Down-Sampling (4B)

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Please send corrections (or suggestions) to youngwlim@hotmail.com.

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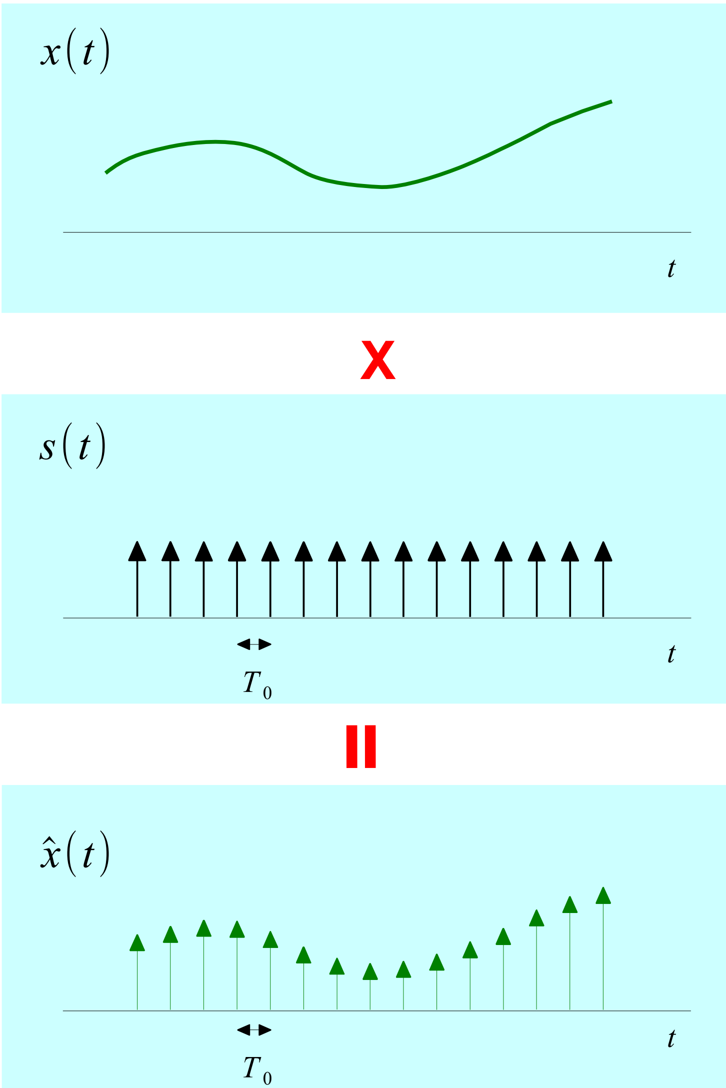
Spectrum Replication (1)

Ideal Sampling



$$\hat{x}(t) = \sum_{n=-\infty}^{+\infty} x(nT_0)\,\delta(t-nT_0)$$

$$s(t) = \sum_{n=-\infty}^{+\infty} \delta(t-nT_0)$$

$$= \frac{1}{T_0}\sum_{m=-\infty}^{+\infty} e^{+j2\pi m f_s t}$$

$$\hat{x}(t) = \frac{1}{T_0}\sum_{n=-\infty}^{+\infty} x(t)\, e^{+j2\pi m f_s t}$$

Shift Property

$$\hat{X}(f) = \frac{1}{T_0}\sum_{n=-\infty}^{+\infty} X(f-m f_s)$$

Spectrum Replication (2)

$$S(f) = \frac{1}{T}\sum_{m=-\infty}^{+\infty}\delta(f-mf_s)$$

Convolution in Frequency

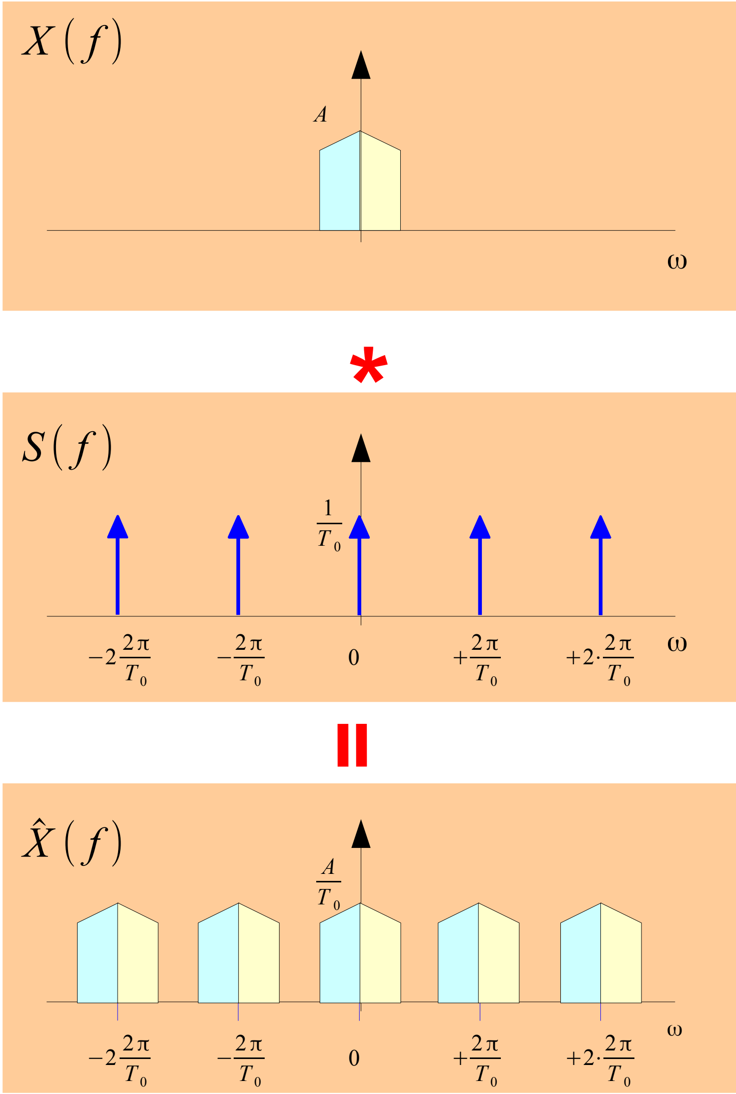
$$\hat{X}(f) = X(f) * S(f)$$

$$= \int_{-\infty}^{+\infty} X(f-f')S(f')\,df'$$

$$= \frac{1}{T_0}\sum_{m=-\infty}^{+\infty}\int_{-\infty}^{+\infty} X(f-f')\delta(f'-mf_s)\,df'$$

$$\hat{X}(f) = \frac{1}{T_0}\sum_{n=-\infty}^{+\infty} X(f-mf_s)$$

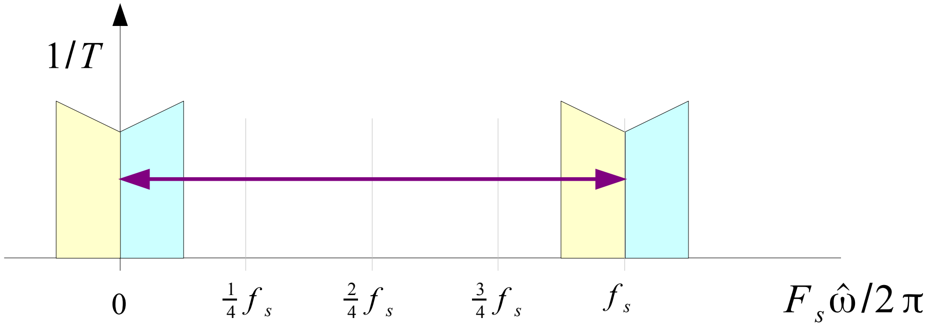
Frequency Domain

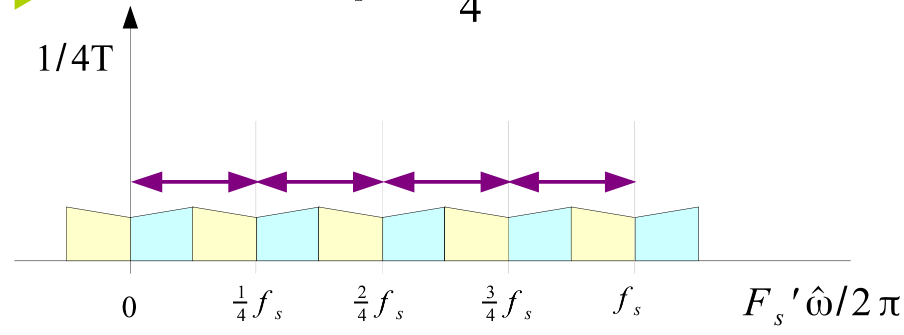


Decreasing Sampling Frequency

DOWN

$F_s = f_s$ $\qquad$ $F_s' = \frac{f_s}{4}$



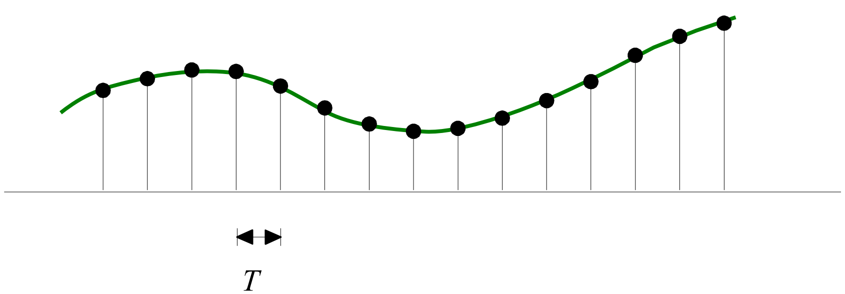


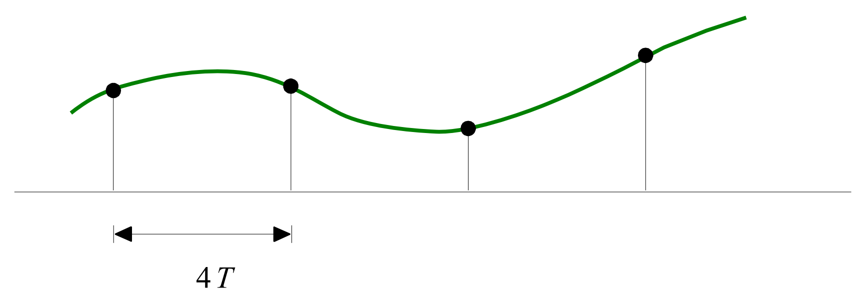
Sampling Frequency $F_s = f_s$

Sampling Time $T = \frac{1}{f_s}$

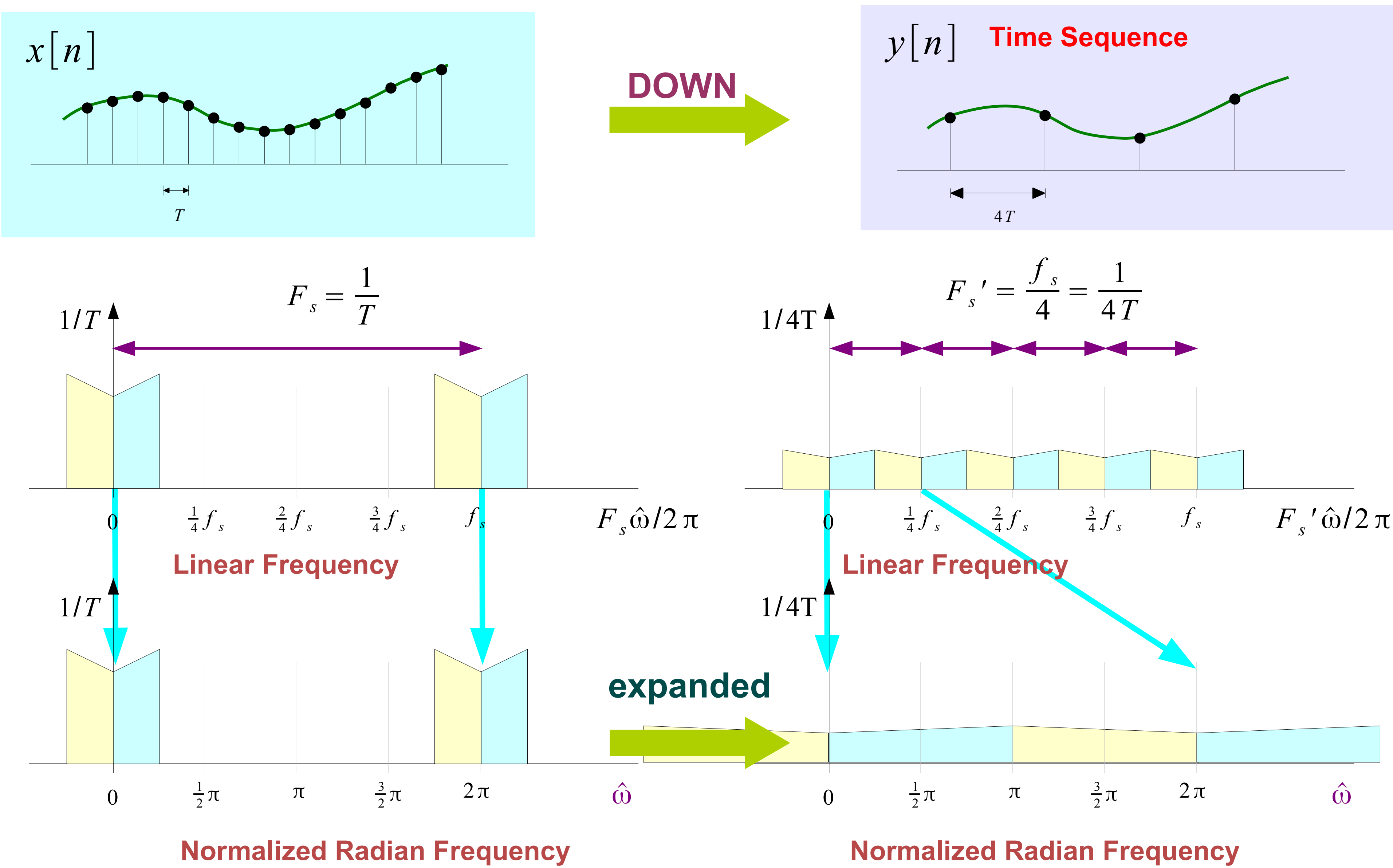
Sampling Frequency $F_s' = \frac{f_s}{4}$

