

Matrix Transformation (2A)

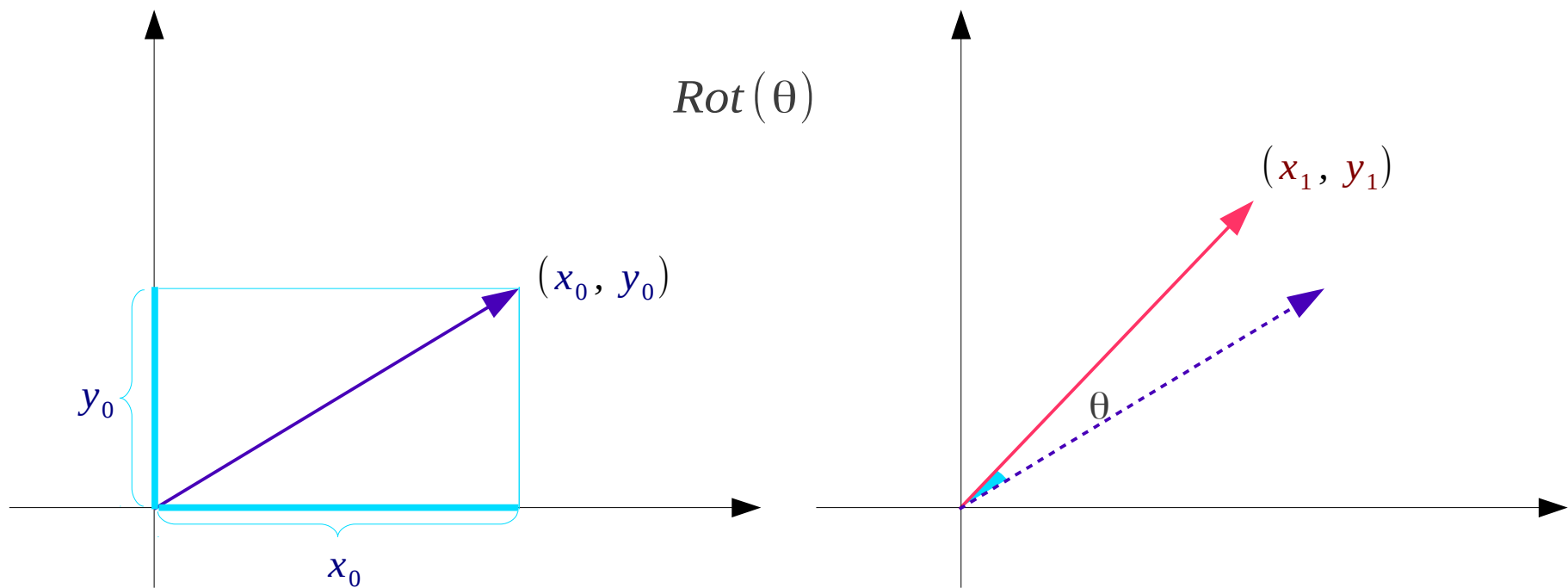
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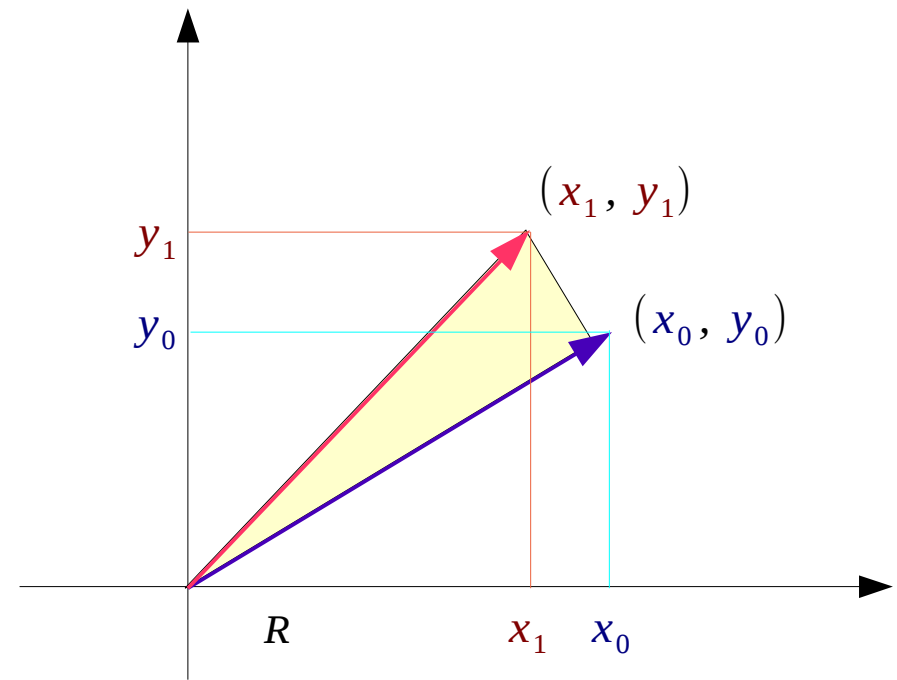
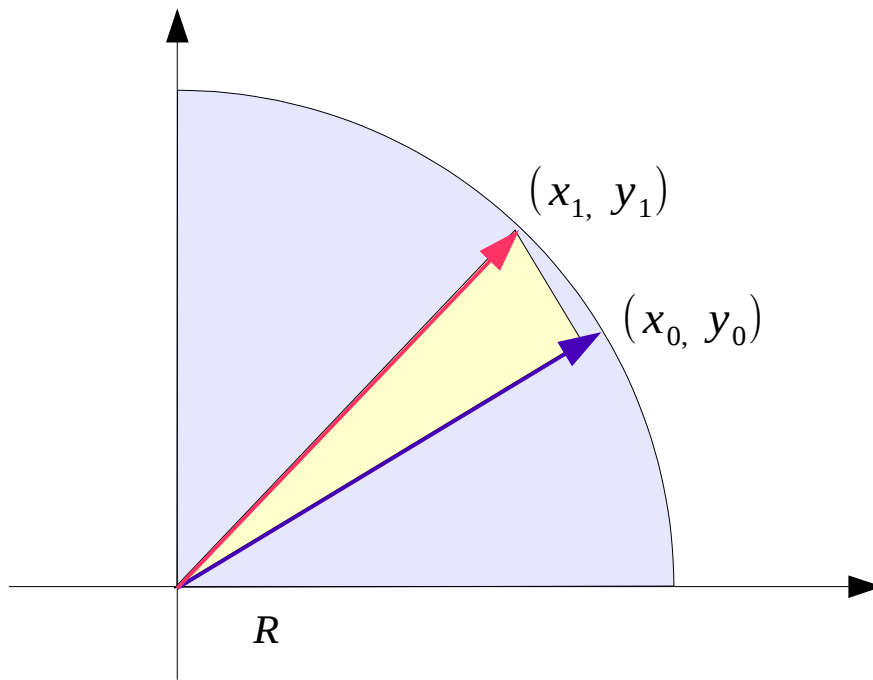
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Vector Rotation (1)



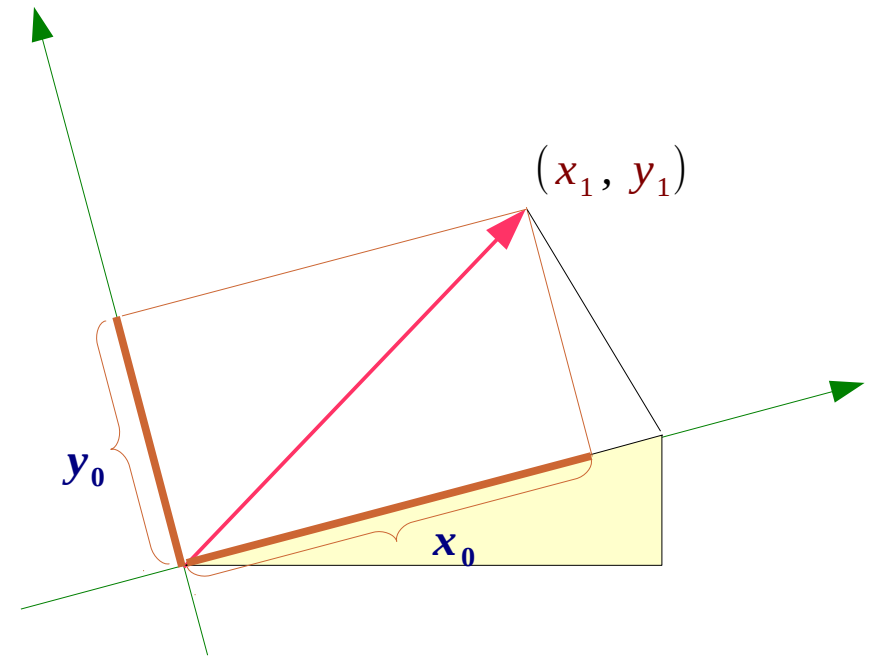
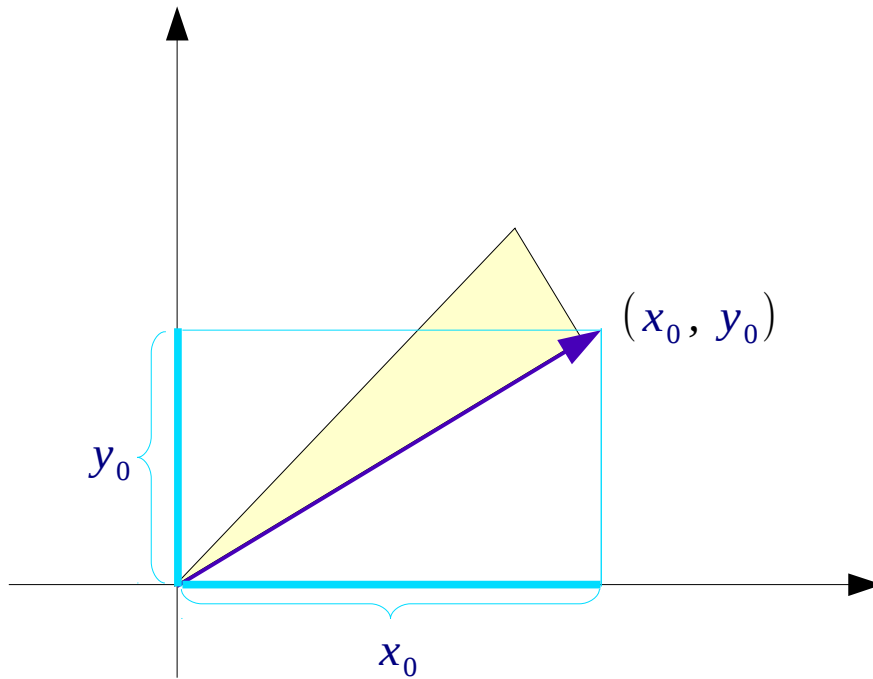
Vector Rotation (2)



$$x_1 = x_0 \cos \theta - y_0 \sin \theta$$

$$y_1 = x_0 \sin \theta + y_0 \cos \theta$$

Vector Rotation (3)



