\omega} = \omega T_s} = X(e^{j\omega T_s}) = \frac{1}{T_s} \sum_{k=-\infty}^{+\infty} X_c(j(\omega - k\omega_s))$$

$$= \frac{1}{T_s} \sum_{k=-\infty}^{+\infty} X_c(j(\omega - \frac{2\pi k}{T_s}))$$

$$X(e^{j\hat{\omega}}) = \sum_{n=-\infty}^{+\infty} x[n] e^{-j\hat{\omega} n}$$

DTFT of a sampled signal

DTFT and DFT

DTFT of a sampled signal

$$X(e^{j\hat\omega}) = \sum_{n=-\infty}^{+\infty} x[n]\, e^{-j\hat\omega n}$$

$$\hat\omega = \omega T_s$$

Frequency Samples

$$\hat\omega \rightarrow \hat\omega_k \qquad 0 \le \hat\omega_k < 2\pi \qquad 0 \le k < N \qquad 0 \le n < L$$

$$X(e^{j\hat\omega_k}) = \sum_{n=0}^{L-1} x[n]\, e^{-j\hat\omega_k n}$$

$$\hat\omega_k = \left(\frac{2\pi}{N}\right) k$$

DFT of a sampled signal

$$X[k] =$$

$$X(e^{j(2\pi/N)k}) = \sum_{n=0}^{L-1} x[n]\, e^{-j(2\pi/N)kn}$$

DTFT sampled in frequency

CTFT and DFT

DFT of a sampled signal

$$X[k] \quad = \quad X(e^{j(2\pi/N)k}) \quad = \quad \sum_{n=0}^{L-1} x[n] \, e^{-j(2\pi/N)kn}$$

DTFT *sampled in frequency*

$$X(e^{j\omega T_s})\Big|_{\omega \,=\, \frac{2\pi k}{NT_s}} \qquad \textbf{CTFT evaluated at} \quad \omega \;=\; \frac{2\pi k}{NT_s}$$

$$= \; \frac{1}{T_s} \sum_{l=-\infty}^{+\infty} X_c(j(\omega - l\omega_s)) \Bigg|_{\omega \,=\, \frac{2\pi k}{NT_s}}$$

$$= \; \frac{1}{T_s} \sum_{l=-\infty}^{+\infty} X_c(j(\omega - l\frac{2\pi}{T_s})) \Bigg|_{\omega \,=\, \frac{2\pi k}{NT_s}}$$

From DTFT to DFT (1)

DTFT of a sampled signal

$$X(e^{j\hat{\omega}}) = \sum_{n=-\infty}^{+\infty} x[n]\, e^{-j\hat{\omega}n} \qquad\qquad \hat{\omega} = \omega T_s$$

Frequency Samples

$$\hat{\omega} \to \hat{\omega}_k \qquad 0 \leq \hat{\omega}_k < 2\pi \qquad 0 \leq k < N \qquad 0 \leq n < L$$

$$X(e^{j\hat{\omega}_k}) = \sum_{n=0}^{L-1} x[n]\, e^{-j\hat{\omega}_k n} \qquad\qquad \hat{\omega}_k = \left(\frac{2\pi}{N}\right) k$$