Sampling Time $T' = 4T$

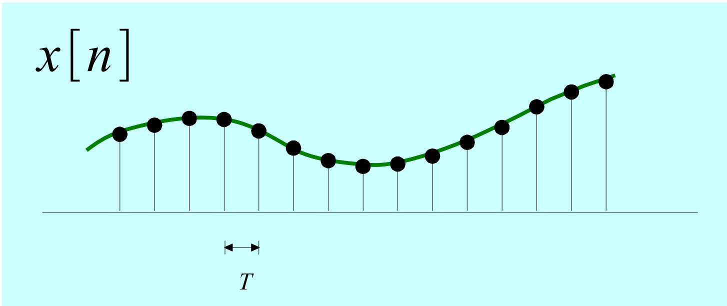




Coarse Sequence & Spectrum



Normalized Radian Frequency

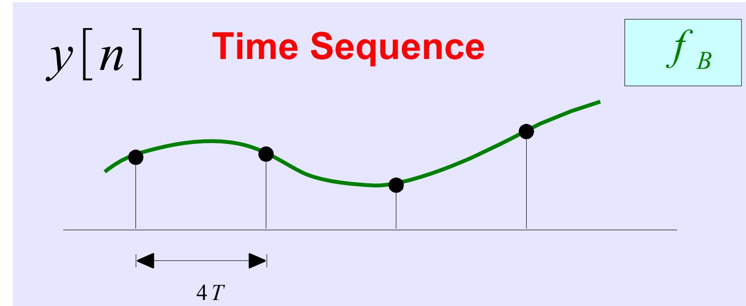


$$\hat{\omega} = \omega \cdot T_s = \frac{\omega}{1/T_s}$$

$$\hat{\omega} = \frac{\omega}{f_s} = 2\pi \frac{f}{f_s}$$

Normalized to f_s

Normalized Radian Frequency

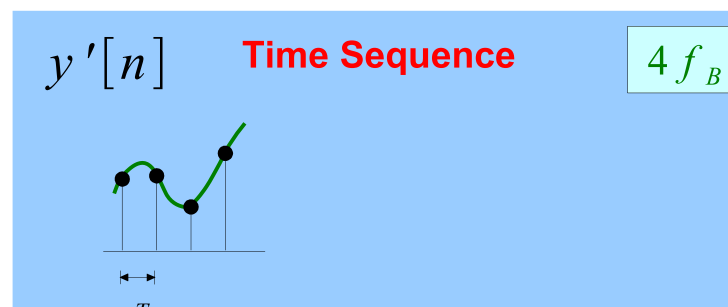


$$\frac{f}{f_s} = \frac{f_B}{1/4T} = f_B \cdot 4T$$

The Same

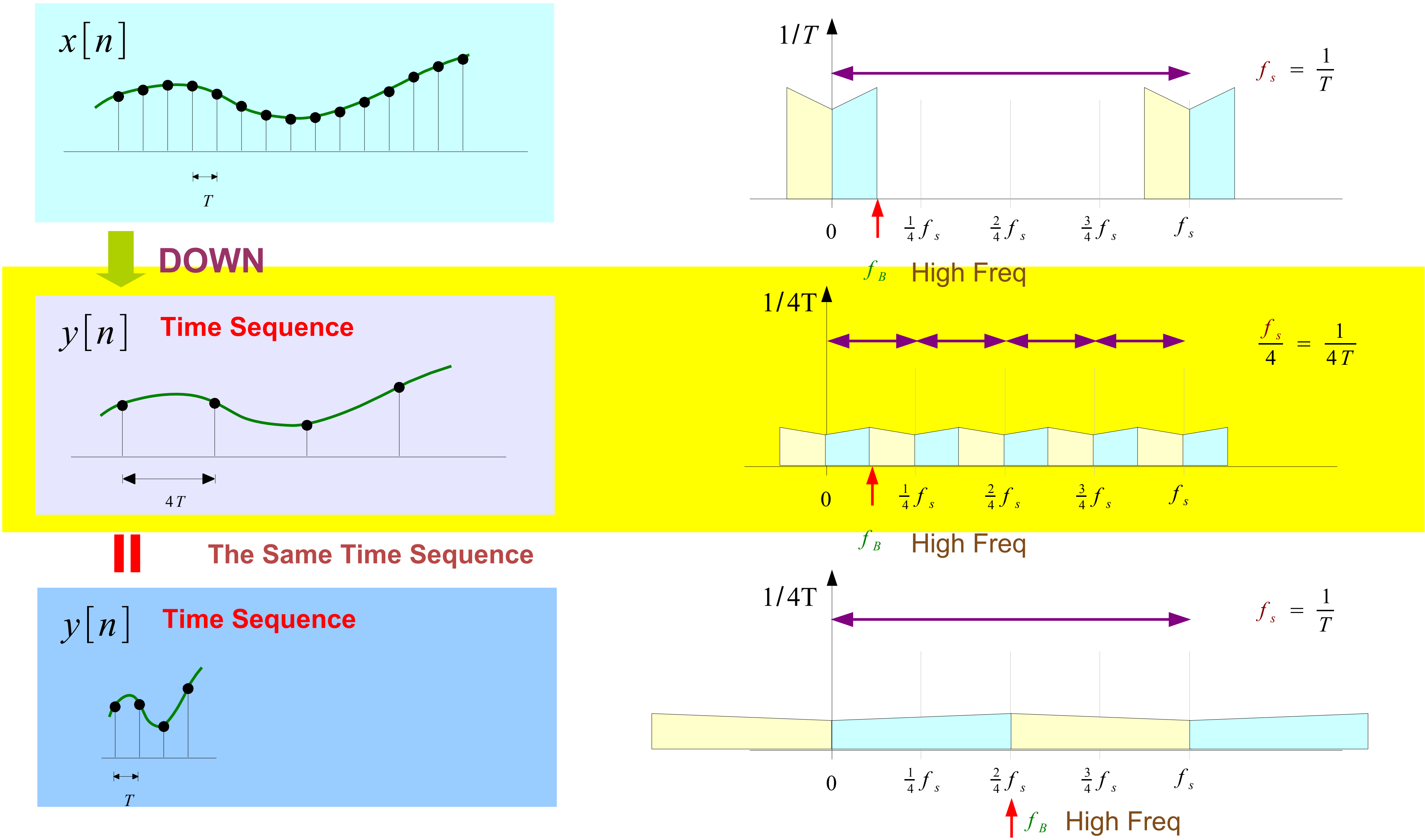
- **Time Sequence**
- **Normalized Radian Frequency**

$$\frac{f}{f_s} = \frac{4 f_B}{1/T} = f_B \cdot 4T$$

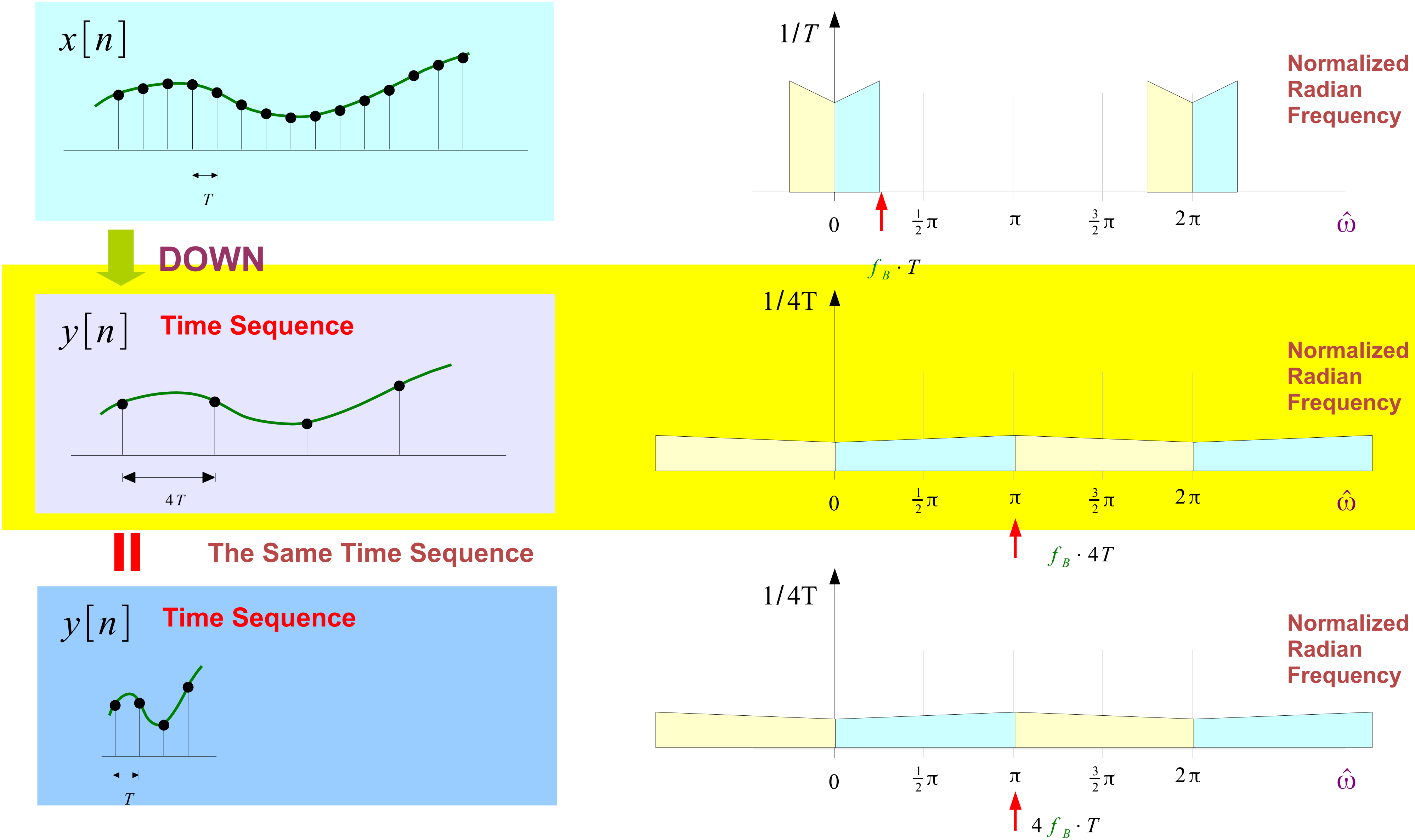


The Highest Frequency:

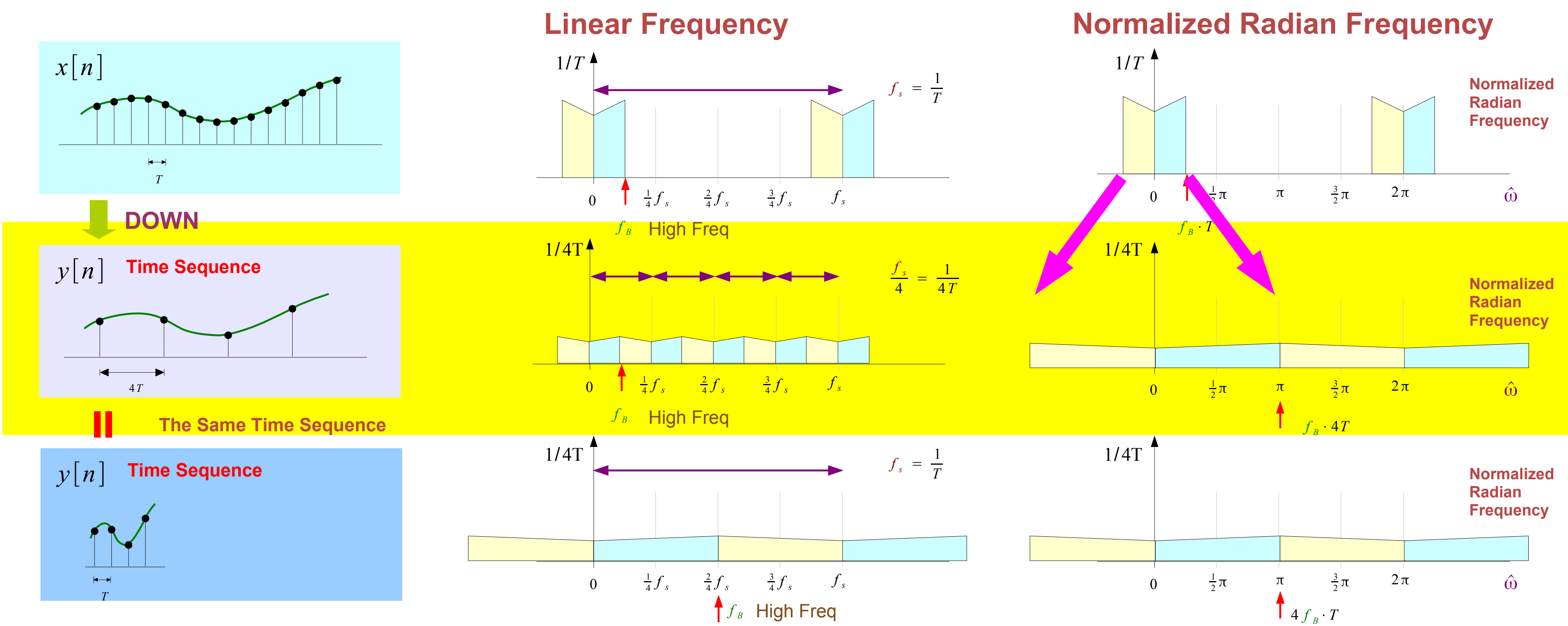
Coarse Sequence Spectrum – Linear Frequency