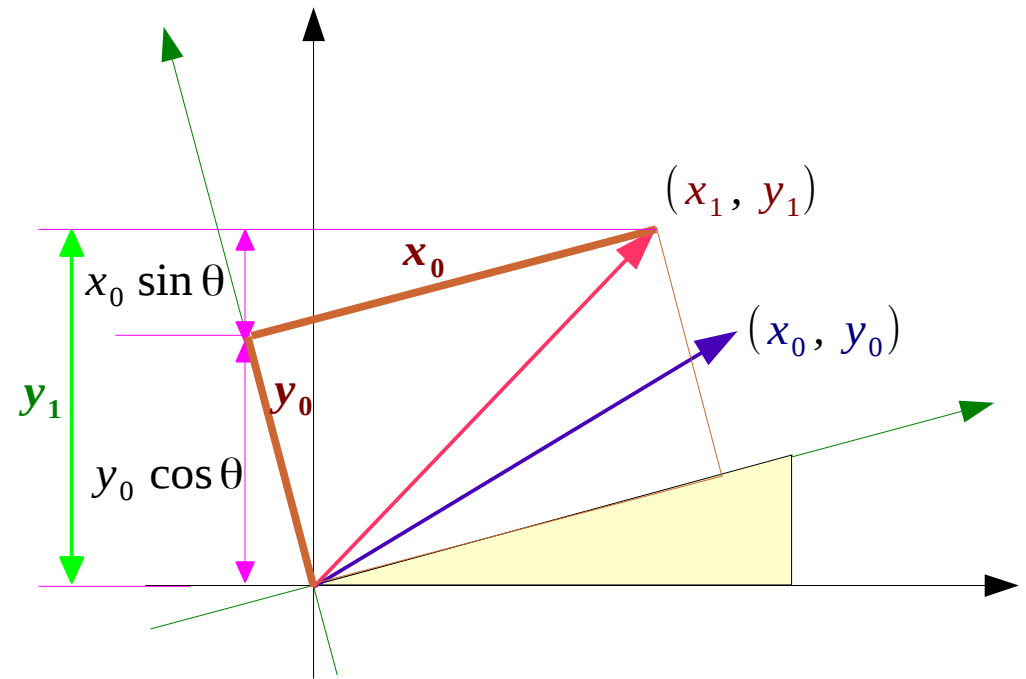
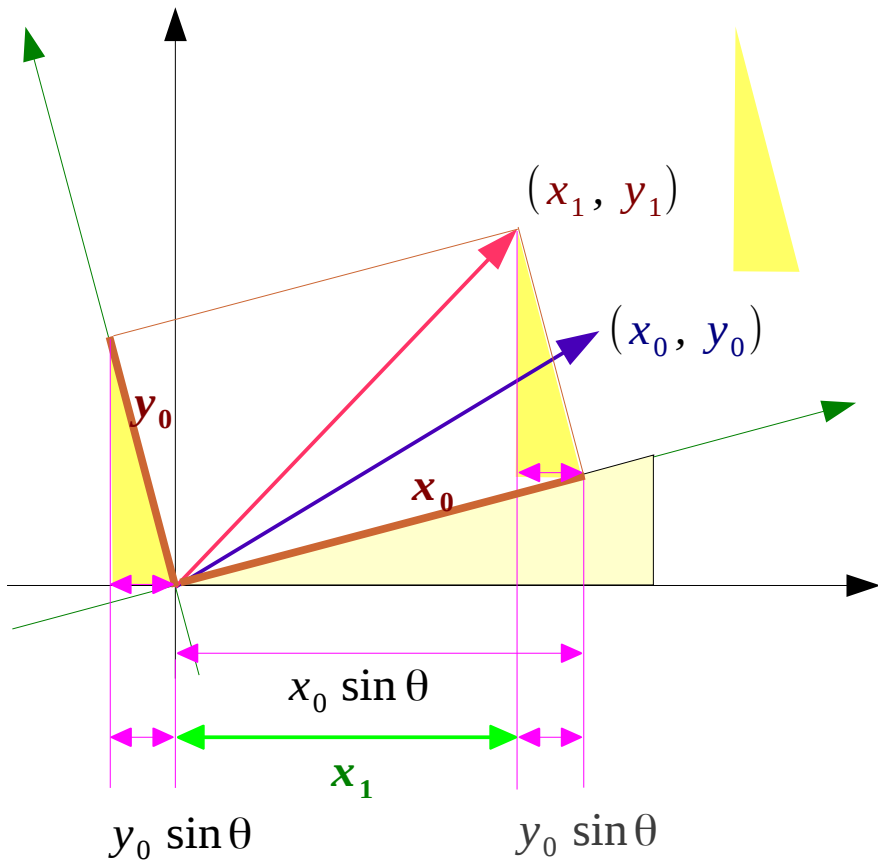
In the rotated coordinate

invariant length x_0, y_0

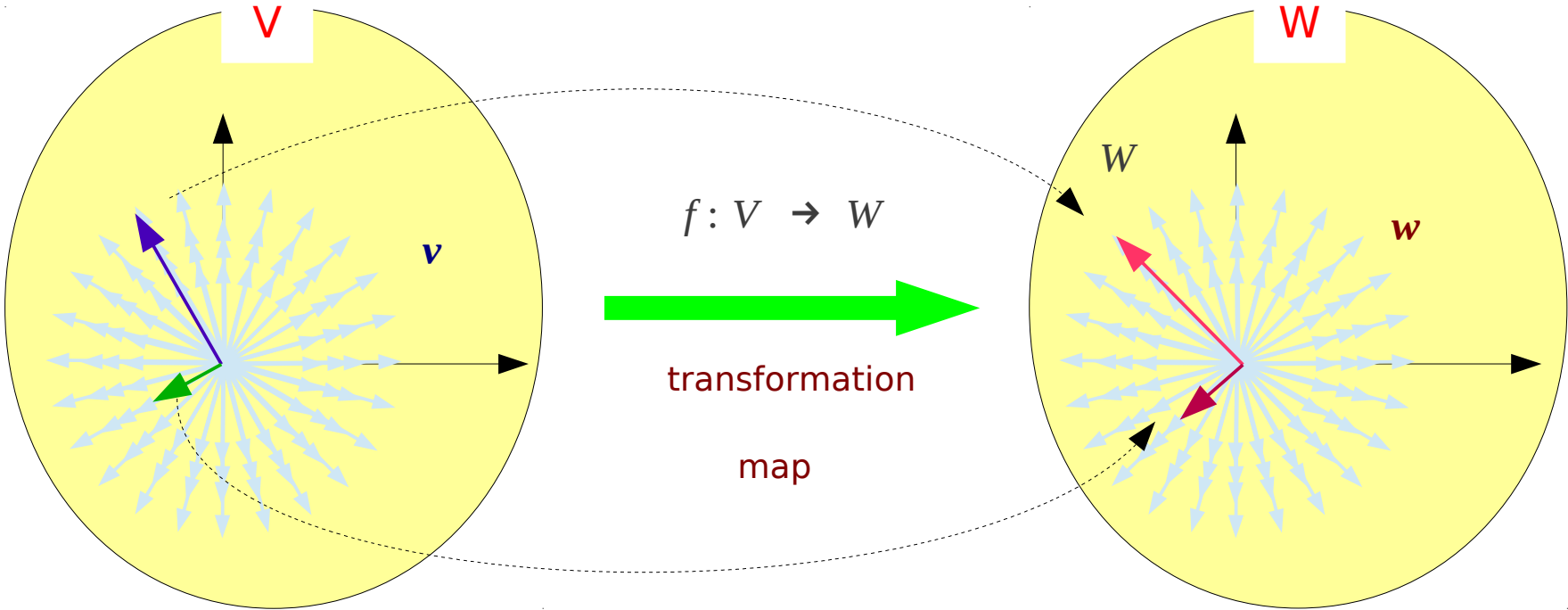
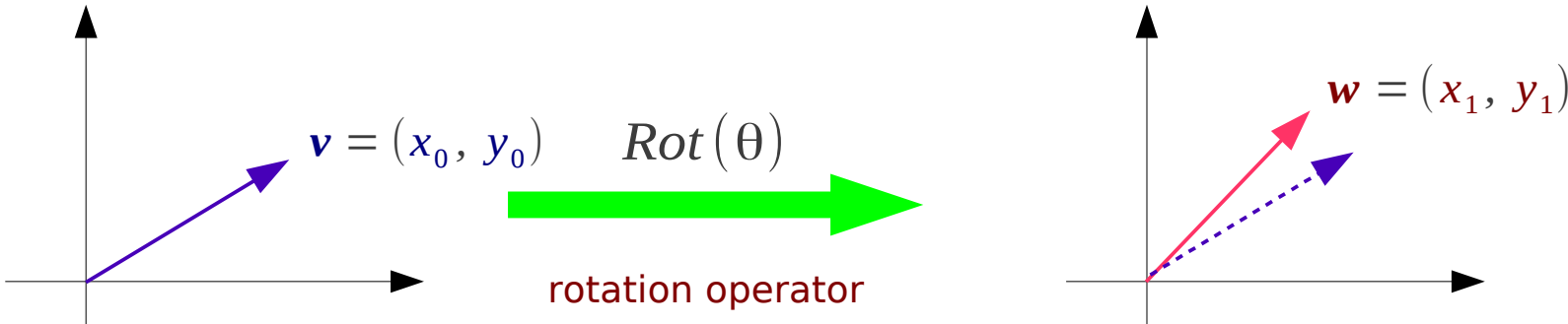
Vector Rotation (4)

$$x_1 = x_0 \cos \theta - y_0 \sin \theta$$

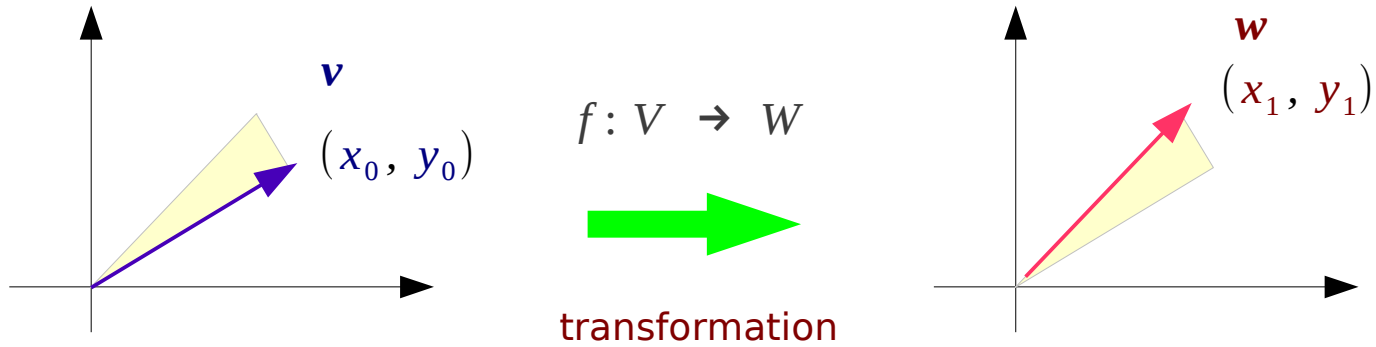
$$y_1 = x_0 \sin \theta + y_0 \cos \theta$$