DFT of a sampled signal

$$X[k] = \qquad X(e^{j(2\pi/N)k}) = \sum_{n=0}^{L-1} x[n]\, e^{-j(2\pi/N)kn}$$

DTFT sampled in frequency

$f(t)$

$x(k\tau) = f(t)\delta(t-k\tau)w(t)$

sampling time $\quad\tau$

window function $\quad 0 \le t \le T$

$$X(\omega) = \int_{-\infty}^{+\infty} x(t)\, e^{-j\omega t}\, dt$$

Fourier Transform

discrete time $\quad t \;\Rightarrow\; k\tau$

discrete freq $\quad \omega \;\Rightarrow\; \omega_k = 2\pi f_k$

$$X(\omega_k) = \sum_{n=0}^{N-1} x(n\tau)\, e^{-j2\pi f_k n\tau}$$

$$e^{-j2\pi f_k n\tau} = e^{-j2\pi k\left(\frac{1}{N\tau}\right)n\tau}$$

$$X(\omega_k) = \sum_{n=0}^{N-1} x(n\tau)\, e^{-j\frac{2\pi}{N}kn}$$

fundamental frequency

$$f_0 = \frac{1}{T} = \frac{1}{N\tau} = \frac{f_s}{N}$$

k-th harmonics

$$f_k = k\cdot\frac{1}{T} = k\cdot\frac{1}{N\tau} = k\cdot\frac{f_s}{N}$$

From FT to DFT (2)

$$f(t)$$

$$x(k\tau) = f(t)\delta(t - k\tau)w(t)$$

sampling time τ

window function $0 \le t \le T$

$$\boxed{X(\omega) = \int_{-\infty}^{+\infty} x(t) e^{-j\omega t} dt}$$

Fourier Transform

discrete time $t \Rightarrow k\tau$

discrete freq $\omega \Rightarrow \omega_k = 2\pi f_k$

fundamental frequency

$$f_0 = \frac{1}{T} = \frac{1}{N\tau} = \frac{f_s}{N}$$

$$X(\omega_k) = \sum_{n=0}^{N-1} x(n\tau) e^{-j\frac{2\pi}{N}kn}$$

k-th harmonics

$$f_k = k\cdot\frac{1}{T} = k\cdot\frac{1}{N\tau} = k\cdot\frac{f_s}{N}$$

$$f_k = k\cdot\frac{f_s}{N}$$

$$X[k] = \sum_{n=0}^{N-1} x[n] e^{-j\frac{2\pi}{N}kn}$$

Discrete Fourier Transform

$$f(t) = \cos(2\pi \cdot 1 \cdot t) - \frac{1}{3}\cos(2\pi \cdot 3 \cdot t) + \frac{1}{5}\cos(2\pi \cdot 5 \cdot t)$$

$F(f)$

$\frac{1}{2} \cdot 1$

$\frac{1}{2} \cdot \frac{1}{3}$

$\frac{1}{2} \cdot \frac{1}{5}$
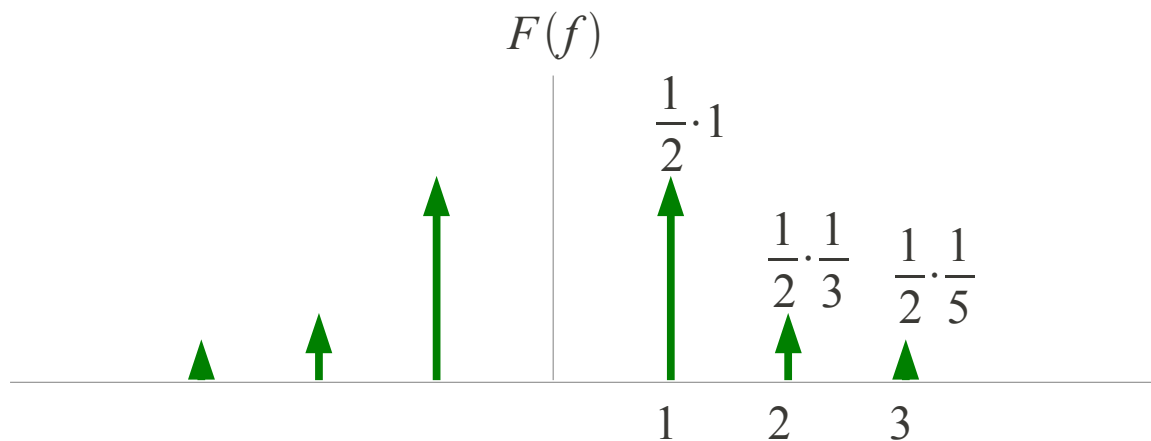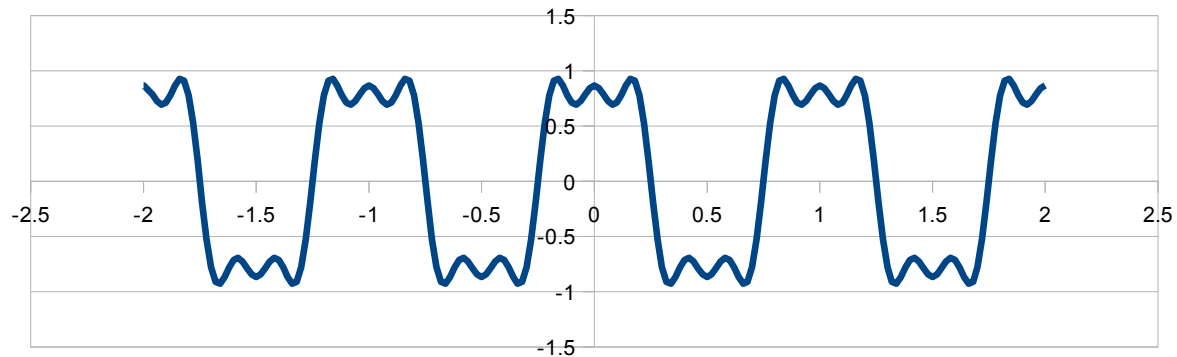
$$p(t) = \sum_{n=-\infty}^{\infty} \delta(t - n\tau)$$