Coarse Sequence Spectrum – Normalized Frequency



Coarse Sequence Spectrum – Linear Frequency

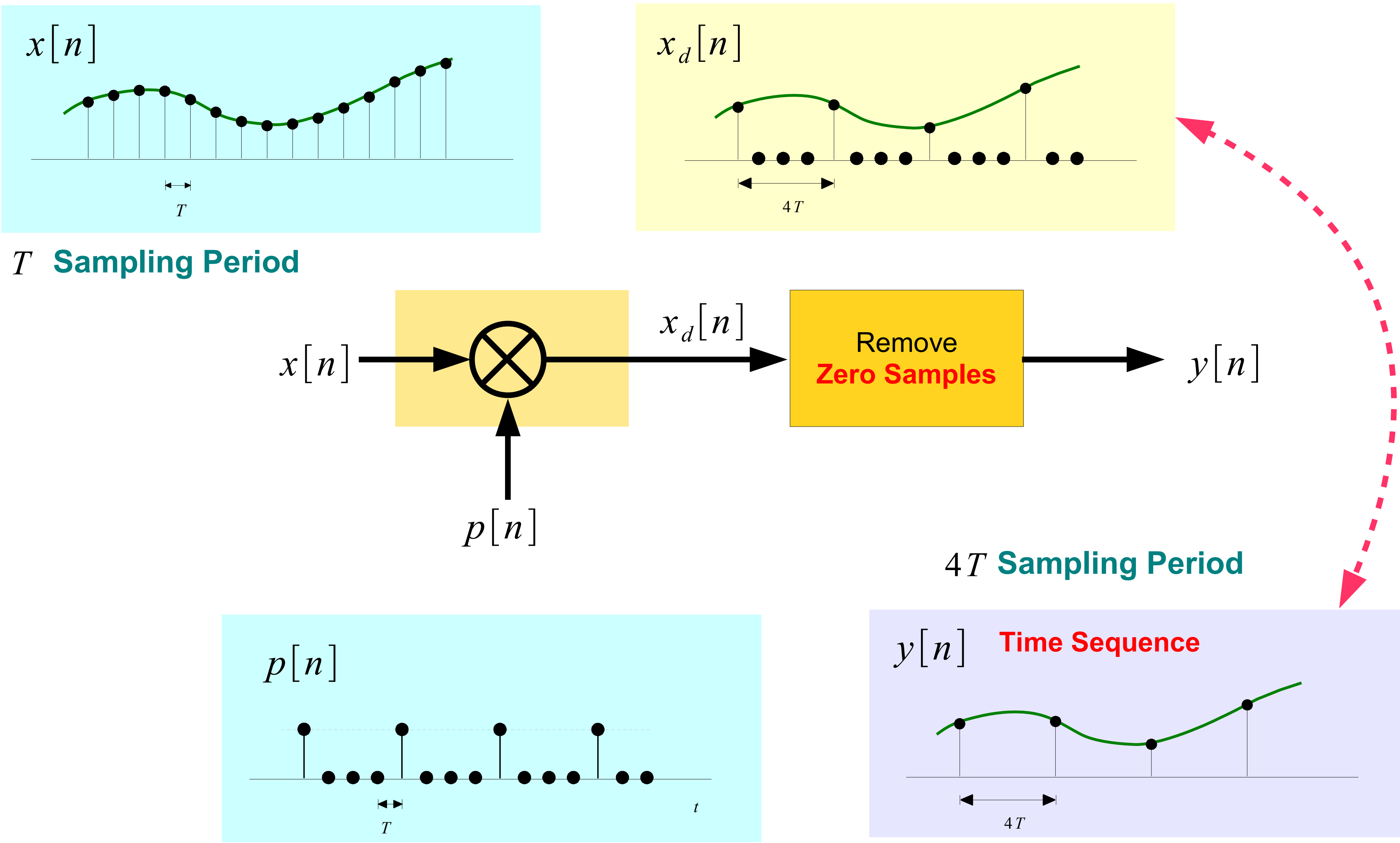


Coarse Sequence Spectrum

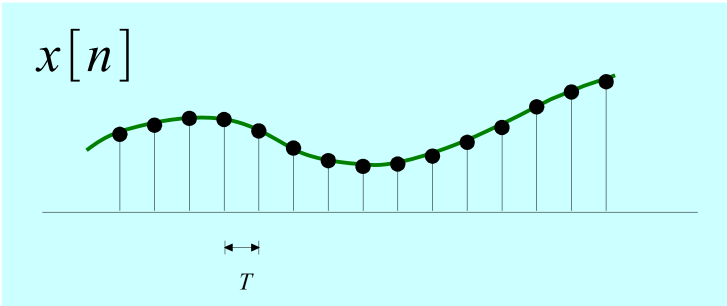
➡ **Expanded**

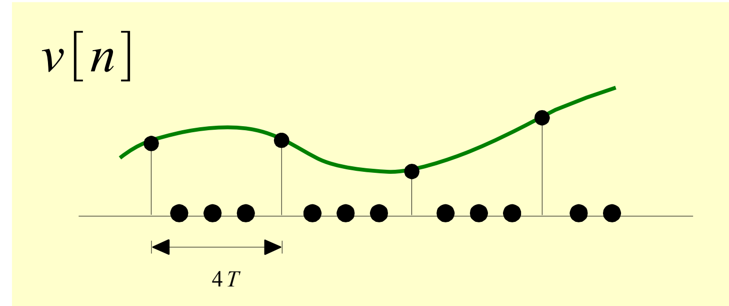
Normalized Radian Frequency

Coarse Sequence Generation

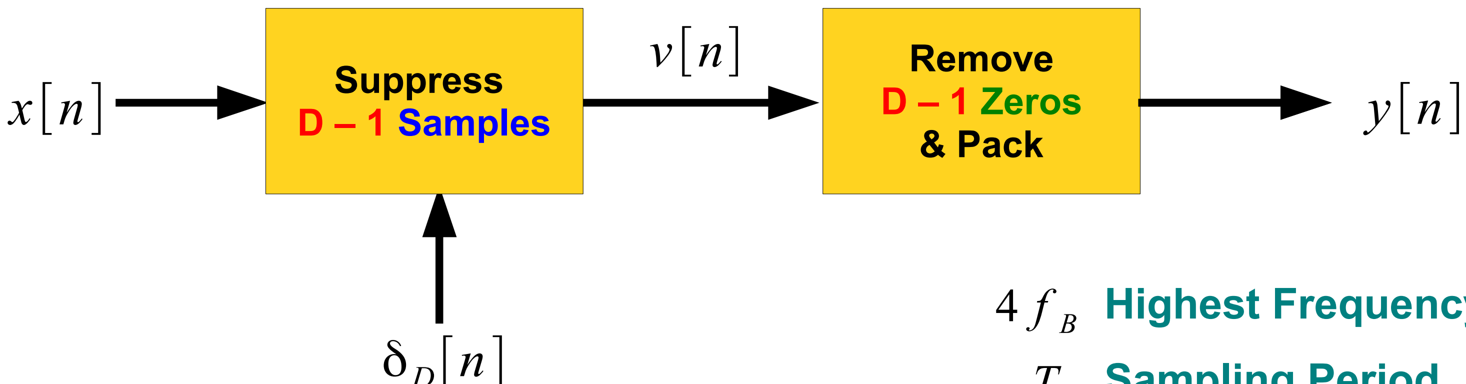


Down Sampling in Two Steps



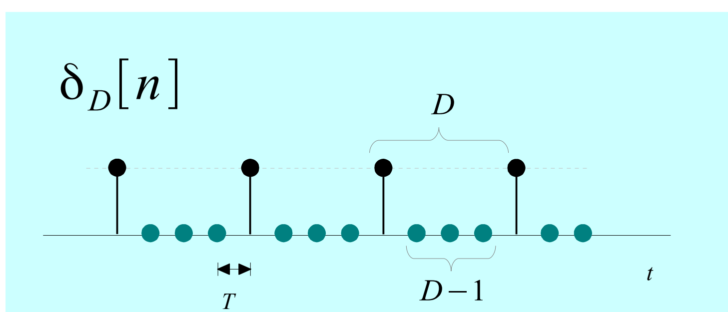


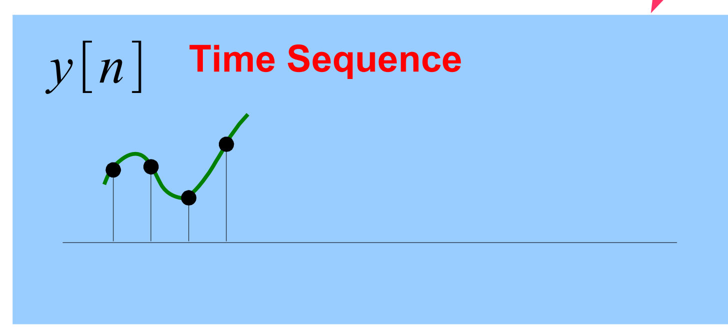
T **Sampling Period**



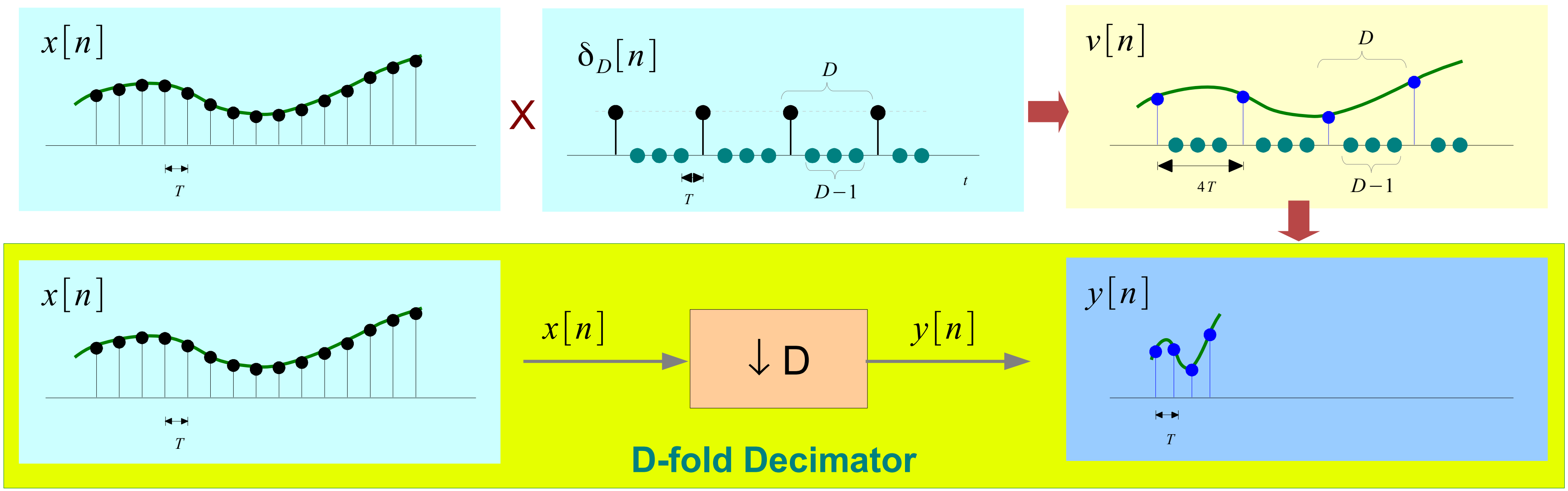
$4\,f_B$ **Highest Frequency**

T **Sampling Period**





Down-Sampling Operator



$$y[n] = S_{1/D}\, x[n] = x[nD]$$

Decrease sampling frequency by a factor of 1/D

Increase sampling period by a factor of D

$D = 2$

$y[0] = x[0 \cdot 2]$

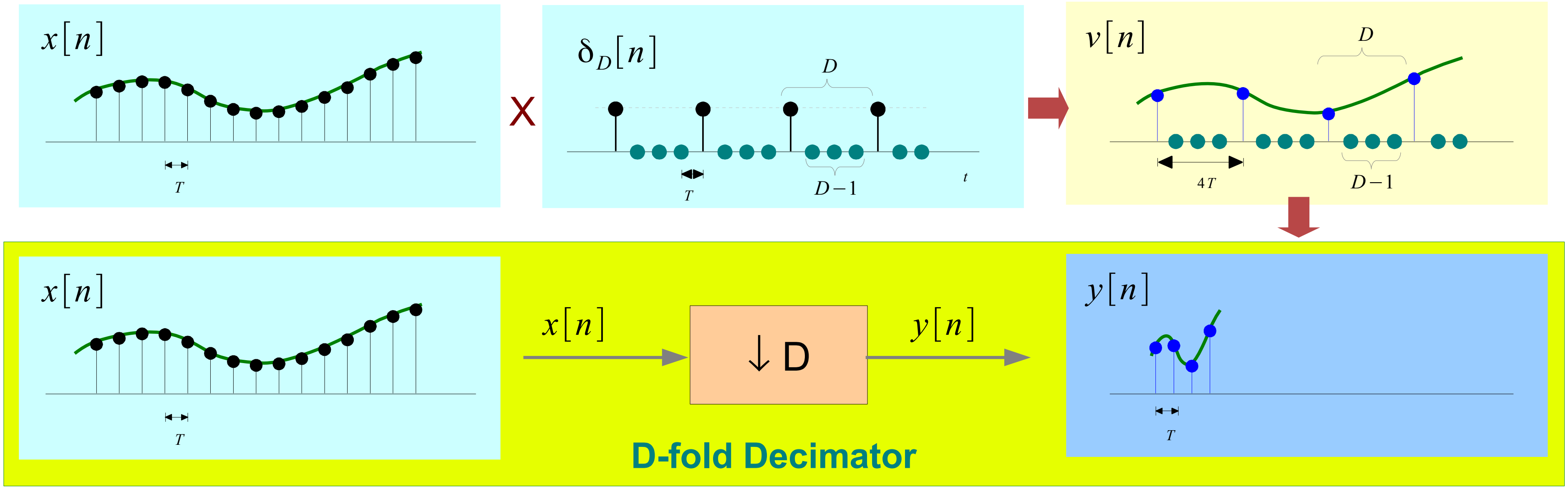
$y[1] = x[1 \cdot 2]$

$y[2] = x[2 \cdot 2]$

$y[3] = x[3 \cdot 2]$

$\cdots$

Down-Sampling Operator



$$y[n] = S_{1/D}\, x[n] = x[nD]$$

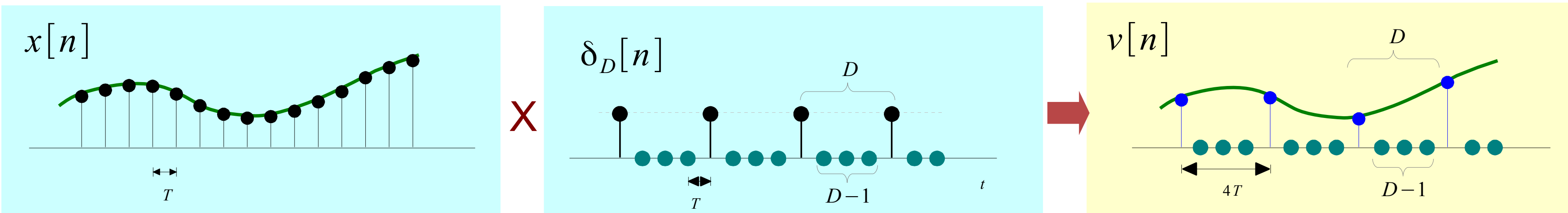
$x[n] = e^{j\hat{\omega} n}$ ➡ $y[n] = e^{j\hat{\omega} D n}$

$-\pi \le \hat{\omega} \le +\pi$ $\quad$ $-\pi \le \hat{\omega} D \le +\pi$ $\quad$ stretched

$-\frac{\pi}{D} \le \hat{\omega}_1 \le +\frac{\pi}{D}$ $\quad$ $-\pi \le \hat{\omega}_1 D \le +\pi$ $\quad$ w/o aliasing

$\hat{\omega}_2 > +\frac{\pi}{D}$ $\quad$ $\hat{\omega}_2 D > +\pi$ $\quad$ with aliasing

Suppressing D-1 Samples (1)



$$\delta_D[n] = \frac{1}{D}\sum_{k=0}^{D-1} e^{-j2\pi k n/D} = \begin{cases} \frac{1}{D} D = 1 & \text{if } \mathbf{mod}(n, D) = 0 \\ \frac{1}{D}\frac{1 - e^{-j2\pi n}}{1 - e^{-j2\pi n/D}} = 0 & \text{otherwise} \end{cases}$$

