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Computation of tight closure

Lecture 1 - Basics

In these lectures we want to demonstrate how tight closure can be understood and computed with the help of geometric and cohomological methods. We recall briefly the definition of tight closure.

Tight closure

Let R be a noetherian domain of positive characteristic, let

$$F : R \longrightarrow R,\ f \longmapsto f^p,$$

be the *Frobenius homomorphism*, and

$$F^e : R \longrightarrow R,\ f \longmapsto f^q,\ q = p^e\,,$$

its eth iteration. Let I be an ideal and set

$$I^{[q]} = \text{ extended ideal of } I \text{ under } F^e$$

Then define the *tight closure* of I to be the ideal

$$I^* := \{f \in R : \text{ there exists } z \neq 0 \text{ such that } zf^q \in I^{[q]} \text{ for all } q = p^e\}\,.$$

If R is not a domain, then one requires that z does not belong to any minimal prime ideal of R. This definition is not well suited for computations. The problem is that it one has to check infinitely many conditions. The tight closure of an ideal in a regular ring is just the ideal itself. The following observations translate the containments $f \in I$ and $f \in I^*$ into statements on certain cohomology classes.

Let $(R, \mathfrak{m})$ be a noetherian local ring of dimension d (we treat the case of a standard-graded ring at the same time) and let $I = (f_1, \ldots, f_n)$ be an $\mathfrak{m}$-primary ideal. Then we have a free (not necessarily minimal) resolution (in fact it is enough that the complex is exact on the punctured spectrum)

$$\cdots \longrightarrow F_3 \longrightarrow F_2 \longrightarrow F_1 \stackrel{f_1,\ldots,f_n}{\longrightarrow} F_0 = R \longrightarrow R/I \longrightarrow 0\,.$$

An element $f \in R$ belongs to I if and only if it is mapped to 0 in R/I. We split up the long exact sequence into several short exact sequences of R-modules, namely into

$$0 \longrightarrow \operatorname{Syz}_1 := \operatorname{Syz}(f_1, \ldots, f_n) \longrightarrow F_1 \stackrel{f_1,\ldots,f_n}{\longrightarrow} F_0 = R \longrightarrow R/I \longrightarrow 0\,,$$

$$0 \longrightarrow \operatorname{Syz}_2 \longrightarrow F_2 \longrightarrow \operatorname{Syz}_1 \longrightarrow 0\,,$$

$$0 \longrightarrow \mathrm{Syz}_3 \longrightarrow F_3 \longrightarrow \mathrm{Syz}_2 \longrightarrow 0\,,$$

etc. These syzygy modules do not have especially nice properties. This changes if we consider the restriction of these sequences to the open subset

$$U := D(\mathfrak{m}) = \mathrm{Spec}\,(R) \setminus \{\mathfrak{m}\}\,.$$

This scheme is called the *punctured spectrum* of R, and restriction means that we consider the restrictions of the coherent sheaves $\widetilde{\mathrm{Syz}}$. Because the support of I is just $\{\mathfrak{m}\}$, the restriction of R/I to U becomes 0, hence we get the short exact sequence

$$0 \longrightarrow \mathrm{Syz}_1 = \mathrm{Syz}\,(f_1, \ldots, f_n) \longrightarrow F_1 = \mathcal{O}_U^n \stackrel{f_1,\ldots,f_n}{\longrightarrow} F_0 = \mathcal{O}_U \longrightarrow 0$$

(we do not distinguish in the symbols between the modules and the sheaves, with the exception of the structure sheaf). That this last mapping is surjective is also clear since the corresponding module-mapping is surjective when localized at f_i and since $U = \bigcup_{i=1}^n D(f_i)$ (and since sheaf surjectivity is a local property). Now for a surjective sheaf homomorphism

$$\mathcal{S} \longrightarrow \mathcal{T}$$

between locally free sheaves the kernel is itself locally free. So in particular $\mathrm{Syz}\,(f_1, \ldots, f_n)$ is locally free (on U). By induction it follows that all Syz_i are locally free.

