Prof. Dr. H. Brenner **Osnabrück WS 2011/2012**

Mathematics for natural sciences I

Exercise sheet 7

Warm-up-exercises

EXERCISE 7.1. Write in $\mathbb{Q}^2$ the vector

$$(2,-7)$$

as a linear combination of the vectors

$$(5,-3) \text{ and } (-11,4)\,.$$

EXERCISE 7.2. Write in $\mathbb{C}^2$ the vector

$$(1,0)$$

as a linear combination of the vectors

$$(3+5i,-3+2i) \text{ and } (1-6i,4-i)\,.$$

EXERCISE 7.3. Let K be a field and let V be a K-vector space. Let v_i, $i \in I$, be a family of vectors in V and let $w \in V$ be another vector. Assume that the family

$$w, v_i, i \in I\,,$$

is a system of generators of V and that w is a linear combination of the v_i, $i \in I$. Prove that also v_i, $i \in I$, is a system of generators of V.

EXERCISE 7.4. Let K be a field and let V be a K-vector space. Prove the following facts.

(1) Let U_j, $j \in J$, be a family of subspaces of V. Prove that also the intersection
$$U = \bigcap_{j \in J} U_j$$
is a subspace.
(2) Let v_i, $i \in I$, be a family of elements of V and consider the subset W of V which is given by all linear combinations of these elements. Show that W is a subspace of V.
(3) The family v_i, $i \in I$, is a system of generators of V if and only if
$$\langle v_i,\, i \in I \rangle = V\,.$$

Exercise 7.5. Show that the three vectors

$$\begin{pmatrix}0\\1\\2\\1\end{pmatrix},\begin{pmatrix}4\\3\\0\\2\end{pmatrix},\begin{pmatrix}1\\7\\0\\-1\end{pmatrix}$$

in $\mathbb{R}^4$ are linearly independent.

Exercise 7.6. Give an example of three vectors in $\mathbb{R}^3$ such that each couple of them is linearly independent, but all three vectors are linearly dependent.

Exercise 7.7. Let K be a field, let V be a K-vector space and let v_i, $i \in I$, be a family of vectors in V. Prove the following facts.

(1) If the family is linearly independent then for each subset $J \subseteq I$ also the family v_i , $i \in J$ is linearly independent.
(2) The empty family is linearly independent.
(3) If the family contains the null vector then it is not linearly independent.
(4) If a vector appears several times in the family, then the family is not linearly independent.
(5) A vector v is linearly independent if and only if $v \neq 0$.
(6) Two vectors v und u are linearly independent if and only if u is not a scalar multiple of v and vice versa.

Exercise 7.8. Let K be a field, let V be a K-vector space and let v_i, $i \in I$, be a family of vectors in V. Let λ_i, $i \in I$ be a family of elements $\neq 0$ in K. Prove that the family v_i, $i \in I$, is linearly independent (a system of generators of V, a basis of V) if and only if the same holds for the family $\lambda_i v_i$, $i \in I$.

Exercise 7.9. Determine a basis for the solution space of the linear equation

$$3x + 4y - 2z + 5w = 0\,.$$

Exercise 7.10. Determine a basis for the solution space of the linear system of equations

$$-2x + 3y - z + 4w = 0 \text{ and } 3z - 2w = 0\,.$$

Exercise 7.11. Prove that in $\mathbb{R}^3$ the three vectors

$$\begin{pmatrix}2\\1\\5\end{pmatrix},\begin{pmatrix}1\\3\\7\end{pmatrix},\begin{pmatrix}4\\1\\2\end{pmatrix}$$

are a basis.

Exercise 7.12. Establish if in $\mathbb{C}^2$ the two vectors

$$\begin{pmatrix} 2+7i \\ 3-i \end{pmatrix} \text{ und } \begin{pmatrix} 15+26i \\ 13-7i \end{pmatrix}$$

form a basis.

Exercise 7.13. Let K be a field. Find a linear system of equations in three variables, whose solution space is exactly

$$\{\lambda \begin{pmatrix} 3 \\ 2 \\ -5 \end{pmatrix} \mid \lambda \in K\}.$$

Hand-in-exercises

Exercise 7.14. (3 points)

Write in $\mathbb{Q}^3$ the vector

$$(2, 5, -3)$$

as a linear combination of the vectors

$$(1, 2, 3), (0, 1, 1) \text{ und } (-1, 2, 4).$$

Prove that it cannot be expressed as a linear combination of two of the three vectors.

Exercise 7.15. (2 points)

Establish if in $\mathbb{R}^3$ the three vectors

$$\begin{pmatrix} 2 \\ 3 \\ -5 \end{pmatrix}, \begin{pmatrix} 9 \\ 2 \\ 6 \end{pmatrix}, \begin{pmatrix} -1 \\ 4 \\ -1 \end{pmatrix}$$

form a basis.

Exercise 7.16. (2 points)

Establish if in $\mathbb{C}^2$ the two vectors

$$\begin{pmatrix} 2-7i \\ -3+2i \end{pmatrix} \text{ and } \begin{pmatrix} 5+6i \\ 3-17i \end{pmatrix}$$

form a basis.

Exercise 7.17. (4 points)

Let $\mathbb{Q}^n$ be the n-dimensional standard vector space over $\mathbb{Q}$ and let $v_1, \ldots, v_n \in \mathbb{Q}^n$ be a family of vectors. Prove that this family is a $\mathbb{Q}$-basis of $\mathbb{Q}^n$ if and only if the same family, considered as a family in $\mathbb{R}^n$, is a $\mathbb{R}$-basis of $\mathbb{R}^n$.

Exercise 7.18. (3 points)

Let K be a field and let

$$\begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix} \in K^n$$

be a nonzero vector. Find a linear system of equations in n variables with $n-1$ equations, whose solution space is exactly

$$\{\lambda \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix} \mid \lambda \in K\}.$$