$$\left(e^{-j2\pi k(n/D)} = e^{-j2\pi k m} = 1\right)$$

$$v[n] = \delta_D[n]x[n] = \frac{1}{D}\sum_{k=0}^{D-1} x[n]e^{-j2\pi k n/D}$$

$$\hat{\omega}_k = 2\pi k/D$$

$$V(z) = \frac{1}{D}\sum_{k=0}^{D-1} X(e^{-j2\pi k/D}\cdot z)$$

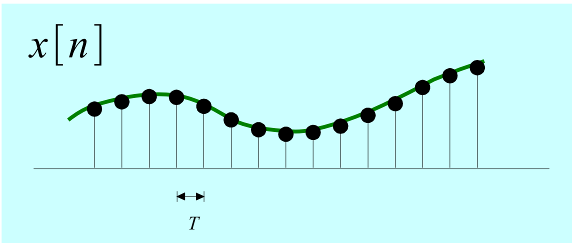
$$Z\{x[n]e^{-j\hat{\omega}_k n}\} = \sum_{n=-\infty}^{+\infty} x[n](e^{j\hat{\omega}_k}\cdot z)^{-n} = X(e^{j\hat{\omega}_k}\cdot z)$$

$$z = e^{j\hat{\omega}}$$

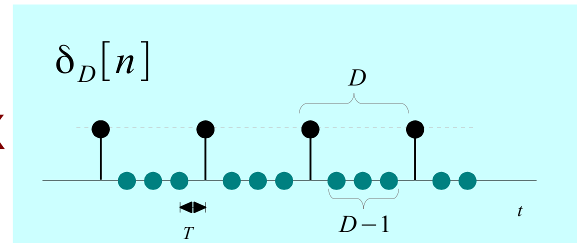
$$V(\hat{\omega}) = \frac{1}{D}\sum_{k=0}^{D-1} X(\hat{\omega} - 2\pi k/D)$$

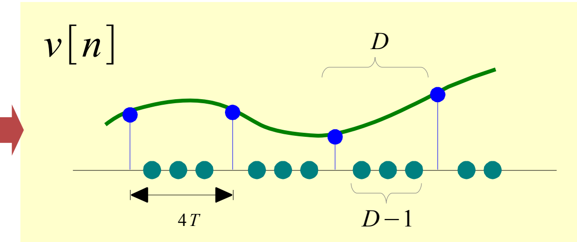
$$X(e^{-j2\pi k/D}\cdot z) = X(e^{-j\omega_k}e^{j\hat{\omega}}) = X(e^{j(\hat{\omega}-\omega_k)})$$

Removing D-1 Samples and Packing (2)



X





$$V(z) = \cdots + v[0]z^{0} + 0 + \cdots + 0 + v[D]z^{-D} + 0 + \cdots + 0 + v[2D]z^{-2D} + \cdots$$

$$V(z) = \sum_{n=-\infty}^{+\infty} v[n]z^{-n} = \sum_{m=-\infty}^{+\infty} v[mD]z^{-mD} = F(z^{D})$$

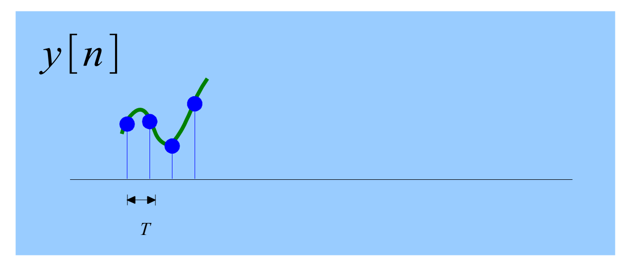
$$V(z) = \frac{1}{D}\sum_{k=0}^{D-1} X(e^{-j2\pi k/D}\cdot z)$$

$$Y(z) = \cdots + v[0]z^{0} + v[D]z^{-1} + v[2D]z^{-2} + v[3D]z^{-3} + \cdots$$

$$= \cdots + y[0]z^{0} + y[1]z^{-1} + y[2]z^{-2} + y[3]z^{-3} + \cdots$$

$$V(\hat{\omega}) = \frac{1}{D}\sum_{k=0}^{D-1} X(\hat{\omega} - 2\pi k/D)$$

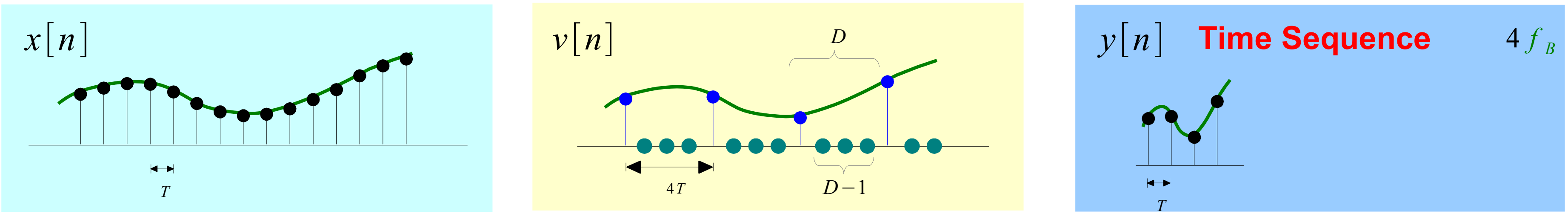
$$V(z^{1/D}) = \sum_{n=-\infty}^{+\infty} v[n]z^{-n/D} = \sum_{m=-\infty}^{+\infty} v[mD]z^{-m}$$



$$Y(z) = V(z^{1/D}) = \frac{1}{D}\sum_{k=0}^{D-1} X(e^{-j2\pi k/D}\cdot z^{1/D})$$

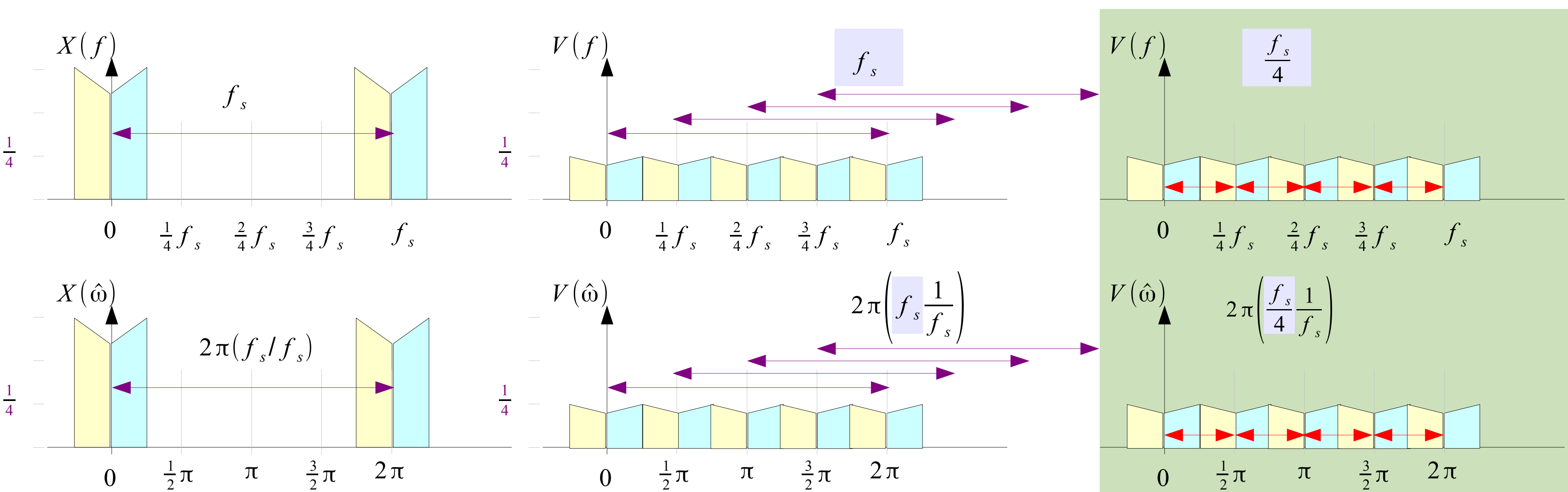
$$Y(\hat{\omega}) = \frac{1}{D}\sum_{k=0}^{D-1} X\left(\frac{\hat{\omega}}{D} - k\frac{2\pi}{D}\right)$$

Down-Sampling Spectra (1)