If R has dimension at least 2 and is normal (or is at least S_2), then

$$\Gamma(U, \mathcal{O}_U) = R\,.$$

Hence $f \in I$ if and only if $f \in I\Gamma(U, \mathcal{O}_U)$, and this property can be checked over U. Because U is itself not an affine scheme, this property is not a local property, but a global property. Locally f belongs to the ideal sheaf given by I on U. The difference between local and global properties are usually controlled by sheaf cohomology (or by local cohomology). The short exact sequence

$$0 \longrightarrow \mathrm{Syz}\,(f_1, \ldots, f_n) \longrightarrow \mathcal{O}_U^n \stackrel{f_1,\ldots,f_n}{\longrightarrow} \mathcal{O}_U \longrightarrow 0$$

from above gives rise to a long exact cohomology sequence

$$\begin{aligned} 0 \longrightarrow \Gamma(U, \mathrm{Syz}\,(f_1, \ldots, f_n)) \longrightarrow R^n \stackrel{f_1,\ldots,f_n}{\longrightarrow} R& \\ \longrightarrow H^1(U, \mathrm{Syz}\,(f_1, \ldots, f_n)) \longrightarrow H^1(U, \mathcal{O}_U)^n \longrightarrow .& \end{aligned}$$

The element $f \in R$ is mapped to some element

$$c = \delta(f) \in H^1(U, \mathrm{Syz}\,(f_1, \ldots, f_n))\,.$$

Now

$$c = 0 \text{ if and only if } f \in I\,,$$

because f comes from the left if and only if it is mapped to 0 on the right.

The short exact sheaf sequence

$$0 \longrightarrow \mathrm{Syz}_2 \longrightarrow F_2 = \mathcal{O}_U^{\beta_2} \longrightarrow \mathrm{Syz}_1 \longrightarrow 0$$

yields again a long exact cohomology sequence, and we write down the part

$$\longrightarrow H^1(U, \mathcal{O}_U)^{\beta_2} \longrightarrow H^1(U, \mathrm{Syz}_1) \stackrel{\delta}{\longrightarrow} H^2(U, \mathrm{Syz}_2) \longrightarrow H^2(U, \mathcal{O}_U)^{\beta_2} \longrightarrow .$$

In particular we get a cohomology class

$$c_2 := \delta(c) = \delta(\delta(f))$$

in $H^2(U, \mathrm{Syz}_2)$. Suppose that $H^1(U, \mathcal{O}_U) = 0$. Then

$$\delta\colon\ H^1(U, \mathrm{Syz}_1) \longrightarrow H^2(U, \mathrm{Syz}_2)$$

is injective and so $f \in I$ if and only if $c_2 = 0$. From the other short exact sheaf sequences

$$0 \longrightarrow \mathrm{Syz}_i \longrightarrow F_i = \mathcal{O}_U^{\beta_i} \longrightarrow \mathrm{Syz}_{i-1} \longrightarrow 0$$

we obtain

$$\longrightarrow H^{i-1}(U, \mathcal{O}_U)^{\beta_i} \longrightarrow H^{i-1}(U, \mathrm{Syz}_{i-1}) \stackrel{\delta}{\longrightarrow} H^i(U, \mathrm{Syz}_i) \longrightarrow H^i(U, \mathcal{O}_U)^{\beta_i} \longrightarrow$$

and hence the inductively defined cohomology classes

$$c_i := \delta^i(f) .$$

Now suppose that R is Cohen-Macaulay of dimension $d \geq 2$. Then

$$H^i(U, \mathcal{O}_U) = H^{i+1}_{\mathfrak{m}}(R) = 0$$

for $1 \leq i \leq d-2$ and therefore

$$H^{i-1}(U, \mathrm{Syz}_{i-1}) \cong H^i(U, \mathrm{Syz}_i)$$

for i between 2 and $d-2$ and

$$H^{d-2}(U, \mathrm{Syz}_{d-2}) \longrightarrow H^{d-1}(U, \mathrm{Syz}_{d-1})$$

is injective ($d \geq 3$). Thus $f \in I$ if and only if $\delta^i(f) = 0$ for any $i = 1, \ldots, d-1$. We will in particular work with

$$c_{d-1} \in H^{d-1}(U, \mathrm{Syz}_{d-1})$$

and call this the *top-dimensional cohomology class* inside the *top-dimensional syzygy sheaf*.