Fourier Series Expansion of Impulse Train

$$p(t) = \sum_{k=-\infty}^{+\infty} a_k \, e^{+jk\omega_s t}$$

Fourier Series Coefficients

$$a_k = \frac{1}{\tau} \int_{-\tau/2}^{+\tau/2} \delta(t) \, e^{-jk\omega_s t} \, dt$$

$$= \frac{1}{\tau} \int_{-\tau/2}^{+\tau/2} \delta(t) \, e^{-jk\omega_s 0} \, dt$$

$$= \frac{1}{\tau} \int_{-\tau/2}^{+\tau/2} \delta(t) \, dt \ = \ \frac{1}{T_s}$$
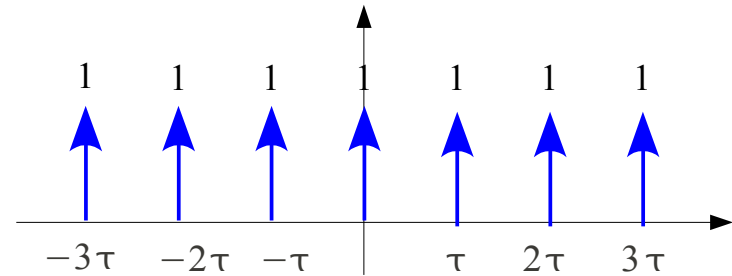
$$\omega_s = \frac{2\pi}{\tau}$$