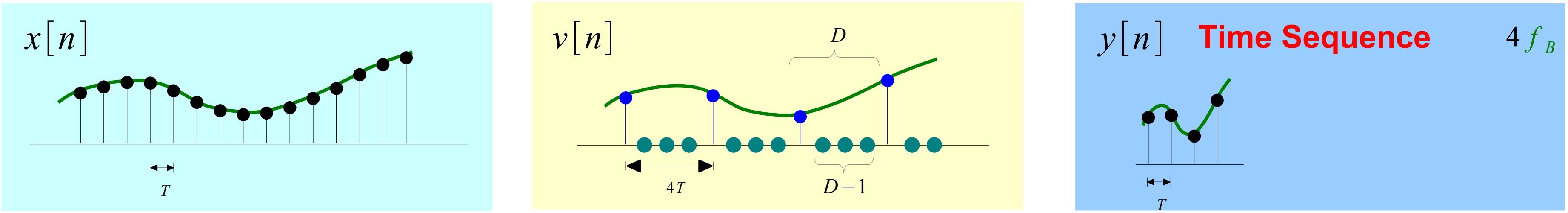


$$X(z) \qquad V(z) = \frac{1}{D}\sum_{k=0}^{D-1} X\left(e^{-j2\pi k/D}\cdot z\right) \qquad Y(z) = \frac{1}{D}\sum_{k=0}^{D-1} X\left(e^{-j2\pi k/D}\cdot z^{1/D}\right)$$

$$X(\hat{\omega}) \qquad V(\hat{\omega}) = \frac{1}{D}\sum_{k=0}^{D-1} X\left(\hat{\omega} - 2\pi k/D\right) \qquad Y(\hat{\omega}) = \frac{1}{D}\sum_{k=0}^{D-1} X\left(\hat{\omega}/D - 2\pi k/D\right)$$

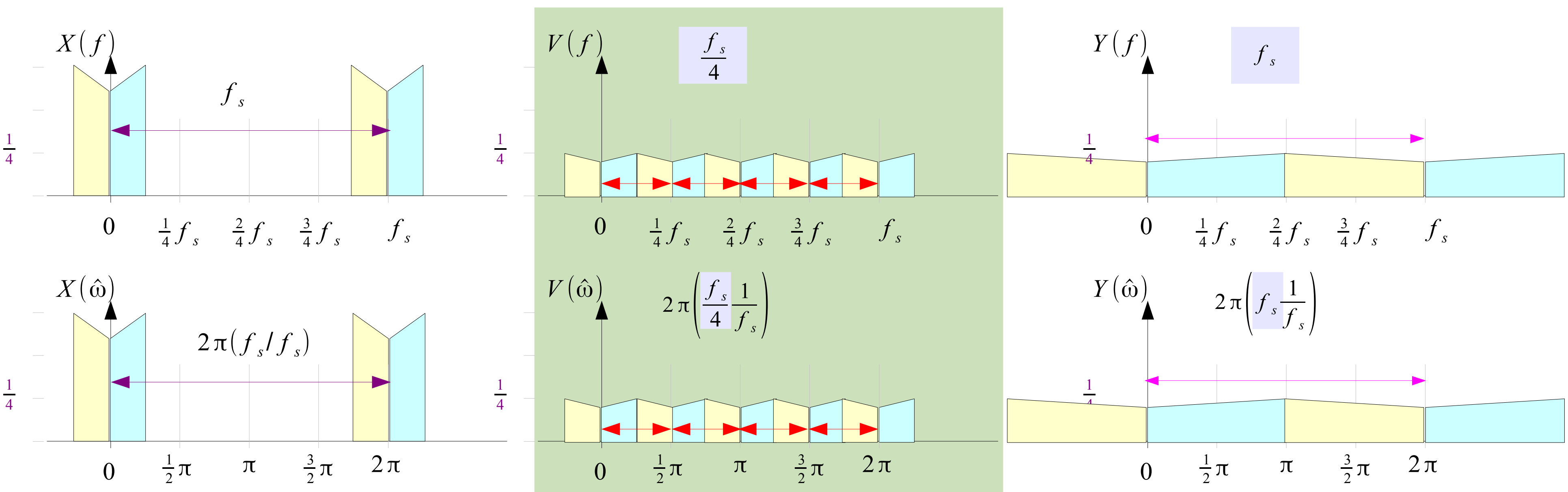


Down-Sampling Spectra (1)

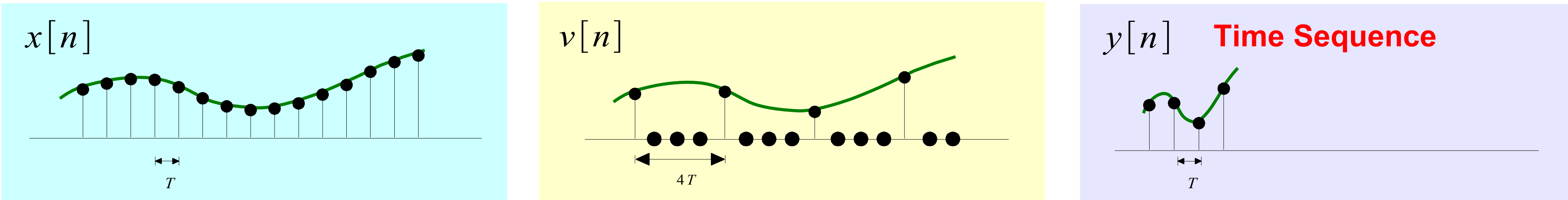


$$X(z) \qquad V(z) = \frac{1}{D}\sum_{k=0}^{D-1} X\left(e^{-j2\pi k/D}\cdot z\right) \qquad Y(z) = \frac{1}{D}\sum_{k=0}^{D-1} X\left(e^{-j2\pi k/D}\cdot z^{1/D}\right)$$

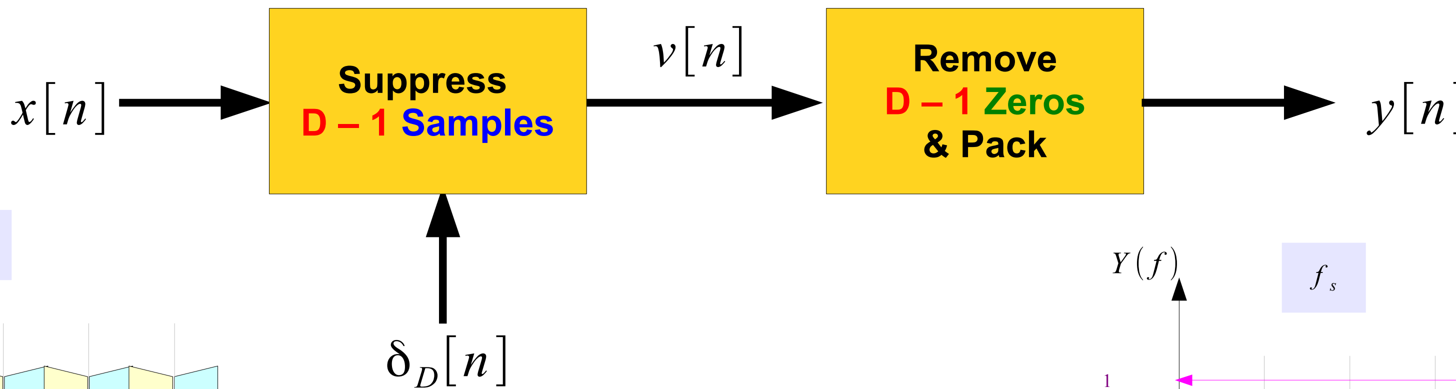
$$X(\hat{\omega}) \qquad V(\hat{\omega}) = \frac{1}{D}\sum_{k=0}^{D-1} X(\hat{\omega} - 2\pi k/D) \qquad Y(\hat{\omega}) = \frac{1}{D}\sum_{k=0}^{D-1} X(\hat{\omega}/D - 2\pi k/D)$$

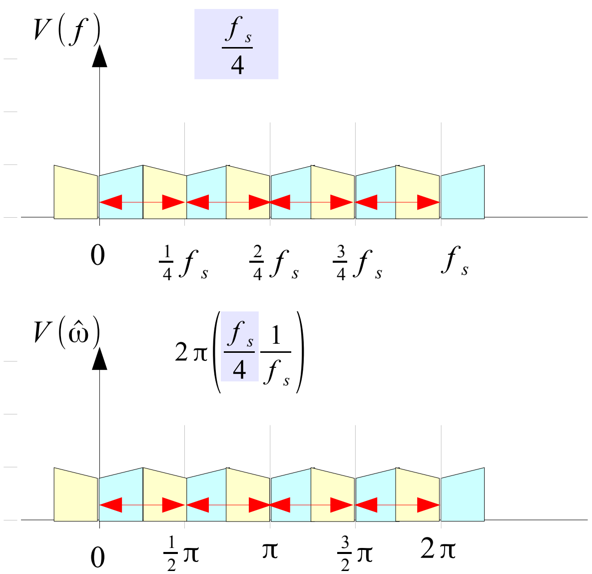


Down Sampling



T **Sampling Period**



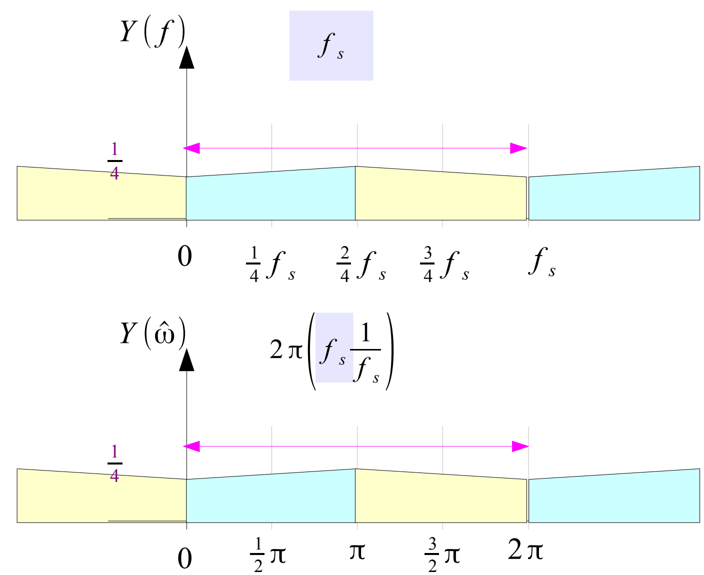


Amplitude Axis Scaling 1 / D

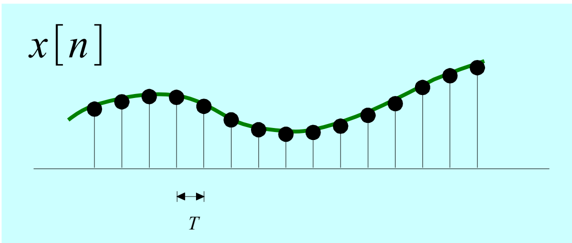
Time Axis Scaling 1 / D

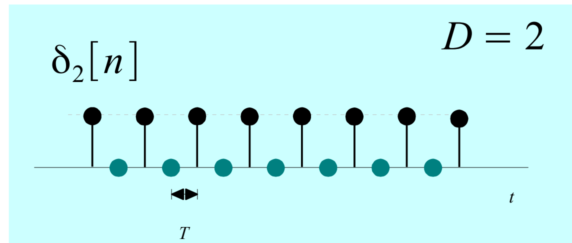
Spectrum Replication f_s / D

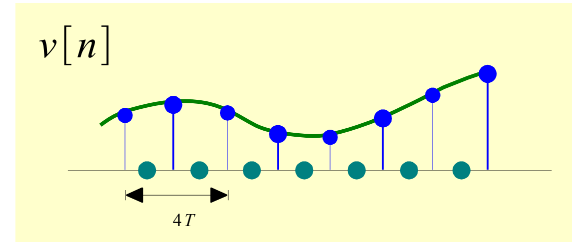
Stretching Spectrum D



Example When D=2 (1)



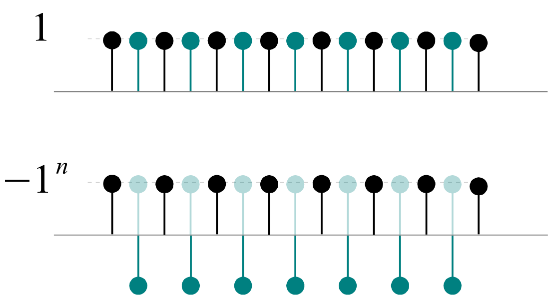




$$x[n] = e^{j\omega n}$$

$$\delta_2[n] = \tfrac{1}{2}\left(1 + (-1)^n\right) = \tfrac{1}{2}\left(1 + e^{-j\pi n}\right)$$

$$\left(e^{-j\pi} = -1\right)$$



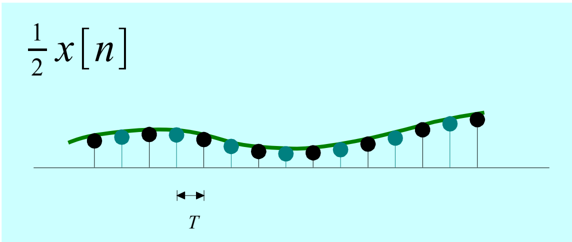
$$v[n] = \tfrac{1}{2}x[n] + \tfrac{1}{2}e^{-j\pi n}x[n] = \tfrac{1}{2}e^{j\omega n} + \tfrac{1}{2}e^{-j\pi n}e^{j\omega n} = \tfrac{1}{2}e^{j\omega n} + \tfrac{1}{2}e^{+j(\omega-\pi)n}$$