EXAMPLE 1.1. Suppose that the (primary) ideal $I = (f_1, \ldots, f_n)$ (R is local and Cohen-Macaulay) has finite projective dimension. Then we have a finite free resolution

$$0 \longrightarrow F_d \longrightarrow F_{d-1} \longrightarrow \cdots \longrightarrow F_1 \stackrel{f_1, \ldots, f_n}{\longrightarrow} F_0 = R \longrightarrow R/I \longrightarrow 0$$

and the length of this resolution is d due to the Auslander-Buchsbaum formula, since the depth of R/I is 0. Then the top-dimensional syzygy module is free, just because

$$\mathrm{Syz}_{d-1} = F_d = R^{\beta_d} .$$

The cohomology class is then described as

$$c_{d-1} = (c_{d-1,j}) \text{ where } c_{d-1,j} \in H^{d-1}(U, \mathcal{O}_U)\,,$$

and it is 0 if and only if all components are 0. These components lie in the (often) well understood cohomology module

$$H^{d-1}(U, \mathcal{O}_U) = H^d_{\mathfrak{m}}(R)\,.$$

If R is even regular, then every ideal has a finite free resolution and the ideal containment problem $f \in I$ reduces to the computation of (the components of) c_{d-1} and deciding whether they are 0 or not.

EXAMPLE 1.2. Let R be a Cohen-Macaulay local ring of dimension $d \geq 2$ and let $I = (f_1, \ldots, f_d)$ be a (full) parameter ideal. We consider the Koszul-resolution of these parameters. This is a finite resolution and the top-dimensional syzygy sheaf is just the structure sheaf. A direct computation using Čech-cohomology shows that for an element $f \in R$ we get

$$c_{d-1} = \delta^{d-1}(f) = \frac{f}{f_1 \cdots f_d} \in H^d_{\mathfrak{m}}(R)\,.$$

Cohomological criteria for tight closure

The following theorem says that tight closure (for a Cohen-Macaulay ring of dimension at least 2) is a "cohomological closure operation", i.e. it depends only on the induced cohomological class over the punctured spectrum U. This is the base for understanding tight closure on U and (in the graded case) on $\operatorname{Proj}(R)$.

THEOREM 1.3. *Suppose that R is a Cohen-Macaulay ring of positive characteristic p and of dimension $d \geq 2$. Let I be an $\mathfrak{m}$-primary ideal. Let $F_\bullet$ be a free (not necessarily minimal) resolution of I, let Syz_j be the corresponding syzygy sheaves, let $f \in R$ be an element and let $c_j \in H^j(D(\mathfrak{m}), \operatorname{Syz}_j)$ be the corresponding cohomology classes. Then for each j, $1 \leq j \leq d-1$, we have the equivalence that $f \in I^*$ if and only if c_j is tightly 0 in the sense that there exists z not in any minimal prime ideal such that*

$$zF^e(c_j) = 0$$

in $H^j(D(\mathfrak{m}), F^{e*}\operatorname{Syz}_j)$ for all e.[1]

Proof. We consider the short exact sheaf sequences

$$0 \longrightarrow \operatorname{Syz}_i \longrightarrow \mathcal{O}_U^{\beta_i} \longrightarrow \operatorname{Syz}_{i-1} \longrightarrow 0$$

on $U = D(\mathfrak{m})$ coming from the resolution for $i = 1, \ldots, d-1$ (Syz_0 is just the structure sheaf). Because all these sheaves are locally free, taking the

[1] We work with the Frobenius pull-back of the sheaves and of the class.

absolute Frobenius (and all its iterations) is exact, therefore we get short exact sequences[2]

$$0 \longrightarrow F^{e*}\,\mathrm{Syz}_i \longrightarrow \mathcal{O}_U^{\beta_i} \longrightarrow F^{e*}\,\mathrm{Syz}_{i-1} \longrightarrow 0$$

and cohomology pull-backs $F^{e*}(c_j) \in H^j(U, F^{e*}\,\mathrm{Syz}_j)$. Note also that for $i = 1$ and $I = (f_1, \ldots, f_n)$ we get

$$0 \longrightarrow F^{e*}\,\mathrm{Syz}_1 \longrightarrow \mathcal{O}_U^n \stackrel{f_1^q,\ldots,f_n^q}{\longrightarrow} \mathcal{O}_U \longrightarrow 0\,,$$

so the image of this map inside $R = \Gamma(U, \mathcal{O}_U)$ is exactly $I^{[q]}$. By the universal property of the absolute Frobenius and of the connecting homomorphisms in cohomology we have

$$F^{e*}(c_j) \;=\; F^{e*}(\delta^j(f)) \;=\; \delta^j(F^{e*}f) \;=\; \delta^j(f^q)$$

and also

$$zF^{e*}(c_j) \;=\; \delta^j(zf^q).$$

Because of the injectivity of δ in the given range we have that zf^q belongs to the ideal $I^{[q]}$ if and only if $zF^{e*}(c_1) = 0$ if and only if $zF^{e*}(c_j) = 0$. □

