Holger Brenner (Osnabrück) **Ann Arbor 2012**

Computation of tight closure

Lecture 3

In this lecture we want to discuss how tight closure inclusion or non-inclusion behaves when we change the data a little bit. We may change the characteristic, or some parameters in the equation defining the rings, or some parameters in the generators defining the ideals. If $\operatorname{Syz}(f_1, \ldots, f_n)(m)$ is strongly semistable and of positive (or negative) degree, then it is ample (or anti-ample). This is an open property, so deforming some parameters will not change tight closure inclusion or exclusion. However, if $\operatorname{Syz}(f_1, \ldots, f_n)(m)$ is strongly semistable of degree 0, then small pertubations may destroy strong semistability and hence affect tight closure inclusion or exclusion.

Affineness under deformations

We consider a base scheme B and a morphism

$$Z \longrightarrow B$$

together with an open subscheme $W \subseteq Z$. For every base point $b \in B$ we get the open subset

$$W_b \subseteq Z_b$$

inside the fiber Z_b. It is a natural question to ask how properties of W_b vary with b. In particular we may ask how the cohomological dimension of W_b varies and how the affineness[1] may vary.

In the algebraic setting we have a D-algebra S and an ideal $\mathfrak{a} \subseteq S$ which defines for every prime ideal $\mathfrak{p} \in \operatorname{Spec}(D)$ the extended ideal $\mathfrak{a}_{\mathfrak{p}}$ in $S \otimes_D \kappa(\mathfrak{p})$.

This question is already interesting when B is a one-dimensional integral scheme, in particular in the following two situations.

(1) $B = \operatorname{Spec}(\mathbb{Z})$. Then we talk about an *arithmetic deformation* and want to know how affineness varies with the characteristic and how the relation is to characteristic zero.

(2) $B = \mathbb{A}^1_K = \operatorname{Spec}(K[t])$, where K is a field. Then we talk about a *geometric deformation* and want to know how affineness varies with the parameter t, in particular how the behaviour over the special

[1] The cohomological dimension of a scheme X is the maximal number i such that $H^i(X, \mathcal{F}) \neq 0$ for some quasicoherent sheaf $\mathcal{F}$. A noetherian scheme is affine if and only if its cohomological dimension is 0. Tight closure can be characterized by the cohomological dimension of torsors.

points where the residue class field is algebraic over K is related to the behaviour over the generic point.

It is fairly easy to show that if the open subset in the generic fiber is affine, then also the open subsets are affine for almost all special points.

We deal with this question where W is a torsor over a family of smooth projective curves (or a torsor over a punctured twodimensional spectrum). The arithmetic as well as the geometric variant of this question are directly related to questions in tight closure theory. Because of the above mentioned degree criteria in the strongly semistable case, a weird behavior of the affineness property of torsors is only possible if we have a weird behavior of strong semistability.

Arithmetic deformations

We start with the arithmetic situation, the following example is due to Brenner and Katzman.

EXAMPLE 3.1. Consider $\mathbb{Z}[x, y, z]/(x^7 + y^7 + z^7)$ and consider the ideal $I = (x^4, y^4, z^4)$ and the element $f = x^3y^3$. Consider reductions $\mathbb{Z} \to \mathbb{Z}/(p)$. Then

$$f \in I^* \text{ holds in } \mathbb{Z}/(p)[x, y, z]/(x^7 + y^7 + z^7) \text{ for } p = 3 \mod 7$$

and

$$f \notin I^* \text{ holds in } \mathbb{Z}/(p)[x, y, z]/(x^7 + y^7 + z^7) \text{ for } p = 2 \mod 7\,.$$

In particular, the bundle $\mathrm{Syz}\,(x^4, y^4, z^4)$ is semistable in the generic fiber, but not strongly semistable for any reduction $p = 2 \mod 7$. The corresponding torsor is an affine scheme for infinitely many prime reductions and not an affine scheme for infinitely many prime reductions.

In terms of affineness (or local cohomology) this example has the following properties: the ideal

$$(x, y, z) \subseteq \mathbb{Z}/(p)[x, y, z, s_1, s_2, s_3]/(x^7 + y^7 + z^7, s_1x^4 + s_2y^4 + s_3z^4 + x^3y^3)$$

has cohomological dimension 1 if $p = 3 \mod 7$ and has cohomological dimension 0 (equivalently, $D(x, y, z)$ is an affine scheme) if $p = 2 \mod 7$.

Geometric deformations - A counterexample to the localization problem

Let $S \subseteq R$ be a multiplicative system and I an ideal in R. Then the *localization problem* of tight closure is the question whether the identity

$$(I^*)_S = (IR_S)^*$$

holds.