Transformation



Matrix Transformation



$$\begin{aligned} x_1 &= x_0 \cos \theta - y_0 \sin \theta \\ y_1 &= x_0 \sin \theta + y_0 \cos \theta \end{aligned}$$

$$\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} = \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix} \begin{bmatrix} x_0 \\ y_0 \end{bmatrix}$$

$$\mathbf{A} = \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix}$$

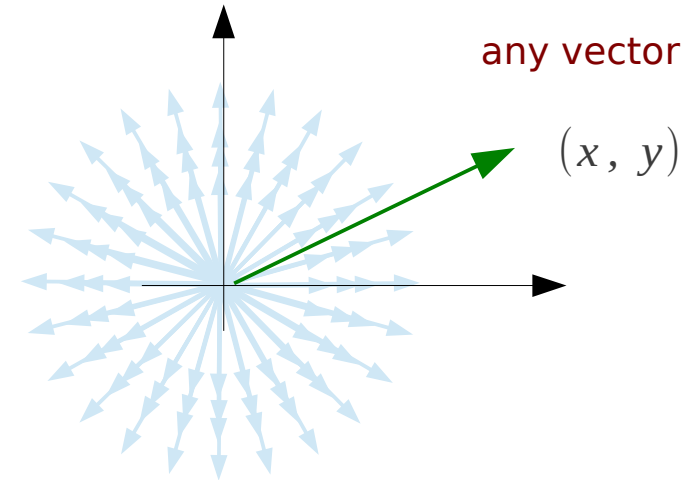
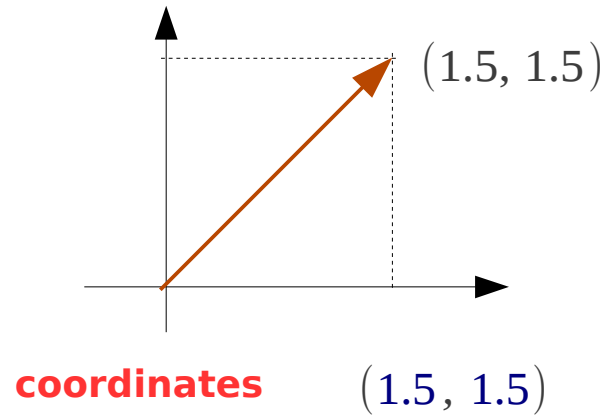
$$\mathbf{w} = \mathbf{A} \mathbf{x}$$

$$\mathbf{w} = T_{\mathbf{A}}(\mathbf{x})$$

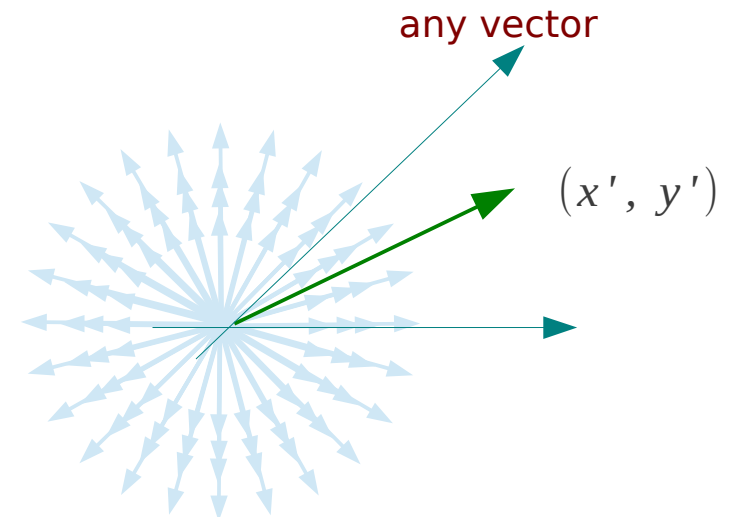
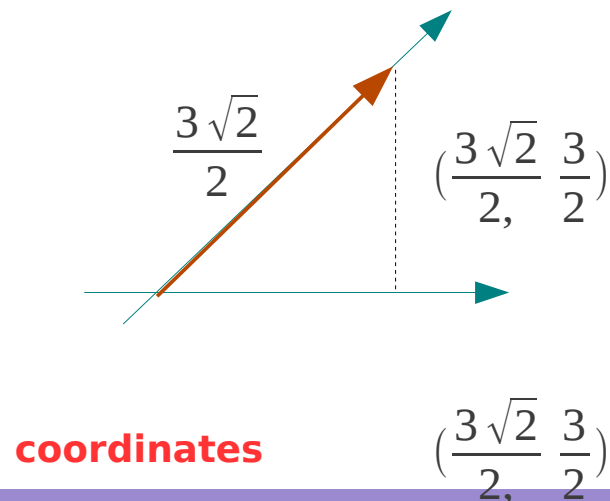
$$\mathbf{x} \xrightarrow{T_{\mathbf{A}}} \mathbf{w}$$

Coordinates and Coordinates Systems

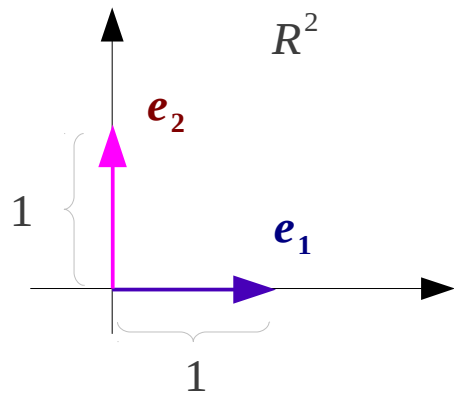
Rectangular Coordinate System



Non-Rectangular Coordinate System



Standard Basis

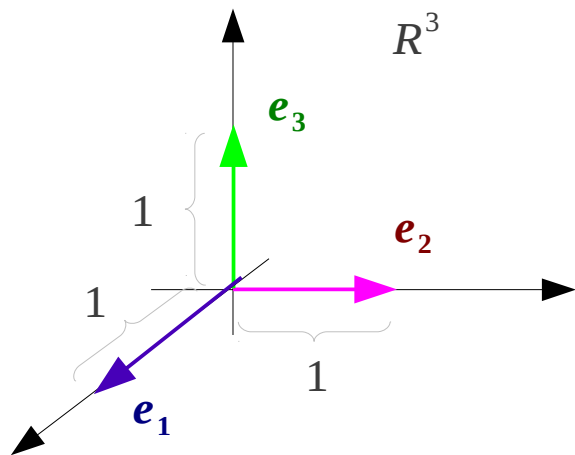
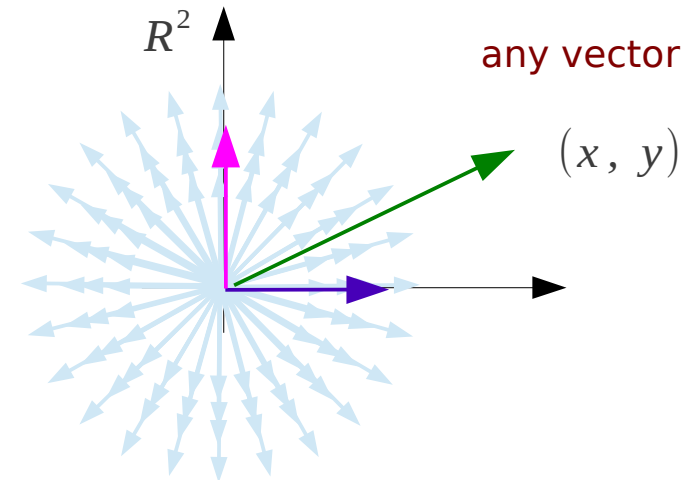


standard basis $\{e_1, e_2\}$

$$e_1 = (1, 0)$$

$$e_2 = (0, 1)$$

spans R^2



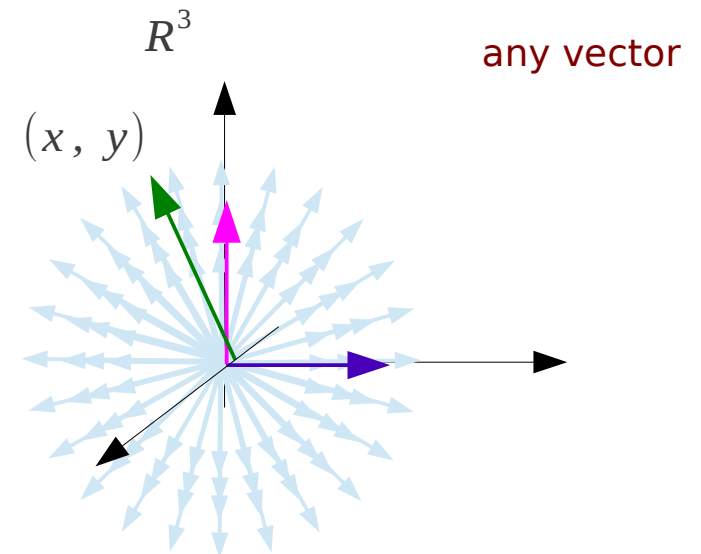
standard basis $\{e_1, e_2, e_3\}$

$$e_1 = (1, 0, 0)$$

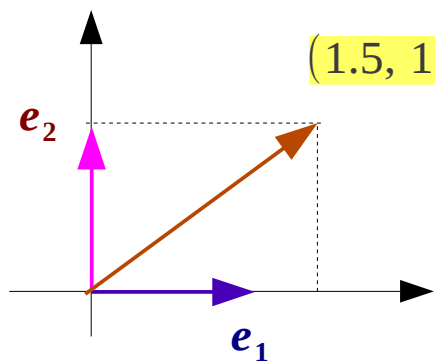
$$e_2 = (0, 1, 0)$$

$$e_3 = (0, 0, 1)$$

spans R^3



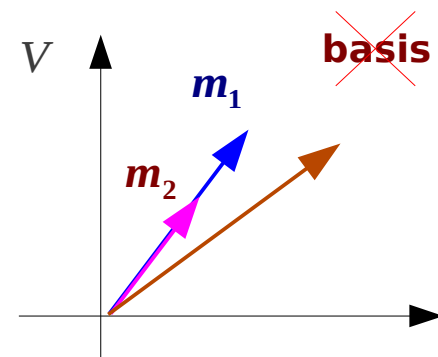
Basis and Coordinates



$$\begin{aligned}
 (1.5, 1.0) &= 1.5 \mathbf{e}_1 + 1.0 \mathbf{e}_2 \\
 &= 1.5 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + 1.0 \begin{bmatrix} 0 \\ 1 \end{bmatrix} \\
 &= \begin{bmatrix} 1.5 & 1.0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}
 \end{aligned}$$