In general it is difficult to control the sequence $F^{e*}\,\mathrm{Syz_j}$, $e \in \mathbb{N}$, of locally free sheaves. It is one of the goals of these lectures to discuss situations where it can be controlled. The easiest case is when $\mathrm{Syz_j}$ is free (which is only possible for $j = d - 1$). In this case we can deduce two well-known theorems in tight closure theory. The presented proofs are different from the classical proofs and give a hint how we will argue in the next lectures.

The standard proof of the following theorem uses the fact that the Frobenius is flat for regular rings. We use instead that every ideal in a regular ring has a finite free resolution or, equivalently, that the top-dimensional syzygy sheaf is free.

Theorem 1.4. *Suppose that R is a regular local ring of positive characteristic p and of dimension d. Then for every ideal I we have $I^* = I$.*

Proof. We assume $d \geq 2$, lower dimensions may be treated directly. Because of $I^* \subseteq \bigcap_{n\in\mathbb{N}}(I + \mathfrak{m}^n)^*$ we can also reduce to the case of a primary ideal I. Suppose that $f \notin I$, and let $c_{d-1} \in H^{d-1}(D(\mathfrak{m}), \mathcal{O}_U)^m = H^d_{\mathfrak{m}}(R)^m$ be the corresponding non-zero class arising from a finite free resolution. At least one component, say $c' \in H^d_{\mathfrak{m}}(R)$ is then also non-zero, and we can write it in terms of Čech-cohomology as

$$c' = \frac{h}{x_1^{n_1} \cdots x_d^{n_d}}\,,$$

[2]Note that these sequences come also from the Frobenius pull-backs of the resolution complex by restriction to U. The Frobenius pull-backs of the resolution complex are however not exact anymore. Hence it is better to work only on U. So it is also allowed that the "resolution" we start with is only exact on U.

where $x_1, \ldots, x_n$ is a regular system of parameters of R and $n_j \geq 1$. We have to show that there is no $z \neq 0$ such that $zF^{e*}(c) = 0$ for all $e \in \mathbb{N}$. Multiplying the class with some element of R we may assume that h is a unit.[3] We have (with $q = p^e$)

$$F^{e*}(c') \;=\; \frac{h^q}{x_1^{qn_1} \cdots x_d^{qn_d}}$$

and its annihilator is $(x_1^{qn_1}, \ldots, x_d^{qn_d})$. But then

$$\bigcap_{e \in \mathbb{N}} (x_1^{qn_1}, \ldots, x_d^{qn_d}) \;\subseteq\; \bigcap_{e \in \mathbb{N}} (x_1^{n_1}, \ldots, x_d^{n_d})^q \;=\; 0.$$

□

The standard proof of the following fact is based on the Briançon-Skoda theorem. It is also true without the Cohen-Macaulay condition.

THEOREM 1.5. *Suppose that R is a graded Cohen-Macaulay ring of positive characteristic p and of dimension d. Let $f_1, \ldots, f_d$ be a homogeneous system of parameters. Then $R_{\geq \deg(f_1)+\ldots+\deg(f_d)} \subseteq (f_1, \ldots, f_d)^*$.*

Proof. We assume $d \geq 2$, lower dimensions may be treated directly. We consider the Koszul-resolution of the parameter ideal $(f_1, \ldots, f_d)$. A homogeneous element h of degree m gives rise to the graded cohomology class

$$c = c_{d-1} = \frac{h}{f_1 \cdots f_d} \in H^d_{\mathfrak{m}}(R)\,.$$

Under the condition that $\deg(h) \geq \sum_{j=1}^{d} \deg(f_i)$ this cohomology class has nonnegative degree. It is known that $H^d_{\mathfrak{m}}(R)$ is 0 for sufficiently large degree, i.e. there is a number[4] a such that $(H^d_{\mathfrak{m}}(R))_i = 0$ for all $i > a$. Now choose a homogeneous element $z \in R_{>a}$ which does not belong to any minimal prime ideal. Then the degree of

$$zF^{e*}(c) \;=\; z\frac{h^q}{f_1^q \cdots f_d^q}$$

is at least $\geq \deg(z) > a$, so this class must be zero and therefore h belongs to the tight closure of the parameter ideal. □

A theorem of Hara states that the "converse" of this theorem is also true for prime numbers $p \gg 0$, i.e. that an element of degree smaller than the sum of the degrees of the parameters can belong to the tight closure only if it belongs already to the ideal itself.