Here the inclusion $\subseteq$ is always true and $\supseteq$ is the problem. The problem means explicitly:

if $f \in (IR_S)^*$, can we find an $h \in S$ such that $hf \in I^*$ holds in R?

Proposition 3.2. *Let $\mathbb{Z}/(p) \subset D$ be a one-dimensional domain and $D \subseteq R$ of finite type, and I an ideal in R. Suppose that localization holds and that*

$$f \in I^* \text{ holds in } R \otimes_D Q(D) = R_{D^*} = R_{Q(D)}$$

($S = D^* = D \setminus \{0\}$ is the multiplicative system). Then $f \in I^*$ holds in $R \otimes_D \kappa(\mathfrak{p})$ for almost all $\mathfrak{p}$ in Spec D.

In order to get a counterexample for the localization property we will look now at geometric deformations:

$$D = \mathbb{F}_p[t] \subset \mathbb{F}_p[t][x, y, z]/(g) = S$$

where t has degree 0 and x, y, z have degree 1 and g is homogeneous. Then (for every field $\mathbb{F}_p[t] \to K$)

$$S \otimes_{\mathbb{F}_p[t]} K$$

is a two-dimensional standard-graded ring over K. For residue class fields of points of $\mathbb{A}^1_{\mathbb{F}_p} = \text{Spec } \mathbb{F}_p[t]$ we have basically two possibilities.

- $K = \mathbb{F}_p(t)$, the function field. This is the *generic* or *transcendental* case.
- $K = \mathbb{F}_q$, the *special* or *algebraic* or *finite* case.

How does $f \in I^*$ vary with K? To analyze the behavior of tight closure in such a family we can use what we know in the two-dimensional standard-graded situation.

In order to establish an example where tight closure does not behave uniformly under a geometric deformation we first need a situation where strong semistability does not behave uniformly. Such an example was given, in terms of Hilbert-Kunz theory, by Paul Monsky in 1997.

Example 3.3. Let

$$g = z^4 + z^2xy + z(x^3 + y^3) + (t + t^2)x^2y^2 \,.$$

Consider

$$S = \mathbb{F}_2[t, x, y, z]/(g) \,.$$

Then Monsky proved the following results on the *Hilbert-Kunz multiplicity* of the maximal ideal (x, y, z) in $S \otimes_{\mathbb{F}_2[t]} L$, L a field:

$$e_{HK}(S \otimes_{\mathbb{F}_2[t]} L) = \begin{cases} 3 \text{ for } L = \mathbb{F}_2(t) \\ 3 + \frac{1}{4^d} \text{ for } L = \mathbb{F}_q = \mathbb{F}_p(\alpha),\ (t \mapsto \alpha,\ d = \deg(\alpha)) \,. \end{cases}$$

By the geometric interpretation of Hilbert-Kunz theory this means that the restricted cotangent bundle

$$\mathrm{Syz}\,(x, y, z) = (\Omega_{\mathbb{P}^2})_C$$

is strongly semistable in the transcendental case, but not strongly semistable in the algebraic case. In fact, for $d = \deg(\alpha)$, $t \mapsto \alpha$, where $K = \mathbb{F}_2(\alpha)$, the d-th Frobenius pull-back destabilizes.

The maximal ideal (x, y, z) can not be used directly. However, we look at the second Frobenius pull-back which is (characteristic two) just

$$I = (x^4, y^4, z^4)\,.$$

By the degree formula we have to look for an element of degree 6. Let's take

$$f = y^3 z^3\,.$$

This is our example (x^3y^3 does not work). First, by strong semistability in the transcendental case we have

$$f \in I^* \text{ in } R \otimes \mathbb{F}_2(t)$$

by the degree formula. If localization would hold, then f would also belong to the tight closure of I for almost all algebraic instances $\mathbb{F}_q = \mathbb{F}_2(\alpha)$, $t \mapsto \alpha$. Contrary to that we show that for all algebraic instances the element f belongs never to the tight closure of I.

Lemma 3.4. *Let $\mathbb{F}_q = \mathbb{F}_p(\alpha)$, $t \mapsto \alpha$,*
$\deg(\alpha) = d$. Set $Q = 2^{d-1}$. Then

$$xyf^Q \notin I^{[Q]}\,.$$

Proof. This is an elementary but tedious computation. □

Theorem 3.5. *Tight closure does not commute with localization.*

Proof. One knows in our situation that xy is a so-called test element. Hence the previous Lemma shows that $f \notin I^*$. □

Corollary 3.6. Tight closure is not plus closure in graded dimension two for fields with transcendental elements.