basis $\{\mathbf{e}_1, \mathbf{e}_2\}$

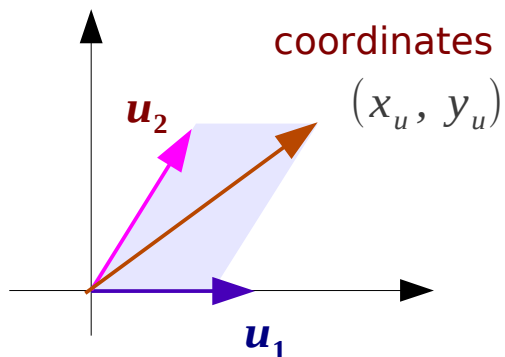
coordinates $(1.5, 1.0)$



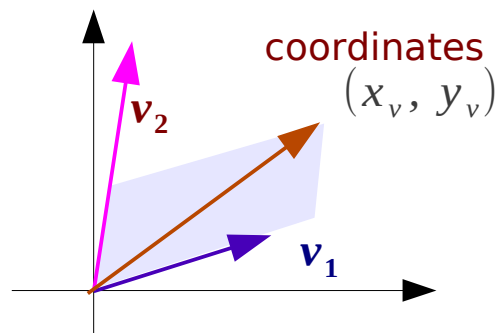
collinear vectors \rightarrow
linearly dependent vectors

many bases but the same number of basis vectors

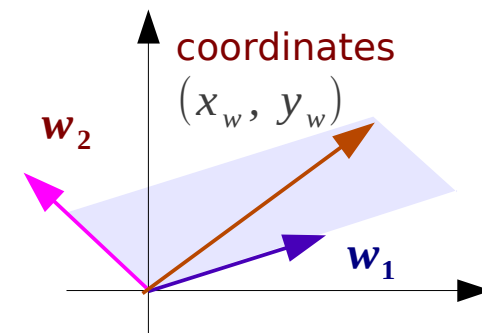
basis $\{\mathbf{u}_1, \mathbf{u}_2\}$



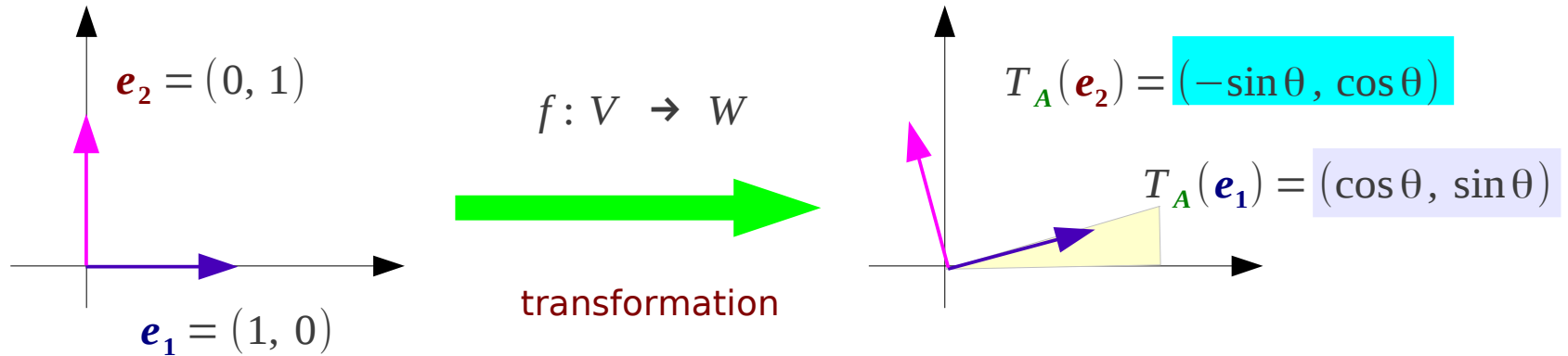
basis $\{\mathbf{v}_1, \mathbf{v}_2\}$



basis $\{\mathbf{w}_1, \mathbf{w}_2\}$



Standard Basis & Standard Matrix



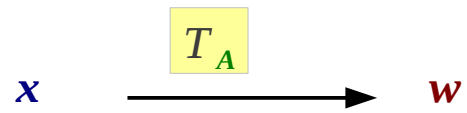
standard basis $\{e_1, e_2\}$

standard matrix $A = [T_A(e_1) \mid T_A(e_2)]$

$$\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} = \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix} \begin{bmatrix} x_0 \\ y_0 \end{bmatrix}$$

$$w = A x$$

$$w = T_A(x)$$



$$A = \left(\begin{array}{c|c|c} T_A(e_1) & T_A(e_2) & T_A(e_n) \end{array} \right)$$

Dimension

In vector space R^2

any one vector

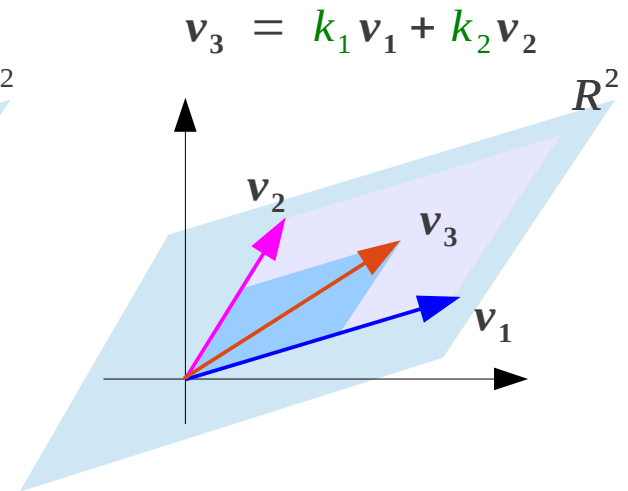
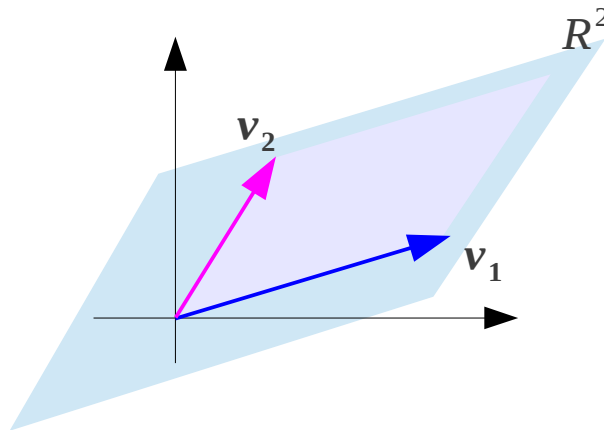
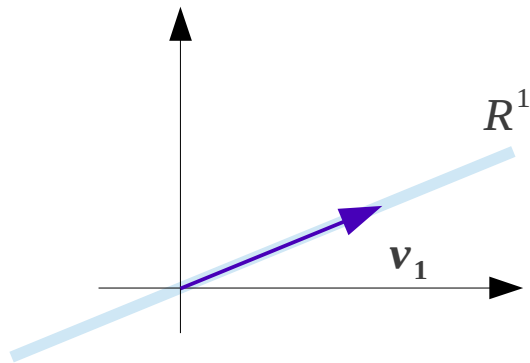
line R^1 linearly independent

any two non-collinear vectors

plane R^2 linearly independent

any three or more vectors

linearly dependent

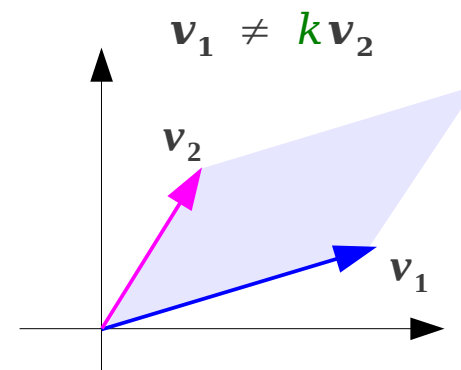


Basis

$S = \{ \mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n \}$ non-empty finite set of vectors in V

S is a basis \iff

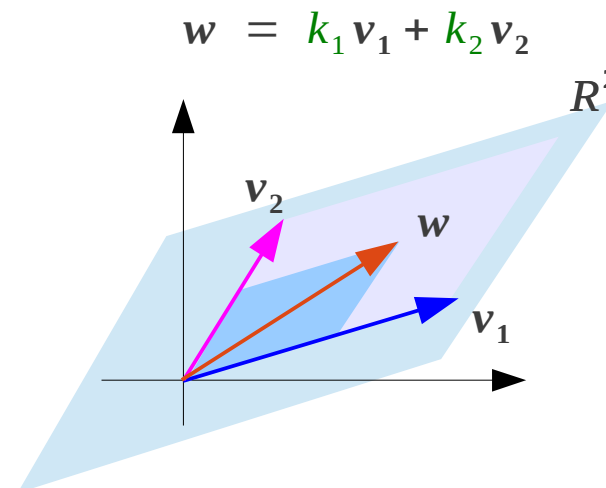
- S linearly independent
- S spans V



$\text{span}(S) = \text{span}\{ \mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n \}$ \iff

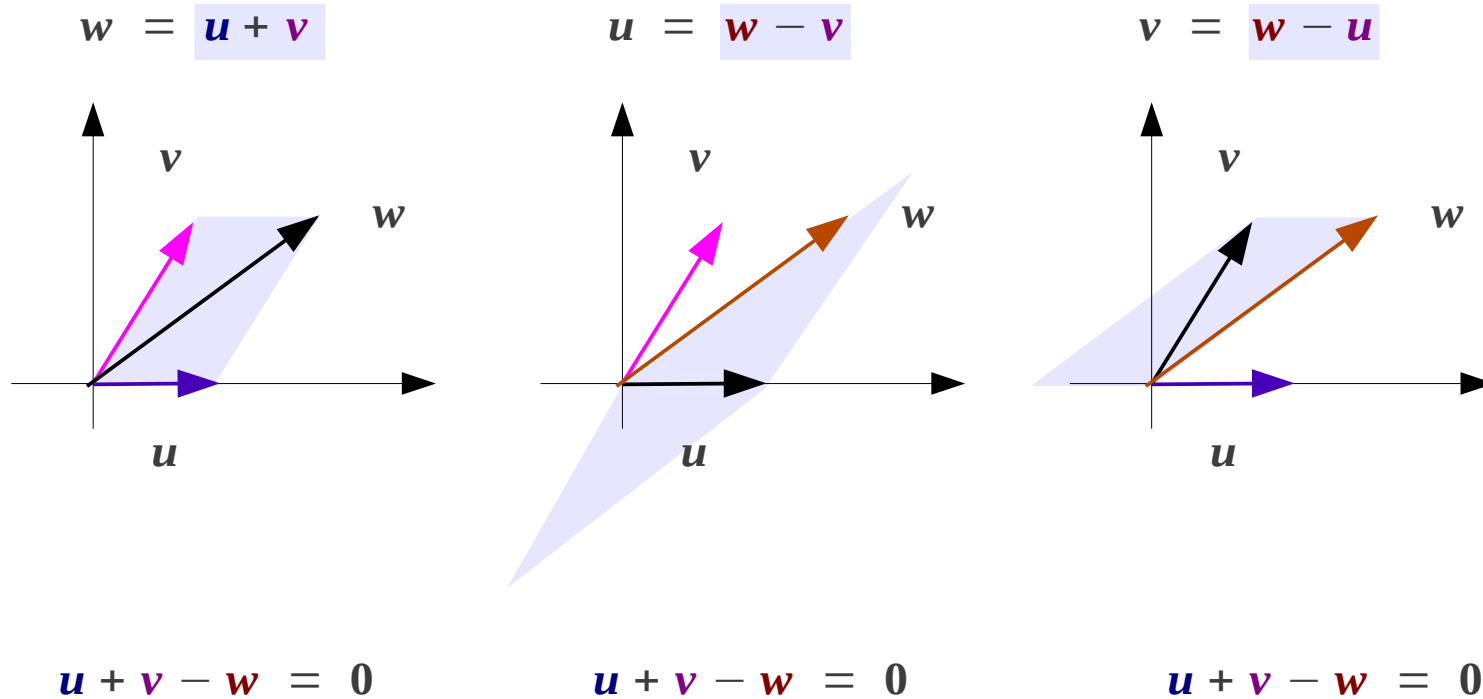
all possible linear combination of the vectors in S

$$\{ \mathbf{w} = k_1 \mathbf{v}_1 + k_2 \mathbf{v}_2 + \dots + k_n \mathbf{v}_n \}$$