[3] First we write the class as a sum of fractions where the numerators are units and the denominators are several monomials. Then we can multiply with a monomial so that only one summand remains.

[4] This number is called the *a-invariant* of R.

A classical example of this inclusion criterion is that

$$z^2 \in (x,y)^*$$

in the Fermat ring $K[x,y,z]/(x^3+y^3+z^3)$ in characteristic $p \neq 3$. The same holds for any equation under the condition that this (hyper)-surface is a normal domain and x and y are parameters. An easy looking question for a non-parameter ideal was raised by M. McDermott, namely whether

$$xyz \in (x^2,y^2,z^2)^* \text{ in } K[x,y,z]/(x^3+y^3+z^3)\,.$$

This was answered positively by A. Singh by a long "equational" argument.

EXAMPLE 1.6. Let $R = K[x,y,z]/(x^3+y^3+z^3)$, where K is a field of positive characteristic $p \neq 3$, $I = (x^2,y^2,z^2)$ and $f = xyz$. We consider the short exact sequence

$$0 \longrightarrow \operatorname{Syz}(x^2,y^2,z^2) \longrightarrow \mathcal{O}_U^3 \stackrel{x^2,y^2,z^2}{\longrightarrow} \mathcal{O}_U \longrightarrow 0$$

and the cohomology class

$$c = \delta(xyz) \in H^1(U, \operatorname{Syz}(x^2,y^2,z^2))\,.$$

We want to show that $zF^{e*}(c) = 0$ for all $e \geq 0$ (here the test element z equals the element z in the ring). It is helpful to work with the graded structure on this syzygy sheaf (or to work on the corresponding elliptic curve $\operatorname{Proj} R$ directly). Now the equation $x^3+y^3+z^3$ can be considered as a syzygy (of total degree 3) for x^2,y^2,z^2 yielding an inclusion

$$0 \longrightarrow \mathcal{O}_U \longrightarrow \operatorname{Syz}(x^2,y^2,z^2)\,.$$

Since this syzygy does not vanish anywhere on U the quotient is invertible and in fact isomorphic to the structure sheaf. Hence we have

$$0 \longrightarrow \mathcal{O}_U \longrightarrow \operatorname{Syz}(x^2,y^2,z^2) \longrightarrow \mathcal{O}_U \longrightarrow 0$$

and the cohomology sequence

$$\longrightarrow H^1(U,\mathcal{O}_U)_s \longrightarrow H^1(U,\operatorname{Syz}(x^2,y^2,z^2))_{s+3} \longrightarrow H^1(U,\mathcal{O}_U)_s \longrightarrow 0\,,$$

where s denotes the sth graded piece. Our cohomology class c lives in $H^1(U,\operatorname{Syz}(x^2,y^2,z^2))_3$, so its Frobenius pull-backs live in $H^1(U,F^{e*}\operatorname{Syz}(x^2, y^2,z^2))_{3q}$, and we can have a look at the cohomology of the pull-backs of the sequence, i.e.

$$\longrightarrow H^1(U,\mathcal{O}_U)_0 \longrightarrow H^1(U,F^{e*}\operatorname{Syz}(x^2,y^2,z^2))_{3q} \longrightarrow H^1(U,\mathcal{O}_U)_0 \longrightarrow 0\,.$$

The class $zF^{e*}(c)$ lives in $H^1(U,F^{e*}\operatorname{Syz}(x^2,y^2,z^2))_{3q+1}$. It is mapped on the right to $H^1(U,\mathcal{O}_U)_1$, which is 0 (because we are working over an elliptic curve), hence it comes from the left, which is $H^1(U,\mathcal{O}_U)_1 = 0$. So $zF^{e*}(c) = 0$ and $f \in (x^2,y^2,z^2)^*$.