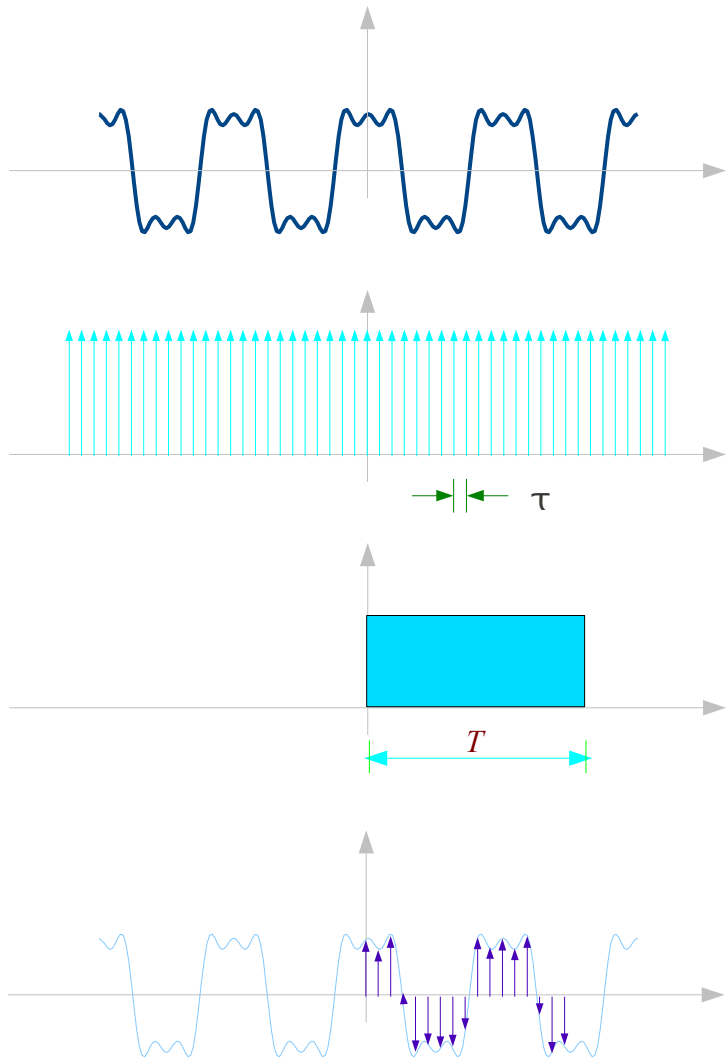$$f_s = \frac{1}{\tau}$$
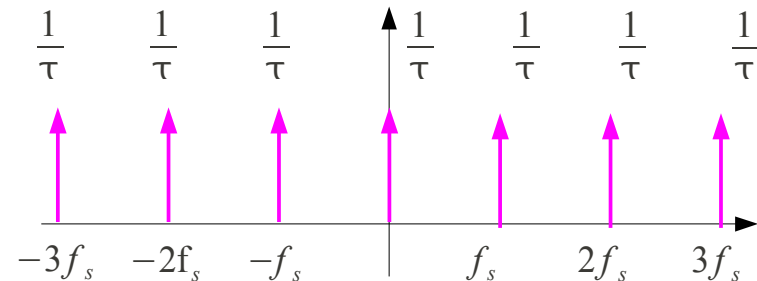
FT → DFT Example (3)

$$S(\omega) = \int_0^T e^{-j\omega t}\, dt = \left[\frac{e^{-j\omega t}}{-j\omega}\right]_0^T$$

$$= \frac{e^{-j\omega T} - 1}{-j\omega}$$

$$= \frac{e^{-j\omega T/2}\left(e^{-j\omega T/2} - e^{+j\omega T/2}\right)}{-j\omega}$$