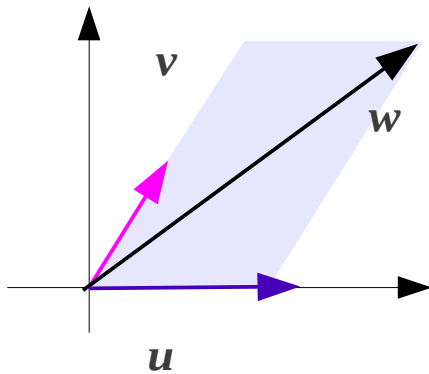
Linear Dependent (1)

$\{u, v, w\}$ linearly dependent



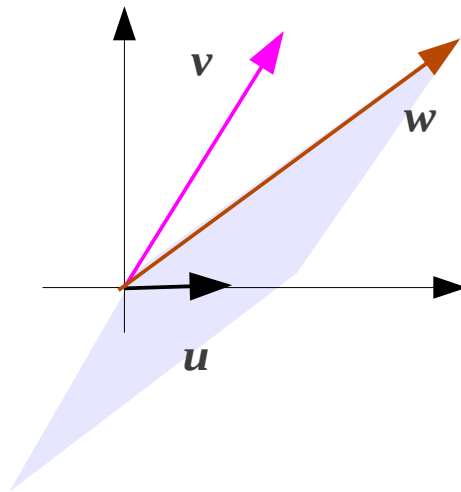
Linear Dependent (2)

$\{u, v, w\}$ linearly dependent



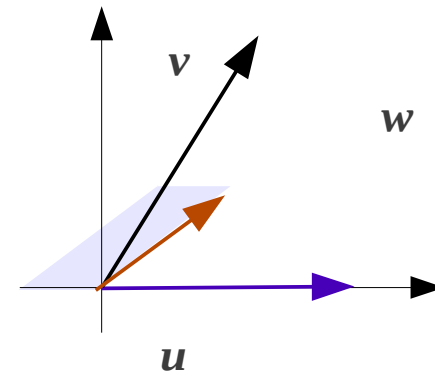
$$k_1 u + k_2 v + k_3 w = 0$$

$$(k_1 = 0) \wedge (k_2 = 0) \wedge (k_3 = 0)$$
$$(k_1 \neq 0) \vee (k_2 \neq 0) \vee (k_3 \neq 0)$$



$$m_1 u + m_2 v + m_3 w = 0$$

$$(m_1 = 0) \wedge (m_2 = 0) \wedge (m_3 = 0)$$
$$(m_1 \neq 0) \vee (m_2 \neq 0) \vee (m_3 \neq 0)$$

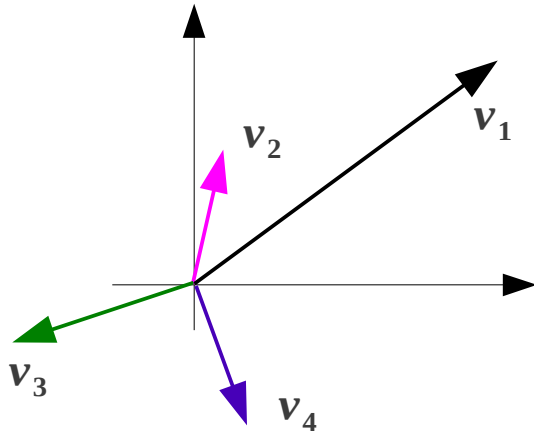


$$n_1 u + n_2 v + n_3 w = 0$$

$$(n_1 = 0) \wedge (n_2 = 0) \wedge (n_3 = 0)$$
$$(n_1 \neq 0) \vee (n_2 \neq 0) \vee (n_3 \neq 0)$$

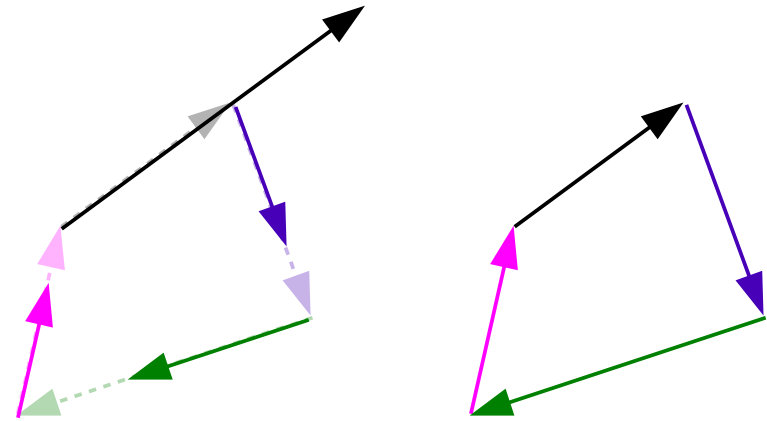
Linear Dependent (3)

$\{v_1, v_2, v_3, v_4\}$ linearly dependent



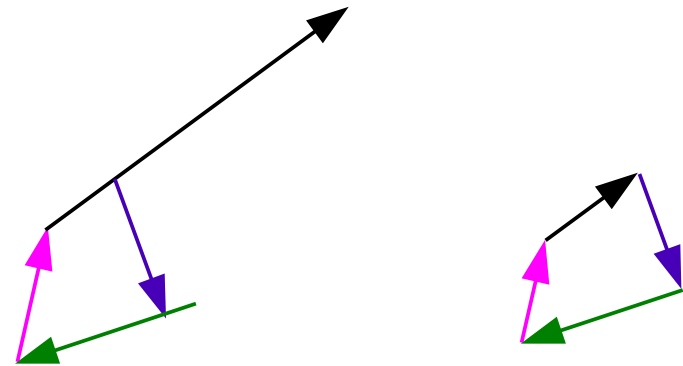
$$0v_1 + m_2v_2 + m_3v_3 + m_4v_4 = \mathbf{0}$$

$$(m_1 = 0) \vee (m_2 \neq 0) \vee (m_3 \neq 0) \vee (m_4 \neq 0)$$



$$k_1v_1 + k_2v_2 + k_3v_3 + k_4v_4 = \mathbf{0}$$

$$(k_1 \neq 0) \vee (k_2 \neq 0) \vee (k_3 \neq 0) \vee (k_4 \neq 0)$$

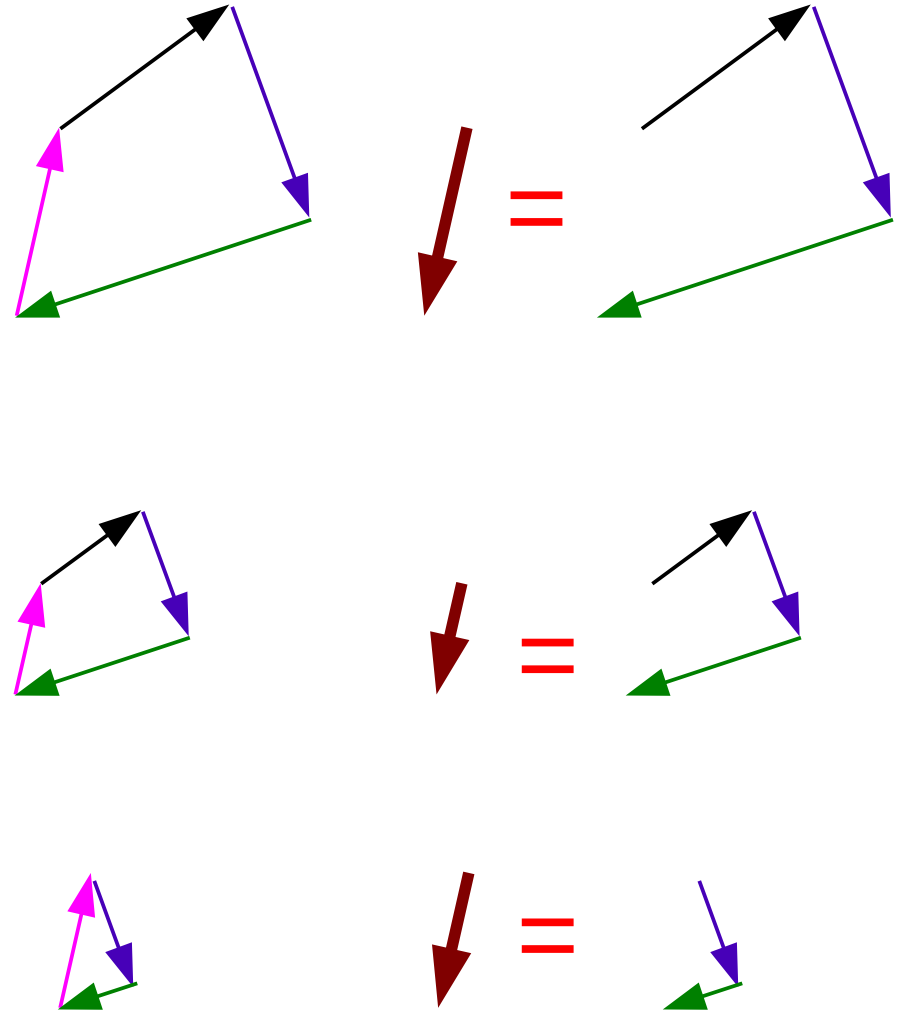
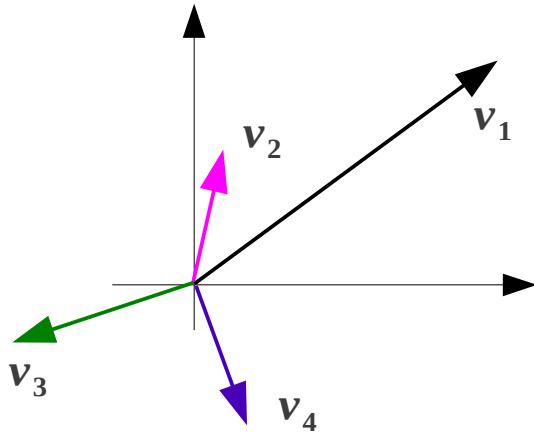


$$m_1v_1 + m_2v_2 + m_3v_3 + m_4v_4 = \mathbf{0}$$

$$(m_1 \neq 0) \vee (m_2 \neq 0) \vee (m_3 \neq 0) \vee (m_4 \neq 0)$$

Linear Dependent (4)