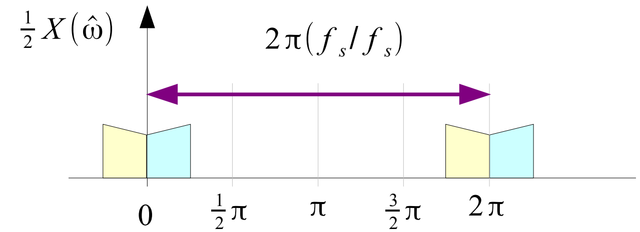
$$V(z) = \tfrac{1}{2}\sum_{n=-\infty}^{+\infty}\left(x[n]z^{-n} + x[n](-z)^{-n}\right) = \tfrac{1}{2}X(z) + \tfrac{1}{2}X(-z)$$

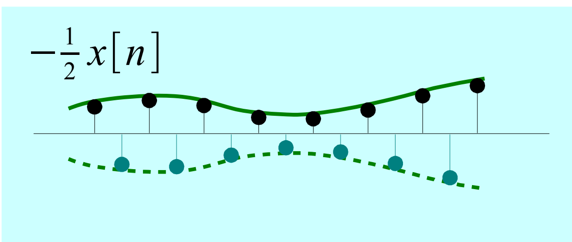
$$V(e^{j\hat{\omega}}) = \tfrac{1}{2}X(e^{j\hat{\omega}}) + \tfrac{1}{2}X(e^{-j\pi}e^{j\hat{\omega}})$$

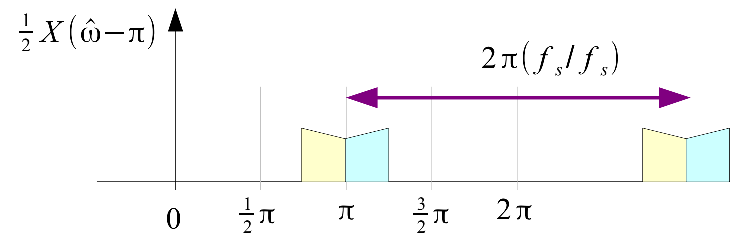
$$V(\hat{\omega}) = \tfrac{1}{2}X(\hat{\omega}) + \tfrac{1}{2}X(\hat{\omega} - \pi)$$

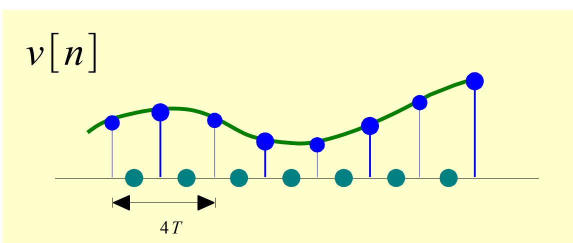
Example When D=2 (2)

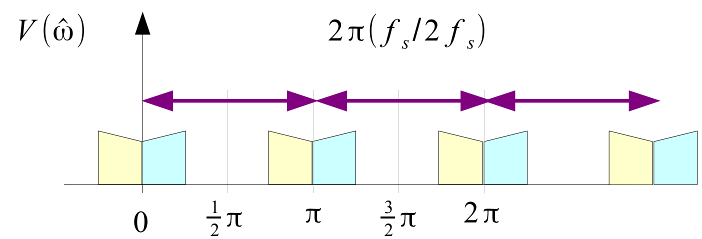












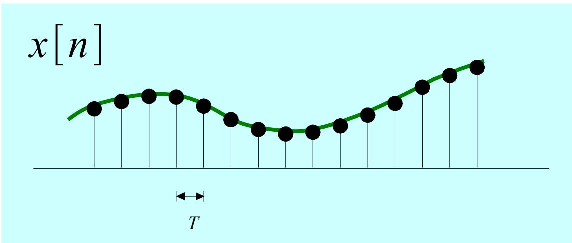
$$v[n] = \tfrac{1}{2}x[n] + \tfrac{1}{2}e^{-j\pi n}x[n]$$

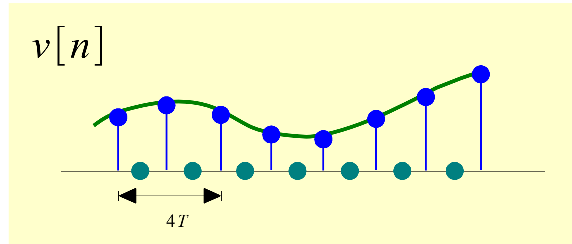
$$V(z) = \tfrac{1}{2}X(z) + \tfrac{1}{2}X(-z)$$

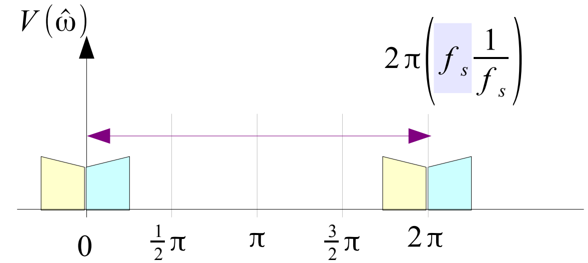
$$V(e^{j\hat{\omega}}) = \tfrac{1}{2}X(e^{j\hat{\omega}}) + \tfrac{1}{2}X(e^{-j\pi}e^{j\hat{\omega}})$$

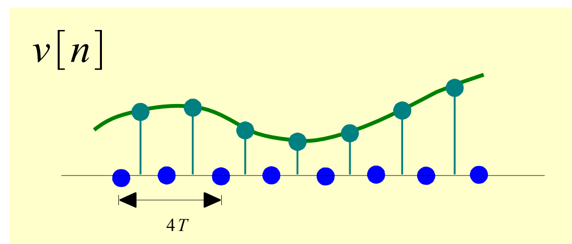
$$V(\hat{\omega}) = \tfrac{1}{2}X(\hat{\omega}) + \tfrac{1}{2}X(\hat{\omega}-\pi)$$

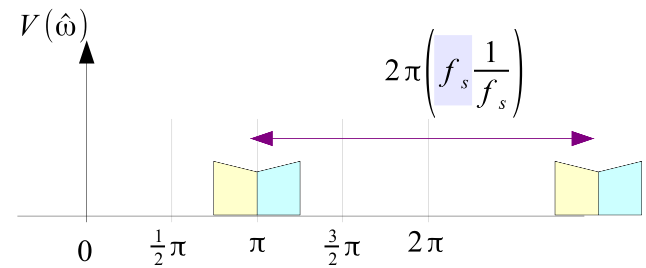
Example When D=2 (3)



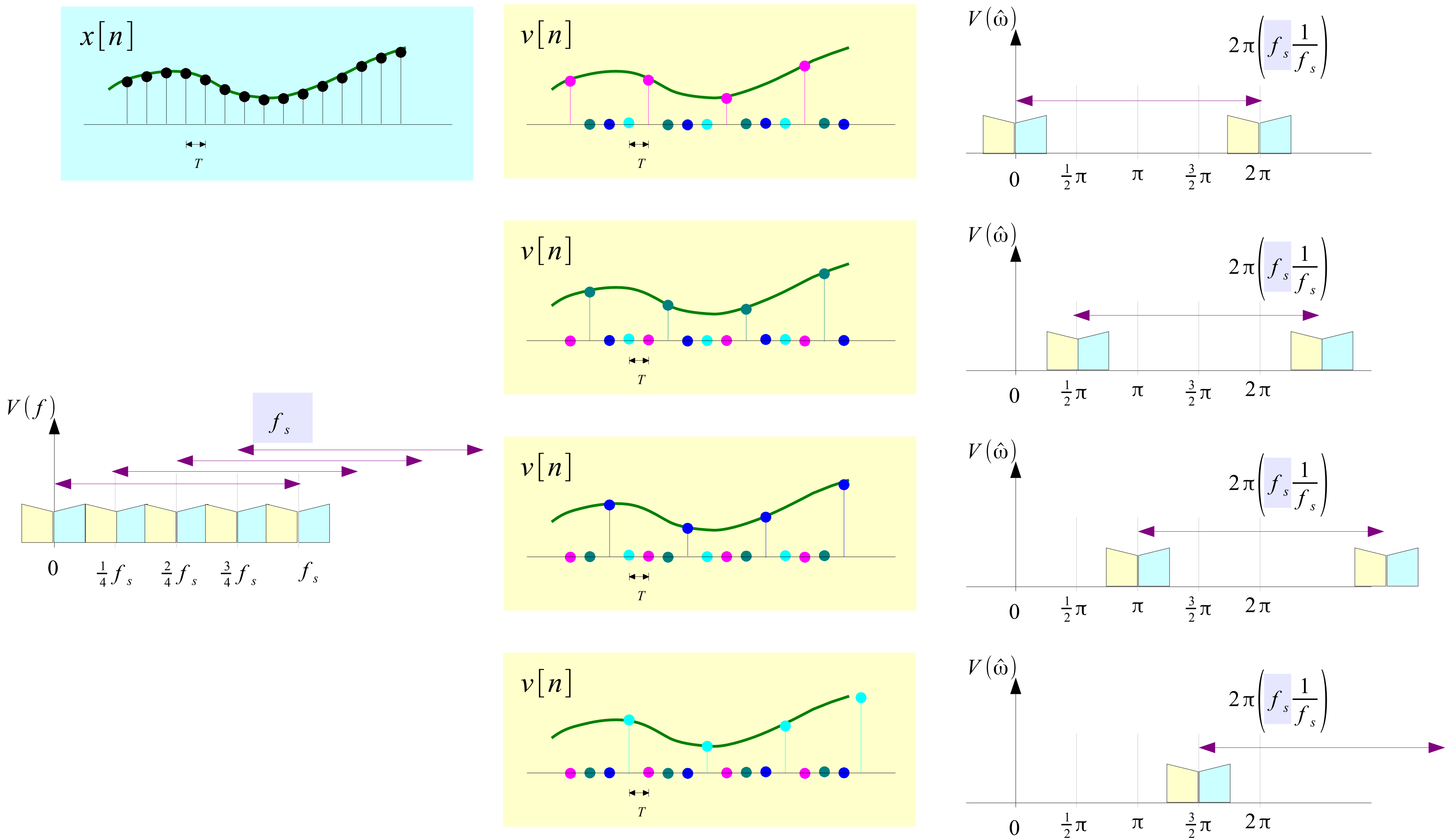




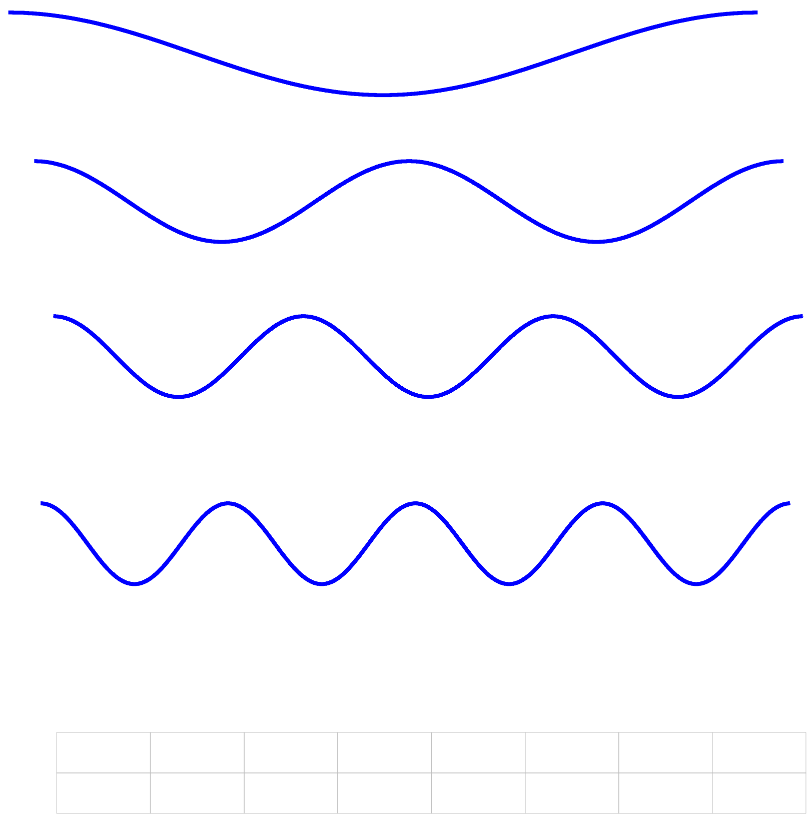


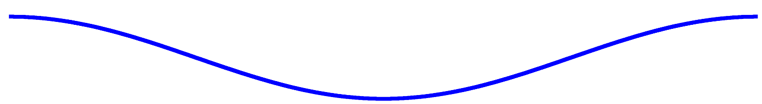


Example When D=2 (3)

