Affine schemes

A scheme U is called *affine* if it is isomorphic to the spectrum of some commutative ring R. If the scheme is of finite type (if we have a variety), then this is equivalent to saying that there exist global functions

$$g_1, \ldots, g_k \in \Gamma(U, \mathcal{O}_U)$$

such that the mapping

$$U \longrightarrow \mathbb{A}^k,\ x \longmapsto (g_1(x), \ldots, g_k(x)),$$

is a closed embedding. The relation to cohomology is given by the following well-known theorem of Serre.

Theorem 5.1. *Let U denote a noetherian scheme. Then the following properties are equivalent.*

1. *U is an affine scheme.*
2. *For every quasicoherent sheaf $\mathcal{F}$ on U and all $i \geq 1$ we have $H^i(U, \mathcal{F}) = 0$.*
3. *For every coherent ideal sheaf $\mathcal{I}$ on U we have $H^1(U, \mathcal{I}) = 0$.*

It is in general a difficult question whether a given scheme U is affine. For example, suppose that $X = \text{Spec}\,(R)$ is an affine scheme and

$$U = D(\mathfrak{a}) \subseteq X$$

is an open subset (such schemes are called quasiaffine) defined by an ideal $\mathfrak{a} \subseteq R$. When is U itself affine? The cohomological criterion above simplifies to the condition that $H^i(U, \mathcal{O}_X) = 0$ for $i \geq 1$.

Of course, if $\mathfrak{a} = (f)$ is a principal ideal (or up to radical a principal ideal), then $U = D(f) \cong \text{Spec}\,(R_f)$ is affine. On the other hand, if $(R, \mathfrak{m})$ is a local ring of dimension ≥ 2, then

$$D(\mathfrak{m}) \subset \text{Spec}\,(R)$$

is not affine, since

$$H^{d-1}(U, \mathcal{O}_X) \;=\; H^d_{\mathfrak{m}}(R)$$

by the relation between sheaf cohomology and local cohomology and a theorem of Grothendieck.

Affineness and superheight

One can show that for an open affine subset $U \subseteq X$ the closed complement $Y = X \setminus U$ must be of pure codimension one (U must be the complement of the support of an effective divisor). In a regular or (locally $\mathbb{Q}$)- factorial domain the complement of every divisor is affine, since the divisor can be described (at least locally geometrically) by one equation. But it is easy to give examples to show that this is not true for normal threedimensional domains. The following example is a standardexample for this phenomenon and is in fact given by a forcing algebra.

EXAMPLE 5.2. Let K be a field and consider the ring

$$R = K[x,y,u,v]/(xu - yv)\,.$$

The ideal $\mathfrak{p} = (x,y)$ is a prime ideal in R of height one. Hence the open subset $U = D(x,y)$ is the complement of an irreducible hypersurface. However, U is not affine. For this we consider the closed subscheme

$$\mathbb{A}^2_K \cong Z = V(u,v) \subseteq \operatorname{Spec}(R)$$

and

$$Z \cap U \subseteq U\,.$$

If U were affine, then also the closed subscheme $Z \cap U \cong \mathbb{A}^2_K \setminus \{(0,0)\}$ would be affine, but this is not true, since the complement of the punctured plane has codimension 2.

The argument employed in this example rests on the following definition and the next theorem.

DEFINITION 5.3. Let R be a noetherian commutative ring and let $I \subseteq R$ be an ideal. The (noetherian) *superheight* is the supremum

$$\sup\left(\operatorname{ht}(IS) : S \text{ is a notherian } R-\text{algebra}\right).$$

THEOREM 5.4. *Let R be a noetherian commutative ring and let $I \subseteq R$ be an ideal and $U = D(I) \subseteq X = \operatorname{Spec}(R)$. Then the following are equivalent.*

(1) *U is an affine scheme.*
(2) *I has superheight ≤ 1 and $\Gamma(U, \mathcal{O}_X)$ is a finitely generated R-algebra.*

It is not true at all that the ring of global sections of an open subset U of the spectrum X of a noetherian ring is of finite type over this ring. This is not even true if X is an affine variety. This problem is directly related to Hilbert's fourteenth problem, which has a negative answer. We will later present examples where U has superheight one, yet is not affine, hence its ring of global sections is not finitely generated.

If R is a two-dimensional local ring with parameters f, g and if B is the forcing algebra for some $\mathfrak{m}$-primary ideal, then the ring of global sections of the torsor is just

$$\Gamma(D(\mathfrak{m}B), \mathcal{O}_B) \;=\; B_f \cap B_g.$$

In the following two examples we use results from tight closure theory to establish (non)-affineness properties of certain torsors.

EXAMPLE 5.5. Let K be a field and consider the Fermat ring

$$R = K[X,Y,Z]/(X^d + Y^d + Z^d)$$

together with the ideal $I = (X,Y)$ and $f = Z^2$. For $d \geq 3$ we have $Z^2 \notin (X,Y)$. This element is however in the tight closure $(X,Y)^*$ of the ideal in positive characteristic (assume that the characteristic p does not divide d) and is therefore also in characteristic 0 inside the tight closure and inside the solid closure. Hence the open subset

$$D(X,Y) \subseteq \operatorname{Spec}(K[X,Y,Z,S,T]/(X^d+Y^d+Z^d, SX+TY-Z^2))$$

is not an affine scheme. In positive characteristic, Z^2 is also contained in the plus closure $(X,Y)^+$ and therefore this open subset contains punctured surfaces (the spectrum of the forcing algebra contains two-dimensional closed subschemes which meet the exceptional fiber $V(X,Y)$ in only one point; the ideal (X,Y) has superheight 2 in the forcing algebra). In characteristic zero however, due to Remark 4.8. the superheight is one and therefore by Theorem 5.4 the algebra $\Gamma(D(X,Y),\mathcal{O}_B)$ is not finitely generated. For $K=\mathbb{C}$ and $d=3$ one can also show that $D(X,Y)_{\mathbb{C}}$ is, considered as a complex space, a Stein space.

Example 5.6. Let K be a field of positive characteristic $p \geq 7$ and consider the ring

$$R = K[X,Y,Z]/(X^5+Y^3+Z^2)$$

together with the ideal $I=(X,Y)$ and $f=Z$. Since R has a rational singularity, it is F-regular, i.e. all ideals are tightly closed. Therefore $Z \notin (X,Y)^*$ and so the torsor

$$D(X,Y) \subseteq \operatorname{Spec}(K[X,Y,Z,S,T]/(X^5+Y^3+Z^2, SX+TY-Z))$$

is an affine scheme. In characteristic zero this can be proved by either using that R is a quotient singularity or by using the natural grading ($\deg(X)=6$, $\deg(Y)=10$, $\deg(Z)=15$) where the corresponding cohomology class $\frac{Z}{XY}$ gets degree -1 and then applying the geometric criteria on the corresponding projective curve (rather the corresponding curve of the standard-homogenization $U^{30}+V^{30}+W^{30}=0$).