$\{v_1, v_2, v_3, v_4\}$ linearly dependent



Linear Independent (1)

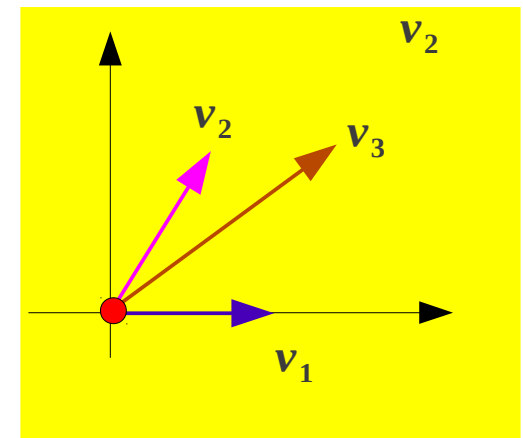
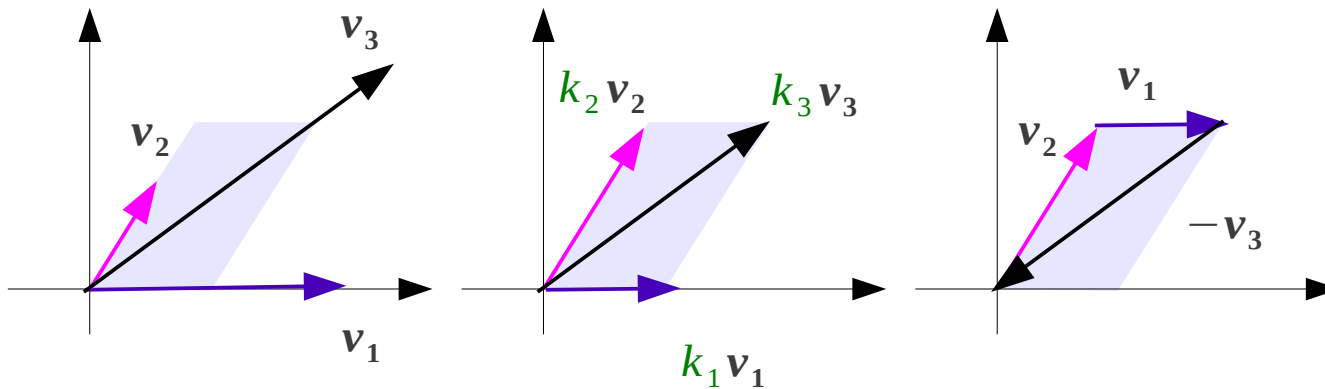
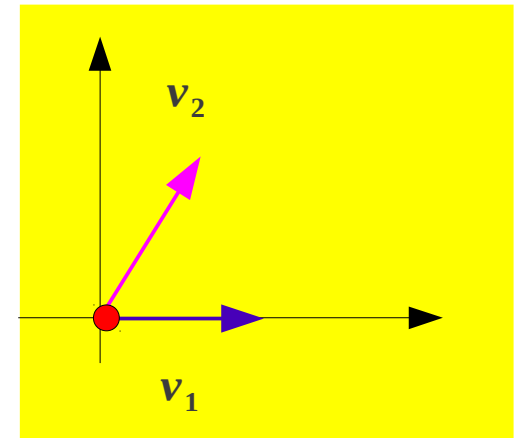
$S = \{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\}$ non-empty set of vectors in V

$$k_1 \mathbf{v}_1 + k_2 \mathbf{v}_2 + \dots + k_n \mathbf{v}_n = \mathbf{0}$$

the solution of the above equation

trivial solution: $k_1 = k_2 = \dots = k_n = 0$

{	if other solution exists	S linearly dependent
	if no other solution exists	S linearly independent



Linear Independent (2)

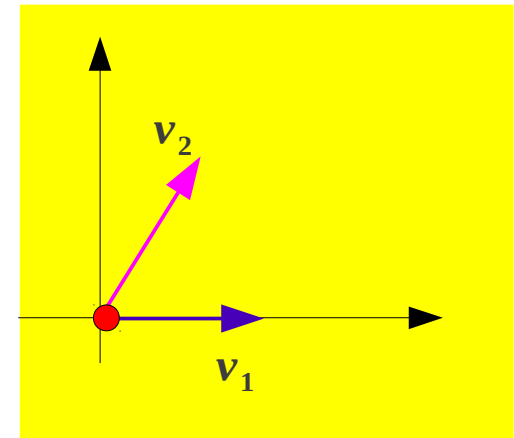
$S = \{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\}$ non-empty set of vectors in V

$$k_1 \mathbf{v}_1 + k_2 \mathbf{v}_2 + \dots + k_n \mathbf{v}_n = \mathbf{0}$$

the solution of the above equation

$$k_1 = k_2 = \dots = k_n = 0$$

$\left\{ \begin{array}{ll} \text{if other solution exists} & \iff S \text{ linearly dependent} \\ \text{if no other solution exists} & \iff S \text{ linearly independent} \end{array} \right.$



$\left\{ \begin{array}{l} \text{at least one vector in } S \text{ is a linear combination of the other vectors in } S \\ \mathbf{v}_i = k_1 \mathbf{v}_1 + k_2 \mathbf{v}_2 + \dots + k_{i-1} \mathbf{v}_{i-1} + k_{i+1} \mathbf{v}_{i+1} + \dots + k_n \mathbf{v}_n \\ \iff S \text{ linearly dependent} \end{array} \right.$

$\left\{ \begin{array}{l} \text{no vector in } S \text{ is a linear combination of the other vectors in } S \\ \mathbf{v}_i \neq k_1 \mathbf{v}_1 + k_2 \mathbf{v}_2 + \dots + k_{i-1} \mathbf{v}_{i-1} + k_{i+1} \mathbf{v}_{i+1} + \dots + k_n \mathbf{v}_n \\ \iff S \text{ linearly independent} \end{array} \right.$

Linear Independent (3)

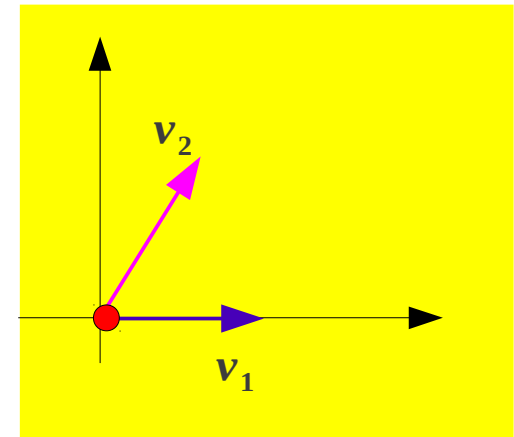
$S = \{ \mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n \}$ non-empty set of vectors in V

$$k_1 \mathbf{v}_1 + k_2 \mathbf{v}_2 + \dots + k_n \mathbf{v}_n = \mathbf{0}$$

the solution of the above equation

$$k_1 = k_2 = \dots = k_n = 0$$

$\left\{ \begin{array}{ll} \text{if other solution exists} & \iff S \text{ linearly dependent} \\ \text{if no other solution exists} & \iff S \text{ linearly independent} \end{array} \right.$



$$S = \{ \mathbf{0} \}$$

linearly dependent

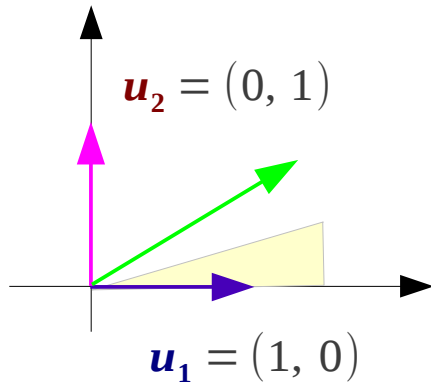
$$S = \{ \mathbf{v}_1 \}$$

linearly independent

$$S = \{ \mathbf{v}_1, \mathbf{v}_2 \} \quad \mathbf{v}_1 \neq k \mathbf{v}_2$$

linearly independent

Change of Basis

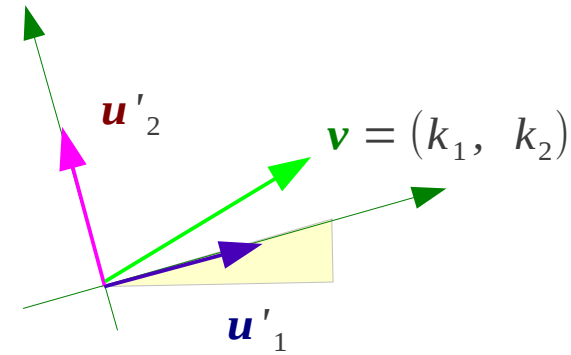


Old Basis $B = \{u_1, u_2\}$

$$[u'_1]_B = \begin{bmatrix} \cos \theta \\ \sin \theta \end{bmatrix} \quad \text{coordinate of } u'_1 \text{ with respect to } B$$

$$[u'_2]_B = \begin{bmatrix} -\sin \theta \\ \cos \theta \end{bmatrix} \quad \text{coordinate of } u'_2 \text{ with respect to } B$$

$$[v]_{B'} = \begin{bmatrix} k_1 \\ k_2 \end{bmatrix} \quad \text{coordinate of } v \text{ with respect to } B'$$



New Basis $B' = \{u'_1, u'_2\}$

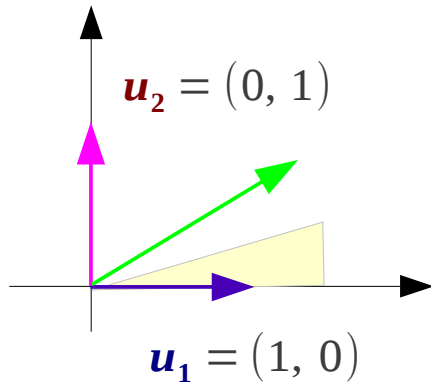
$$u'_1 = \cos \theta u_1 + \sin \theta u_2$$

$$u'_2 = -\sin \theta u_1 + \cos \theta u_2$$

$$\begin{aligned} v &= k_1 u'_1 + k_2 u'_2 \\ &= k_1 (\cos \theta u_1 + \sin \theta u_2) + k_2 (-\sin \theta u_1 + \cos \theta u_2) \\ &= (k_1 \cos \theta - k_2 \sin \theta) u_1 + (k_1 \sin \theta + k_2 \cos \theta) u_2 \end{aligned}$$

$$[v]_B = \begin{bmatrix} k_1 \cos \theta - k_2 \sin \theta \\ k_1 \sin \theta + k_2 \cos \theta \end{bmatrix} \quad \text{coordinate of } v \text{ with respect to } B$$