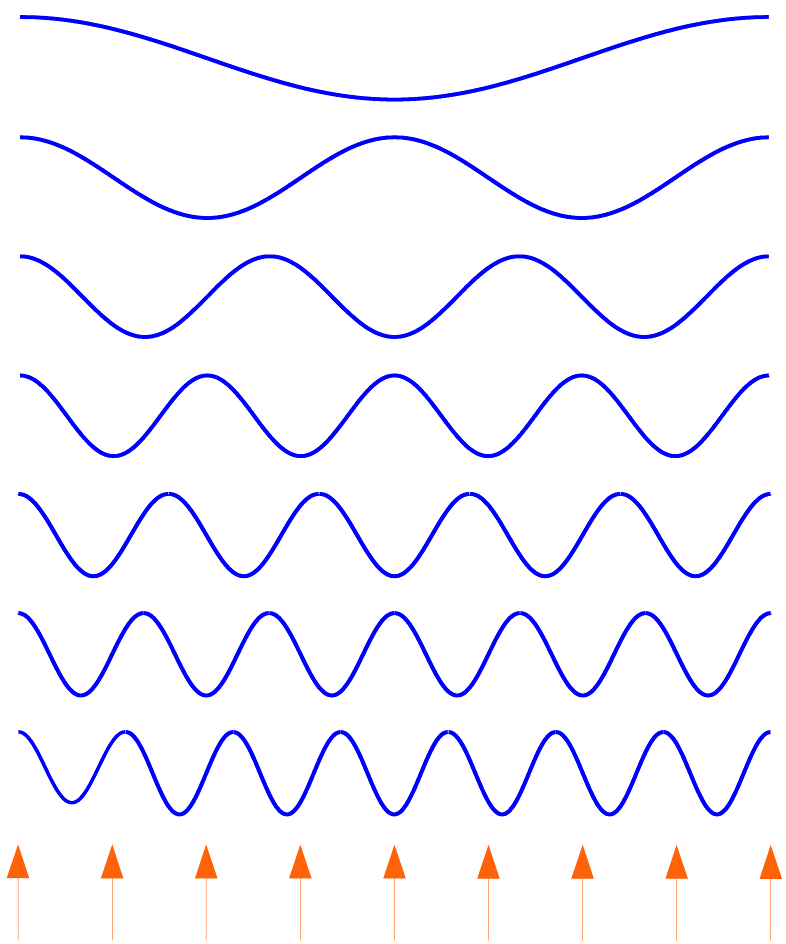


Measuring Rotation Rate





Signals with Harmonic Frequencies (1)



1 Hz *1 cycle / sec*
$$\cos(1\cdot 2\pi t) = \frac{e^{+j(1\cdot 2\pi)t} + e^{-j(1\cdot 2\pi)t}}{2}$$

2 Hz *2 cycles / sec*
$$\cos(2\cdot 2\pi t) = \frac{e^{+j(2\cdot 2\pi)t} + e^{-j(2\cdot 2\pi)t}}{2}$$

3 Hz *3 cycles / sec*
$$\cos(3\cdot 2\pi t) = \frac{e^{+j(3\cdot 2\pi)t} + e^{-j(3\cdot 2\pi)t}}{2}$$

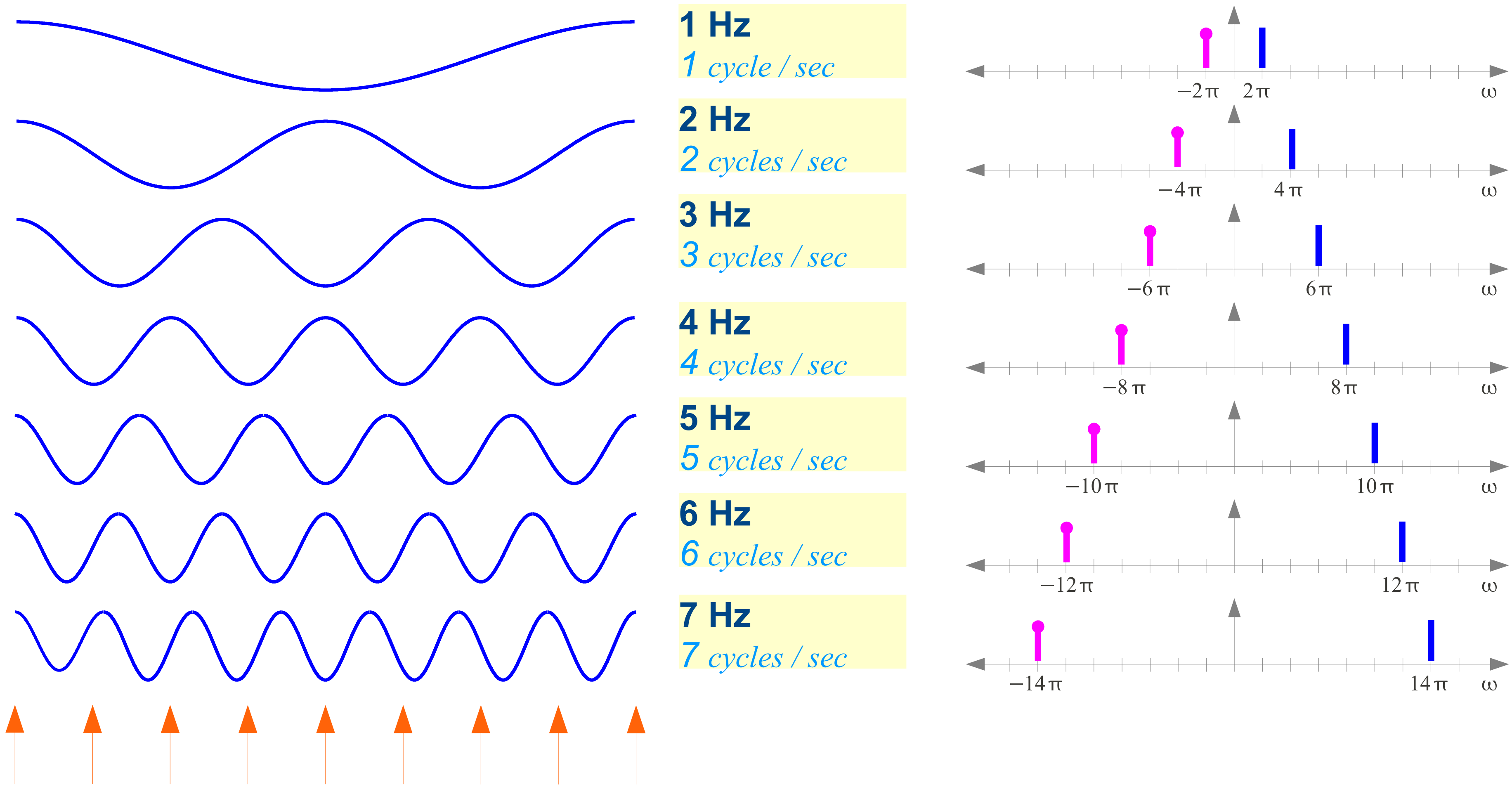
4 Hz *4 cycles / sec*
$$\cos(4\cdot 2\pi t) = \frac{e^{+j(4\cdot 2\pi)t} + e^{-j(4\cdot 2\pi)t}}{2}$$

5 Hz *5 cycles / sec*
$$\cos(5\cdot 2\pi t) = \frac{e^{+j(5\cdot 2\pi)t} + e^{-j(5\cdot 2\pi)t}}{2}$$

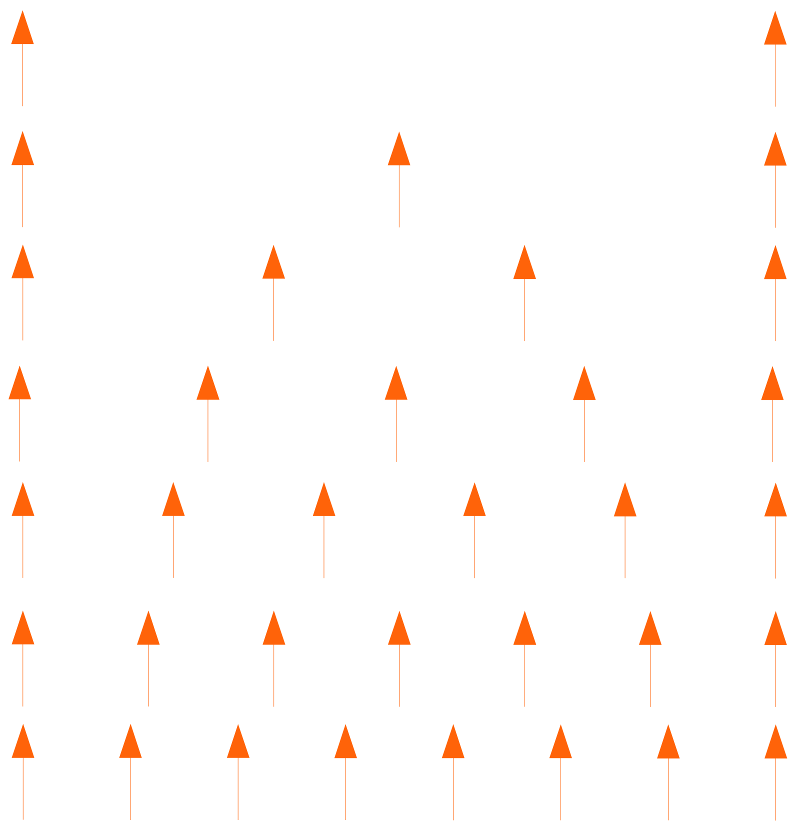
6 Hz *6 cycles / sec*
$$\cos(6\cdot 2\pi t) = \frac{e^{+j(6\cdot 2\pi)t} + e^{-j(6\cdot 2\pi)t}}{2}$$

7 Hz *7 cycles / sec*
$$\cos(7\cdot 2\pi t) = \frac{e^{+j(7\cdot 2\pi)t} + e^{-j(7\cdot 2\pi)t}}{2}$$

Signals with Harmonic Frequencies (2)