Proof. Consider

$$R = \mathbb{F}_2(t)[x, y, z]/(g)\,.$$

In this ring $y^3z^3 \in I^*$, but it can not belong to the plus closure. Else there would be a curve mapping $Y \to C_{\mathbb{F}_2(t)}$ which annihilates the cohomology class c and this would extend to a mapping of relative curves almost everywhere. □

Generic results

Is it more difficult to decide whether an element belongs to the tight closure of an ideal or to the ideal itself? I want to discuss one situation where tight closure behaves easier.

Suppose that we are in a graded situation of a given ring (or a given ring dimension) and have fixed a number (at least the ring dimension) of homogeneous generators and their degrees. Suppose that we want to know the degree bound for (tight closure or ideal) inclusion for generic choice of the ideal generators. Generic means that we write the coefficients of the generators as indeterminates and consider the situation over the (large) affine space corresponding to these indeterminates or over its function field. This problem is already interesting and difficult for the polynomial ring: Suppose we are in $P = K[X, Y, Z]$ and want to study the generic inclusion bound for say $n \geq 4$ generic polynomials $F_1, \ldots, F_n$ all of degree a. What is the minimal degree number m such that $P_{\geq m} \subseteq (F_1, \ldots, F_n)$. The answer is

$$\left\lceil \frac{1}{2(n-1)} \left(3 - 3n + 2an + \sqrt{1 - 2n + n^2 + 4a^2 n} \right) \right\rceil .$$

This rests on the fact that the Fröberg conjecture is solved in dimension 3 by Anick (the Fröberg conjecture gives a precise description of the Hilbert function for an ideal in a polynomial ring which is generically generated. Here we only need to know in which degree the Hilbert function of the residue class ring becomes 0).

The corresponding generic ideal inclusion bound for arbitrary graded rings depends heavily (already in the parameter case) on the ring itself. Surprisingly, the generic ideal inclusion bound for tight closure does not depend on the ring and is only slightly worse than the bound for the polynomial ring. The following theorem is due to Brenner and Fischbacher-Weitz.

THEOREM 3.7. *Let $d \geq 1$ and $a_1, \ldots, a_n$ be natural numbers, $n \geq d + 1$. Let $K[x_0, x_1, \ldots, x_d] \subseteq R$ be a finite extension of standard-graded domains (a graded Noether normalization). Suppose that there exist n homogeneous polynomials $g_1, \ldots, g_n$ in $P = K[x_0, x_1, \ldots, x_d]$ with $\deg(g_i) = a_i$ such that $P_{\geq m} \subseteq (g_1, \ldots, g_n)$. Then*

(1) *$R_{m+d} \subseteq (f_1, \ldots, f_n)^*$ holds in the generic point of the parameter space of homogeneous elements $f_1, \ldots, f_n$ in R of this degree type (the coefficients of the f_i are taken as indeterminates).*
(2) *$R_{m+d+1} \subseteq (f_1, \ldots, f_n)^F \subseteq (f_1, \ldots, f_n)^*$ holds for (open) generic choice of homogeneous elements $f_1, \ldots, f_n$ in R of this degree type.*

EXAMPLE 3.8. Suppose that we are in $K[x, y, z]$ and that $n = 4$ and $a = 10$. Then the generic degree bound for ideal inclusion in the polynomial ring is 19. Therefore by Theorem 3.7 the generic degree bound for tight closure inclusion in a three-dimensional graded ring is 21.

EXAMPLE 3.9. Suppose that $n = d+1$ in the situation of Theorem 3.7. Then generic elements $f_1, \ldots, f_{d+1}$ are parameters. In the polynomial ring $P = K[x_0, x_1, \ldots, x_d]$ we have for parameters of degree $a_1, \ldots, a_{d+1}$ the inclusion

$$P_{\geq \sum_{i=0}^{d} a_i - d} \subseteq (f_1, \ldots, f_{d+1}) \, ,$$

because the graded Koszul resolution ends in $R(-\sum_{i=0}^{d} a_i)$ and

$$(H_{\mathfrak{m}}^{d+1}(P))_k = 0 \text{ for } k \geq -d \, .$$

So the theorem implies for a graded ring R finite over P that $P_{\geq \sum_{i=0}^{d} a_i} \subseteq (f_1, \ldots, f_{d+1})$ holds for generic elements. But by the graded Briançon-Skoda Theorem (see Theorem 1.5). this holds without the generic assumption.