Change of Basis



Old Basis $B = \{u_1, u_2\}$

$$[v]_{B'} = \begin{bmatrix} k_1 \\ k_2 \end{bmatrix}$$

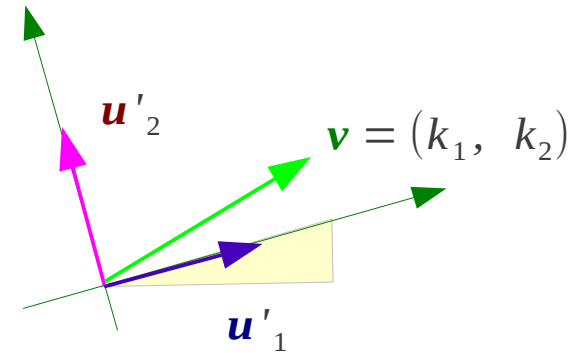
coordinate of v
with respect to B'

$$[u'_1]_B = \begin{bmatrix} \cos \theta \\ \sin \theta \end{bmatrix}$$

coordinate of u'_1
with respect to B

$$[u'_2]_B = \begin{bmatrix} -\sin \theta \\ \cos \theta \end{bmatrix}$$

coordinate of u'_2
with respect to B



New Basis $B' = \{u'_1, u'_2\}$

$$[v]_B = \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \end{bmatrix}$$

coordinate of v
with respect to B

$$[v]_B = \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix} [v]_{B'}$$

$$[v]_B = P_{B' \rightarrow B} [v]_{B'}$$

$$P_{B' \rightarrow B} = \begin{bmatrix} [u'_1]_B & [u'_2]_B \end{bmatrix}$$

Transition Matrix

$$P_{B' \rightarrow B} = \left[\begin{array}{ccc} [\mathbf{u}'_1]_B & [\mathbf{u}'_2]_B & \cdots & [\mathbf{u}'_n]_B \end{array} \right]$$

$[\mathbf{u}'_1]_B$ coordinate of \mathbf{u}'_1
with respect to B

$[\mathbf{u}'_2]_B$ coordinate of \mathbf{u}'_2
with respect to B

$$[\mathbf{v}]_B = P_{B' \rightarrow B} [\mathbf{v}]_{B'}$$

$[\mathbf{v}]_{B'}$ coordinate of \mathbf{v}
with respect to B'



$[\mathbf{v}]_B$ coordinate of \mathbf{v}
with respect to B

$$P_{B \rightarrow B'} = \left[\begin{array}{ccc} [\mathbf{u}_1]_{B'} & [\mathbf{u}_2]_{B'} & \cdots & [\mathbf{u}_n]_{B'} \end{array} \right]$$

$[\mathbf{u}_1]_{B'}$ coordinate of \mathbf{u}_1
with respect to B'

$[\mathbf{u}_2]_{B'}$ coordinate of \mathbf{u}_2
with respect to B'

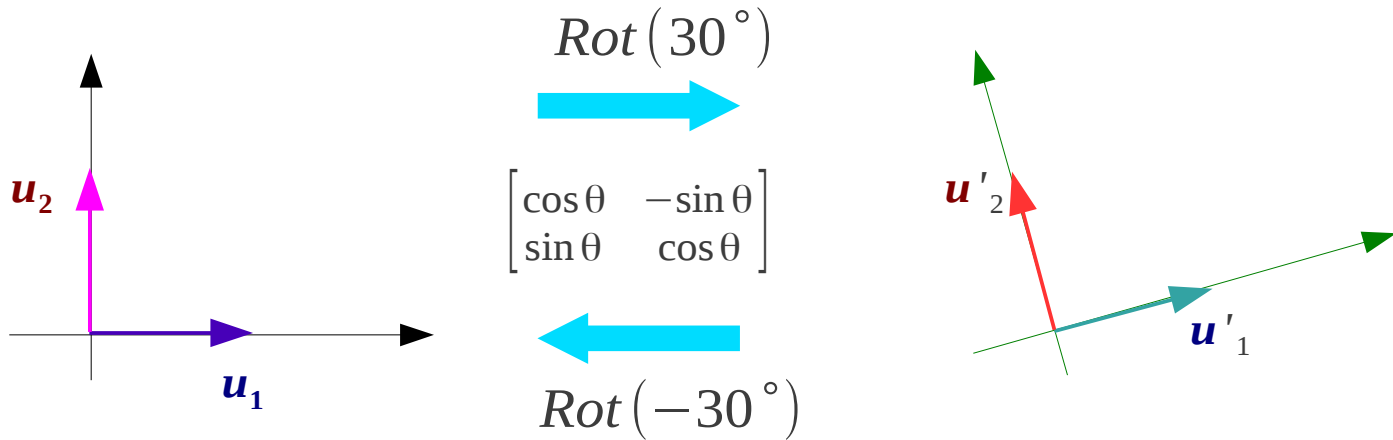
$$[\mathbf{v}]_{B'} = P_{B \rightarrow B'} [\mathbf{v}]_B$$

$[\mathbf{v}]_B$ coordinate of \mathbf{v}
with respect to B



$[\mathbf{v}]_{B'}$ coordinate of \mathbf{v}
with respect to B'

Change of Basis Example (1)



$$P_{B \rightarrow B'} = \begin{bmatrix} [\mathbf{u}_1]_{B'} & [\mathbf{u}_2]_{B'} & \cdots & [\mathbf{u}_n]_{B'} \end{bmatrix}$$

$$[\mathbf{u}_1]_{B'} \quad \text{coordinate of } \mathbf{u}_1 \text{ with respect to } B' \quad \begin{bmatrix} \frac{\sqrt{3}}{2} \\ -\frac{1}{2} \end{bmatrix}$$

$$[\mathbf{u}_2]_{B'} \quad \text{coordinate of } \mathbf{u}_2 \text{ with respect to } B' \quad \begin{bmatrix} \frac{1}{2} \\ \frac{\sqrt{3}}{2} \end{bmatrix}$$

$$P_{B \rightarrow B'} = \begin{bmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix}$$

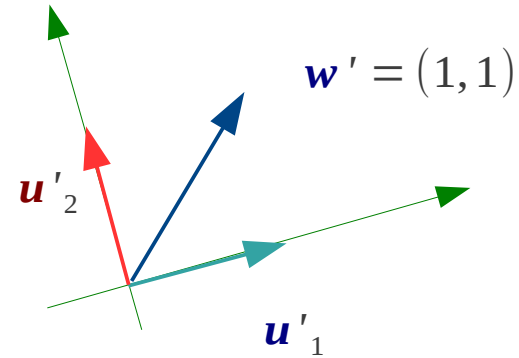
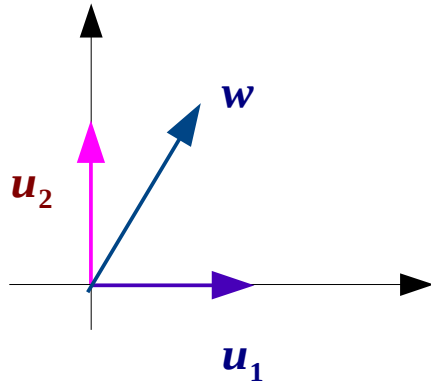
$$P_{B' \rightarrow B} = \begin{bmatrix} [\mathbf{u}'_1]_B & [\mathbf{u}'_2]_B & \cdots & [\mathbf{u}'_n]_B \end{bmatrix}$$

$$[\mathbf{u}'_1]_B \quad \text{coordinate of } \mathbf{u}'_1 \text{ with respect to } B \quad \begin{bmatrix} \frac{\sqrt{3}}{2} \\ \frac{1}{2} \end{bmatrix}$$

$$[\mathbf{u}'_2]_B \quad \text{coordinate of } \mathbf{u}'_2 \text{ with respect to } B \quad \begin{bmatrix} -\frac{1}{2} \\ \frac{\sqrt{3}}{2} \end{bmatrix}$$

$$P_{B' \rightarrow B} = \begin{bmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix}$$

Change of Basis Example (2)



$$[\mathbf{v}]_B = P_{B' \rightarrow B} [\mathbf{v}]_{B'}$$

$$P_{B' \rightarrow B} = \begin{bmatrix} [\mathbf{u}'_1]_B & [\mathbf{u}'_2]_B & \cdots & [\mathbf{u}'_n]_B \end{bmatrix}$$

$[\mathbf{v}]_{B'}$ coordinate of \mathbf{v} with respect to B'

$[\mathbf{u}'_1]_B$ coordinate of \mathbf{u}'_1 with respect to B $\begin{bmatrix} \frac{\sqrt{3}}{2} \\ \frac{1}{2} \end{bmatrix}$

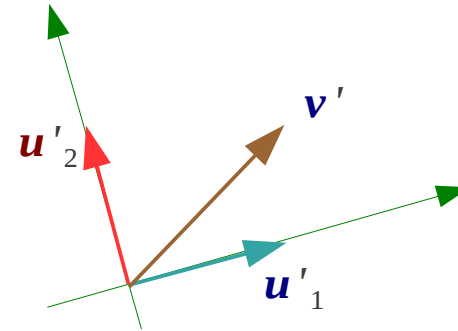
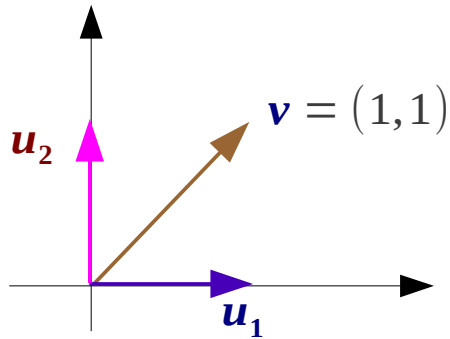
\downarrow
 $[\mathbf{v}]_B$ coordinate of \mathbf{v} with respect to B

$[\mathbf{u}'_2]_B$ coordinate of \mathbf{u}'_2 with respect to B $\begin{bmatrix} -\frac{1}{2} \\ \frac{\sqrt{3}}{2} \end{bmatrix}$

$$\mathbf{w} = \begin{bmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{-1+\sqrt{3}}{2} \\ \frac{1+\sqrt{3}}{2} \end{bmatrix}$$

$$P_{B' \rightarrow B} = \begin{bmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix}$$