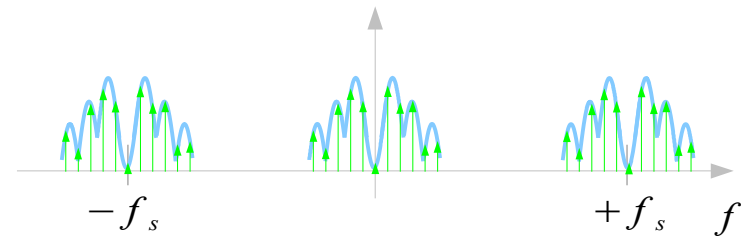$$= \frac{\sin(\omega T/2)}{\omega T/2} \cdot e^{-j\omega T/2}$$
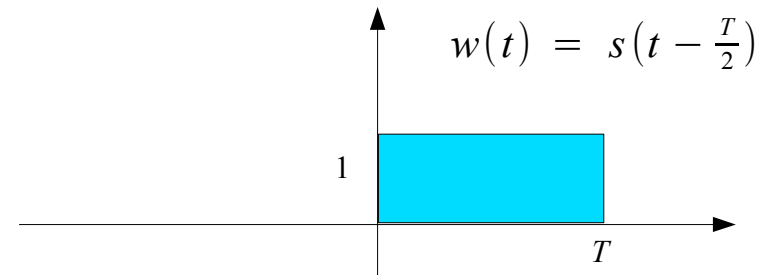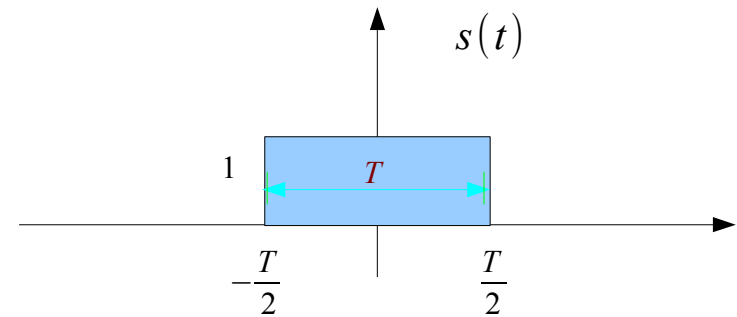
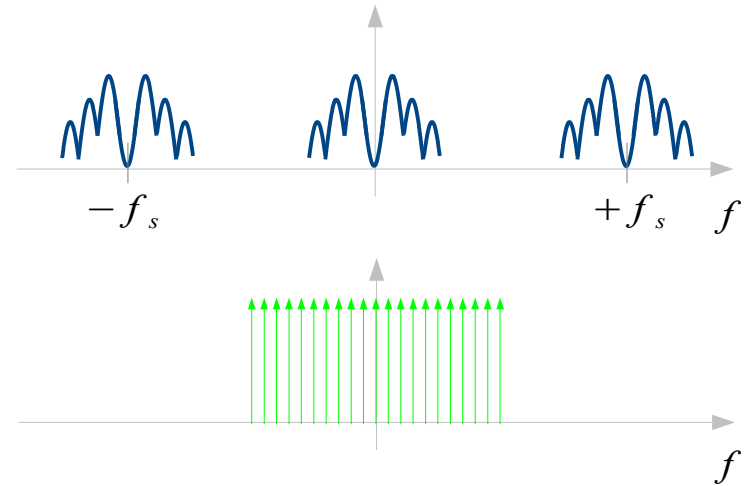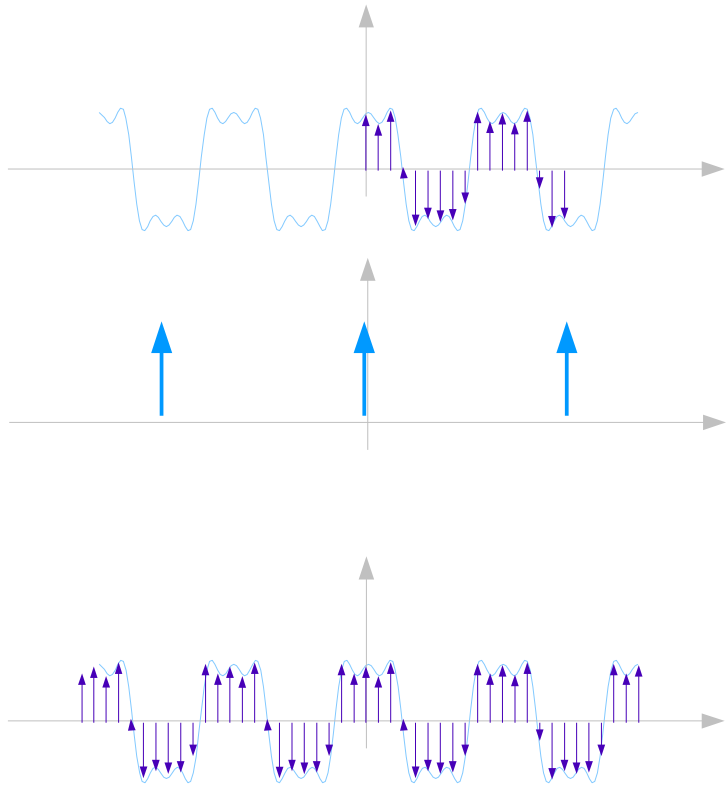$$W(\omega) = e^{-j\omega T/2} \cdot S(\omega)$$