Sampling Frequency



1 Hz
1 sample / sec

2 Hz
2 samples / sec

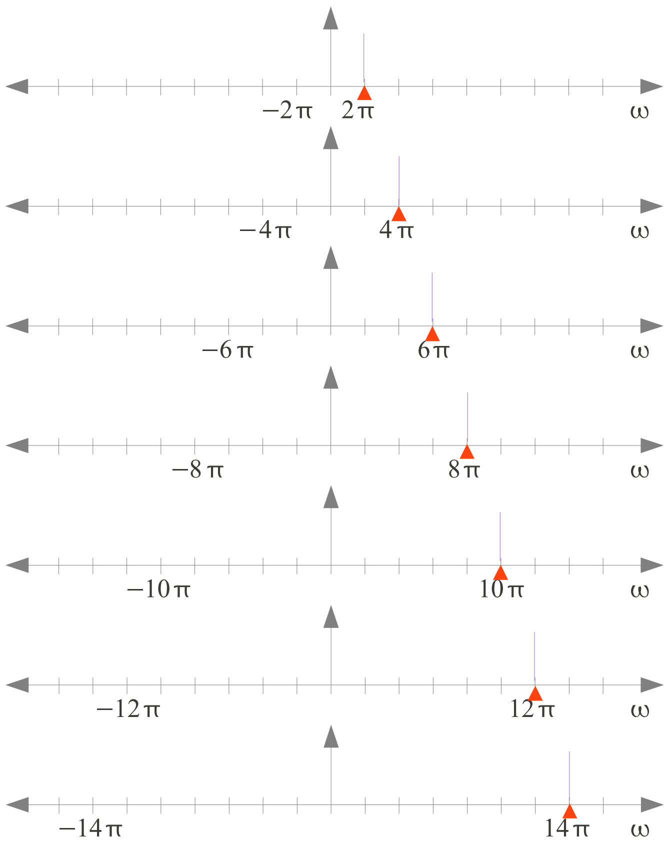
3 Hz
3 samples / sec

4 Hz
4 samples / sec

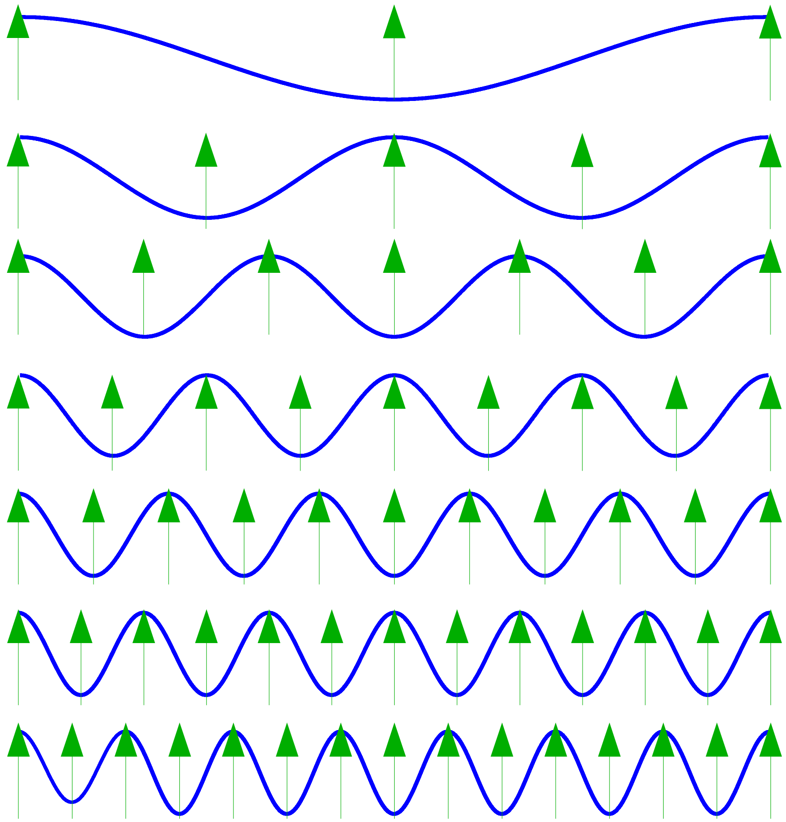
5 Hz
5 samples / sec

6 Hz
6 samples / sec

7 Hz
7 samples / sec

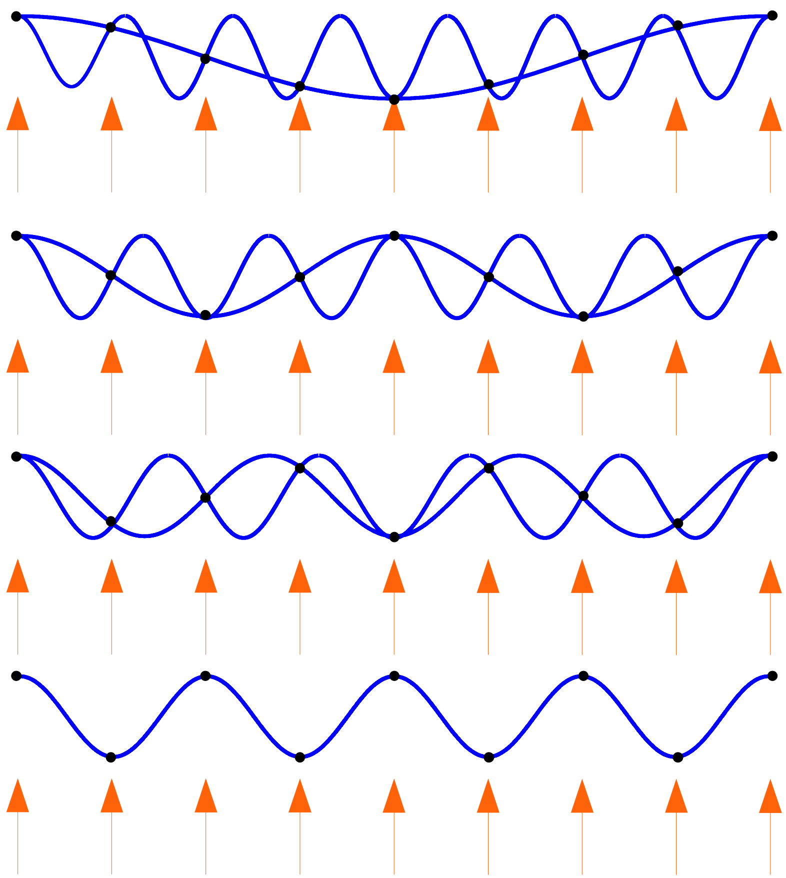


Nyquist Frequency



1 Hz *1 cycle / sec*	*2x1 sample / sec*
2 Hz *2 cycles / sec*	*2x2 samples / sec*
3 Hz *3 cycles / sec*	*2x3 samples / sec*
4 Hz *4 cycles / sec*	*2x4 samples / sec*
5 Hz *5 cycles / sec*	*2x5 samples / sec*
6 Hz *6 cycles / sec*	*2x6 samples / sec*
7 Hz *7 cycles / sec*	*2x7 samples / sec*

Aliasing



1 Hz
7 Hz

2x4 samples / sec

2 Hz
6 Hz

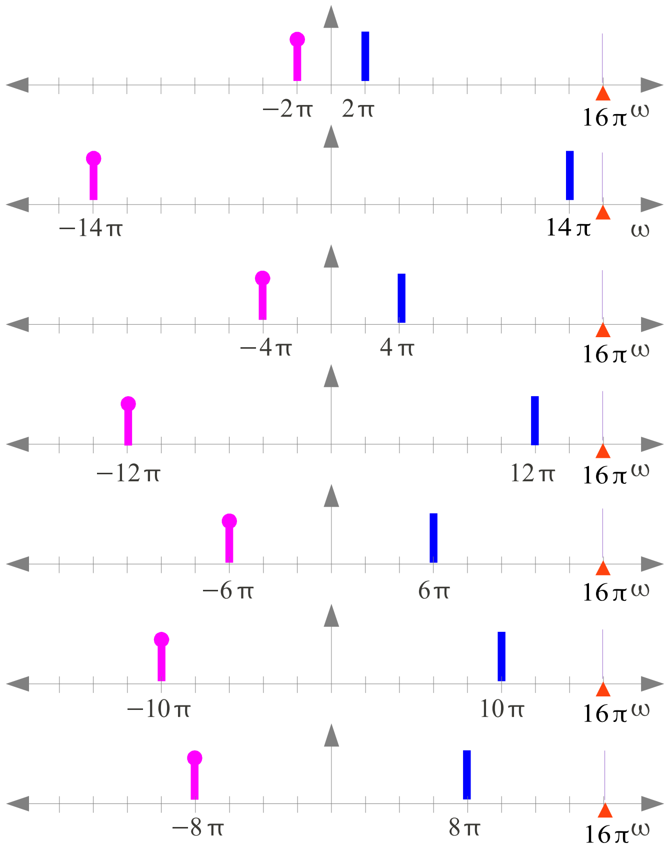
2x4 samples / sec

3 Hz
5 Hz

2x4 samples / sec

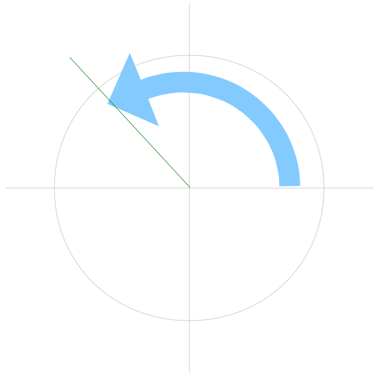
4 Hz

2x4 samples / sec



Sampling

$$\omega_s = 2\pi f_s \ (rad/sec)$$



$$\omega_1 = 2\pi f_1$$

$$\omega_1 = \frac{\omega_s}{2} \ (rad/sec)$$

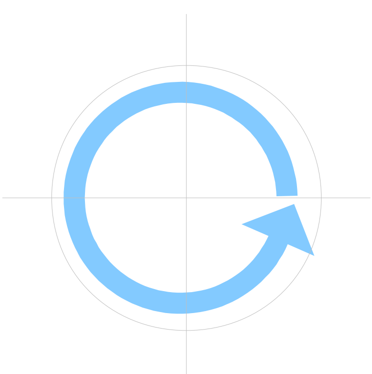
$$f_1 = \frac{f_s}{2} \ (rad/sec)$$

$$\omega_2 = 2\pi f_2$$

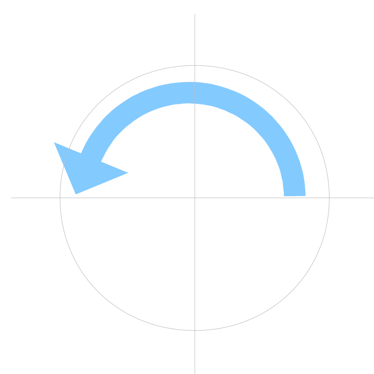
$$\omega_2 = -\frac{\omega_s}{2} \ (rad/sec)$$

$$f_2 = -\frac{f_s}{2} \ (rad/sec)$$

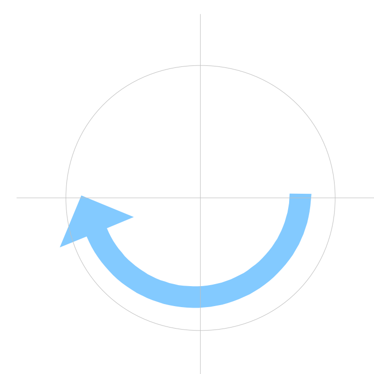
$$2\pi \ (rad) \ / \ T_s \ (sec)$$



$$\pi \ (rad) \ / \ T_s \ (sec)$$

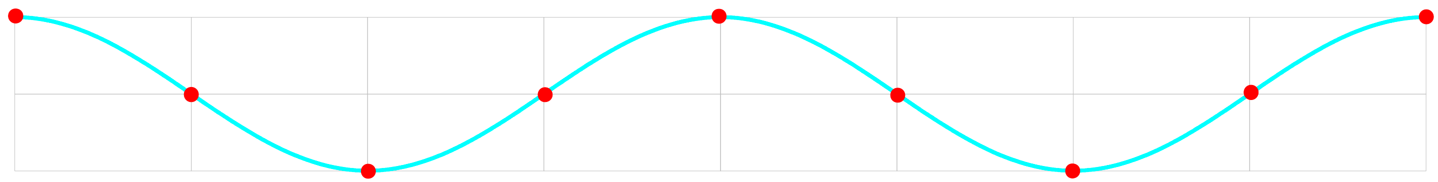


$$-\pi \ (rad) \ / \ T_s \ (sec)$$

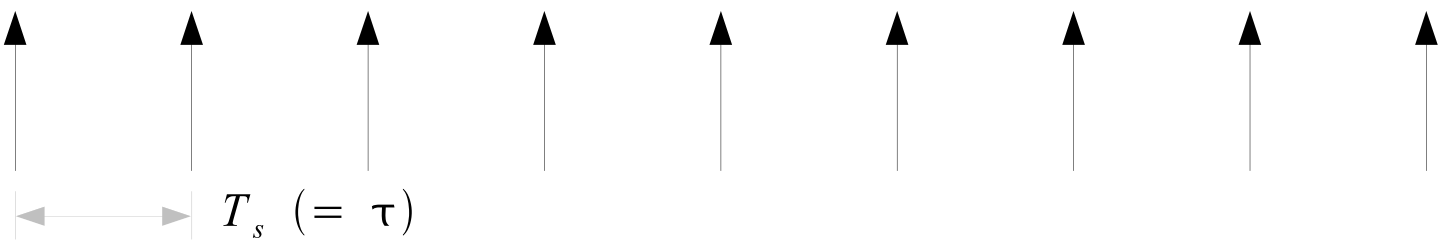


Sampling

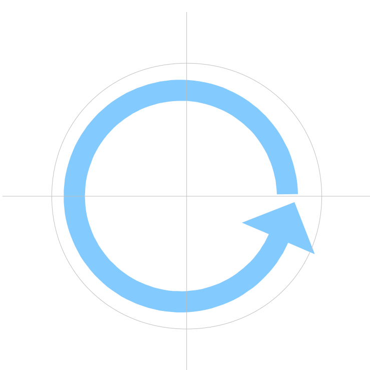
$\omega_1 = 2\pi f_1 \ (rad/sec)$



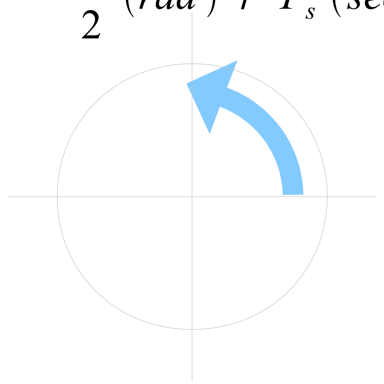
$\omega_s = 2\pi f_s \ (rad/sec)$



$2\pi \ (rad) \ / \ T_s \ (sec)$



$\frac{\pi}{2} \ (rad) \ / \ T_s \ (sec)$



For the period of T_s

Angular displacement $\frac{\pi}{2} \ (rad)$

$$\hat{\omega} = \omega \cdot T_s \ (rad)$$

$$= 2\pi f_1 \cdot T_s \ (rad)$$

$$= 2\pi \frac{f_s}{4} \cdot T_s \ (rad)$$

$$= \frac{\pi}{2} \ (rad)$$

Angular Frequencies in Sampling

continuous-time signals

Signal Frequency

$$f_0 = \frac{1}{T_0}$$

Signal Angular Frequency

$$\omega_0 = 2\pi f_0 \quad (rad/sec)$$

sampling sequence

Sampling Frequency

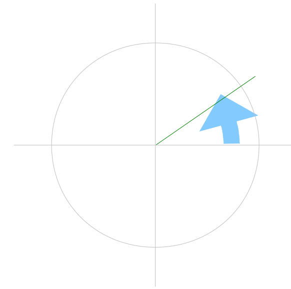
$$f_s = \frac{1}{T_s}$$

Sampling Angular Frequency

$$\omega_s = 2\pi f_s \quad (rad/sec)$$

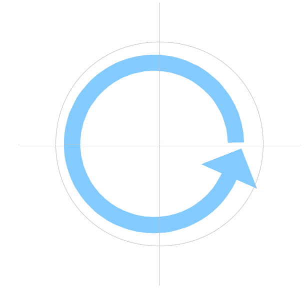
For 1 second

$$\omega_0 = 2\pi f_0 \quad (rad/sec)$$



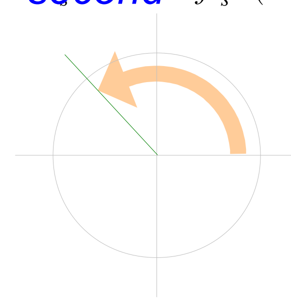
For 1 revolution

$$2\pi (rad) \, / \, T_0 (sec)$$



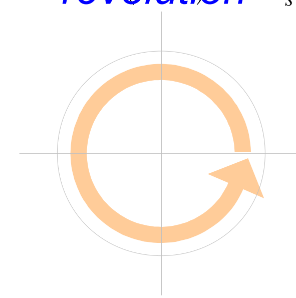
For 1 second

$$\omega_s = 2\pi f_s \quad (rad/sec)$$



For 1 revolution

$$2\pi (rad) \, / \, T_s (sec)$$



References

[1] http://en.wikipedia.org/
[2] J.H. McClellan, et al., Signal Processing First, Pearson Prentice Hall, 2003
[3] A “graphical interpretation” of the DFT and FFT, by Steve Mann
[4] R. Cristi, “Modern Digital Signal Processing”