Change of Basis Example (3)



$$P_{B \rightarrow B'} = \begin{bmatrix} [u_1]_{B'} & [u_2]_{B'} & \cdots & [u_n]_{B'} \end{bmatrix}$$

$$[u_1]_{B'} \quad \text{coordinate of } u_1 \text{ with respect to } B' \quad \begin{bmatrix} \frac{\sqrt{3}}{2} \\ -\frac{1}{2} \end{bmatrix}$$

$$[u_2]_{B'} \quad \text{coordinate of } u_2 \text{ with respect to } B' \quad \begin{bmatrix} \frac{1}{2} \\ \frac{\sqrt{3}}{2} \end{bmatrix}$$

$$P_{B \rightarrow B'} = \begin{bmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix}$$

$$[v]_{B'} = P_{B \rightarrow B'} [v]_B$$

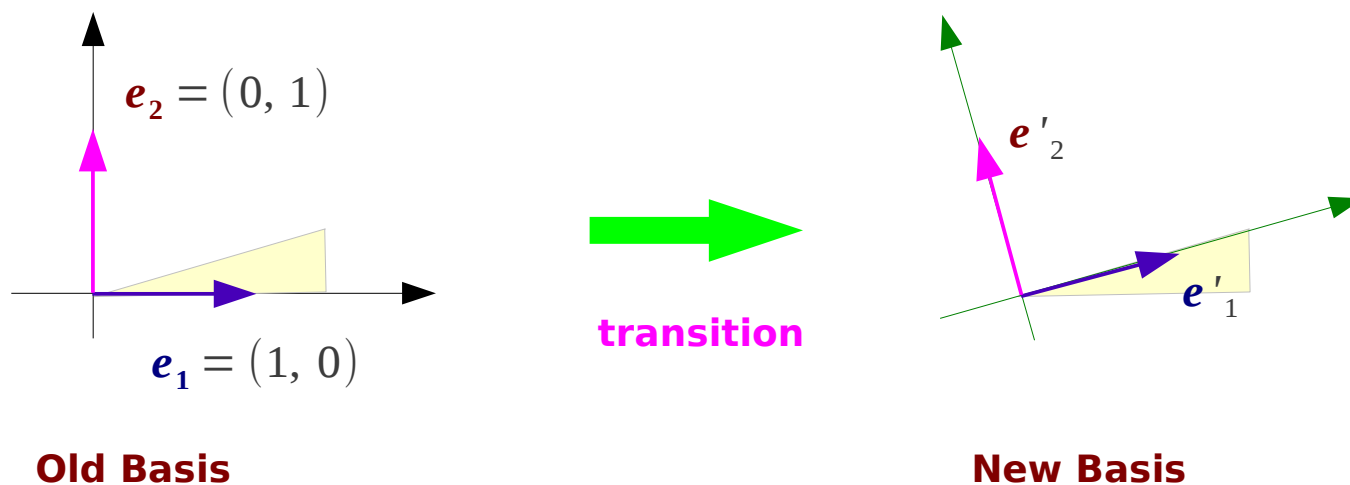
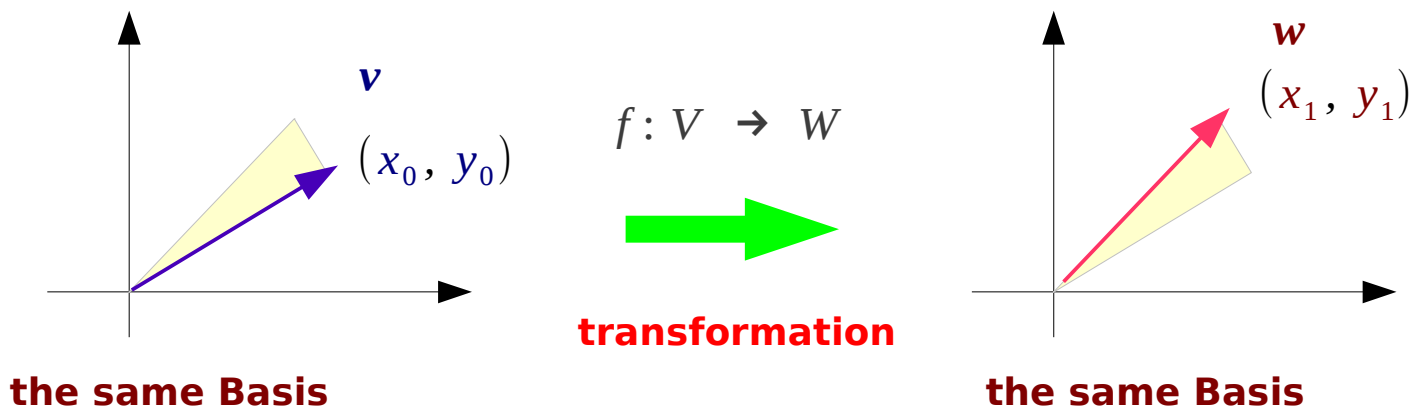
$$[v]_B \quad \text{coordinate of } v \text{ with respect to } B$$

$$\downarrow$$

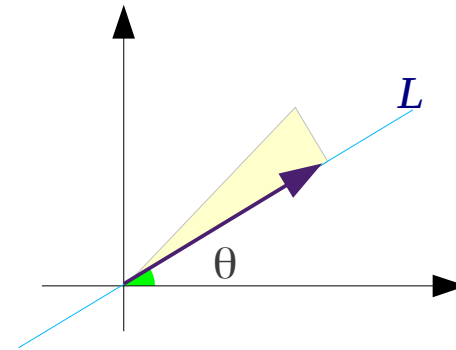
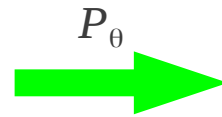
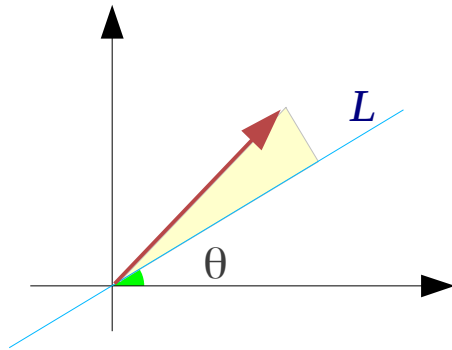
$$[v]_{B'} \quad \text{coordinate of } v \text{ with respect to } B'$$

$$v' = \begin{bmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{1+\sqrt{3}}{2} \\ \frac{-1+\sqrt{3}}{2} \end{bmatrix}$$

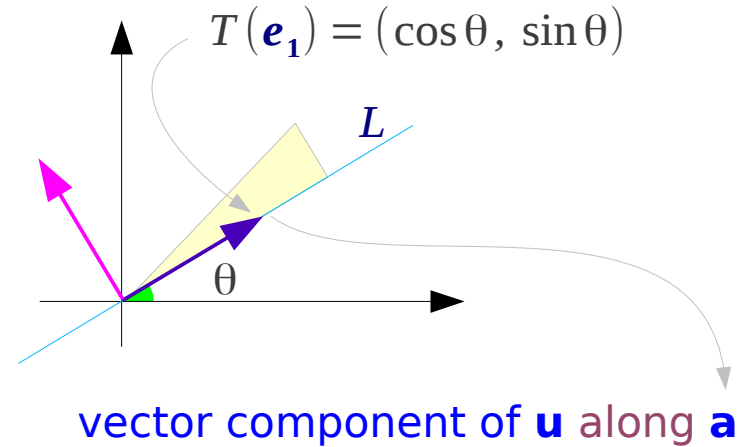
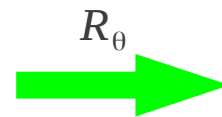
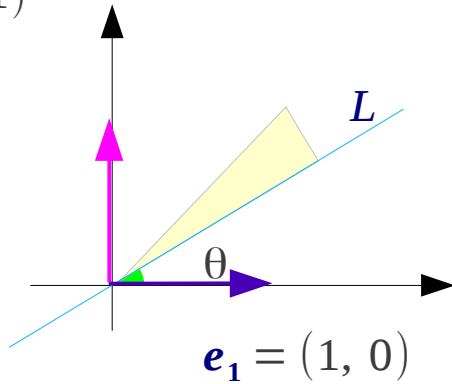
Transformation & Transition Matrix



Projection onto the Lines Through Zero (1)

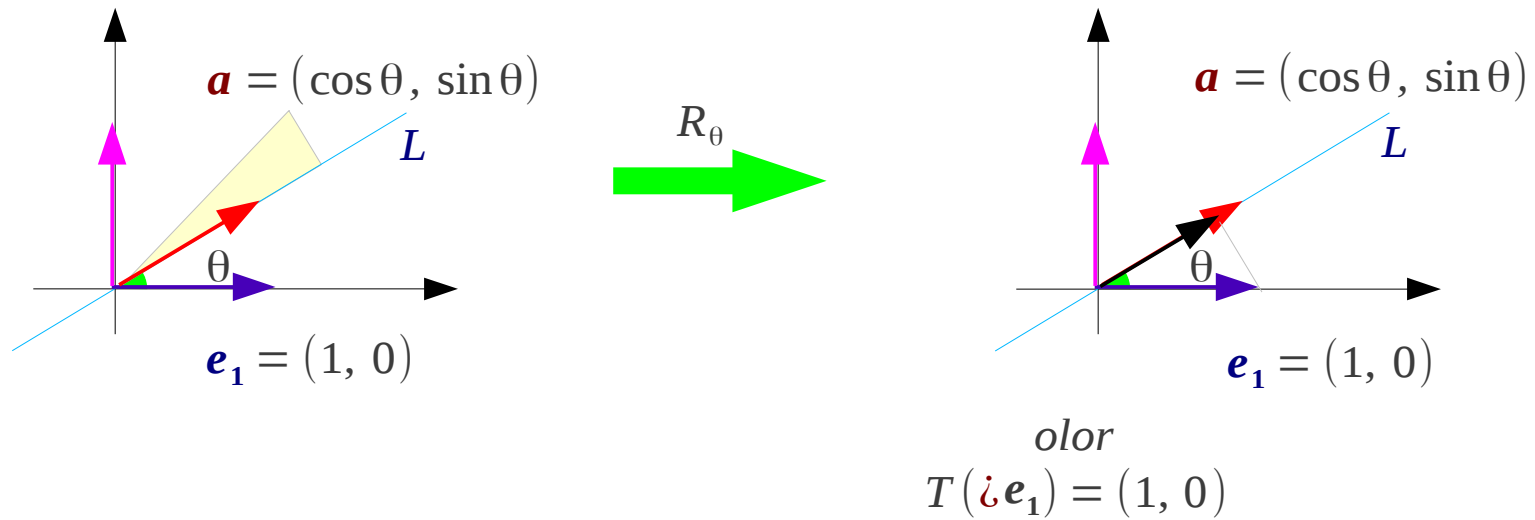


$$\mathbf{e}_2 = (0, 1)$$



$$\text{proj}_a \mathbf{u} = \frac{\mathbf{u} \cdot \mathbf{a}}{\|\mathbf{a}\|^2} \mathbf{a}$$

Projection onto the Lines Through Zero



vector component of \mathbf{u} along \mathbf{a}

$$\text{proj}_{\mathbf{a}} \mathbf{u} = \frac{\mathbf{u} \cdot \mathbf{a}}{\|\mathbf{a}\|^2} \mathbf{a}$$

References

- [1] <http://en.wikipedia.org/>
- [2] <http://planetmath.org/>
- [3] M.L. Boas, “Mathematical Methods in the Physical Sciences”
- [4] D.G. Zill, “Advanced Engineering Mathematics”