$$= \frac{\sin(\omega T/2)}{\omega T/2} \cdot e^{-j\omega T}$$

$s(t)$

$w(t) = s\left(t - \frac{T}{2}\right)$

$|W(\omega)| = |S(\omega)|$
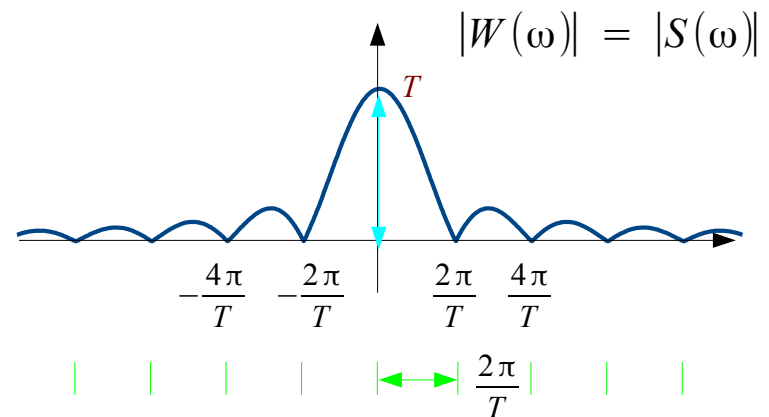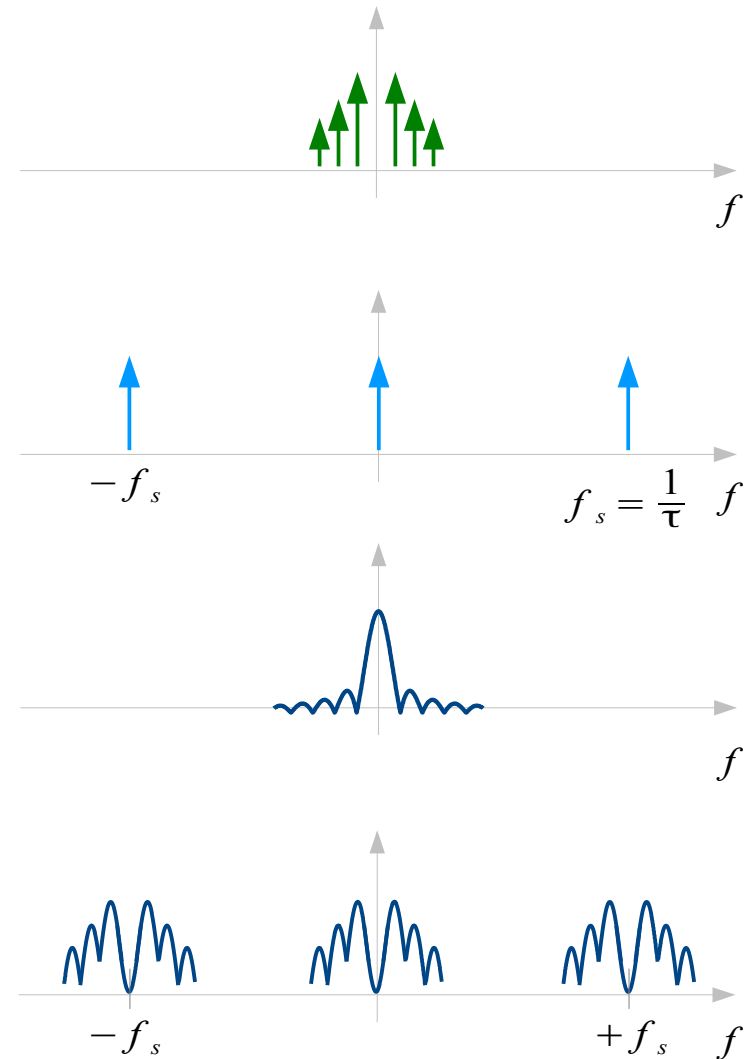
References

[1] http://en.wikipedia.org/
[2] J.H. McClellan, et al., Signal Processing First, Pearson Prentice Hall, 2003
[3] http://www.complextoreal.com/fft3.htm