We consider a linear homogeneous equation

$$f_1t_1 + \ldots + f_nt_n = 0$$

and also a linear inhomogeneous equation

$$f_1t_1 + \ldots + f_nt_n = f_0\,,$$

where $f_1, \ldots, f_n, f_0$ are elements in a field K. The solution set to the homogeneous equation is a vector space V over K of dimension $n-1$ or n (if all f_i are 0). For the solution set T of the inhomogeneous equation there exists an action

$$V \times T \longrightarrow T,\ (v,t) \longmapsto v+t,$$

and if we fix one solution $t_0 \in T$ (supposing that one solution exists), then there exists a bijection

$$V \longrightarrow T,\ v \longmapsto v+t_0.$$

Suppose now that X is a geometric object (a topological space, a manifold, a variety, the spectrum of a ring) and that

$$f_1, \ldots, f_n, f_0 : X \longrightarrow K$$

are functions on X. Then we get the space

$$T = \{(P, t_1, \ldots, t_n) |\, f_1(P)t_1 + \ldots + f_n(P)t_n = f_0(P)\} \subseteq X \times K^n$$

together with the projection to X. For a fixed point $P \in X$, the fiber of T over P is the solution set to the corresponding inhomogeneous equation. For $f_0 = 0$, we get a solution space

$$V \longrightarrow X,$$

where all fibers are vector spaces (maybe of non-constant dimension) and where again V acts on T. Locally, there are bijections $V \cong T$. Let

$$U = \{Q \in X |\, f_i(Q) \neq 0 \text{ for at least one } i\}\,.$$

Then $V|_U$ is a vector bundle and $T|_U$ is a $V|_U$-principal fiber bundle. T is fiberwise an affine space over the base and locally an affine space over U, so locally it is an easy object. We are interested in global properties of T and of $T|_U$.

Group schemes and their actions

We have seen in the first lecture that a vector bundle is in particular a group scheme, i.e. there is a scheme morphism (the addition)

$$\alpha : V \times_X V \longrightarrow V,$$

a morphism

$$0 : X \longrightarrow V$$

(the zero section) and a negative morphism

$$- : V \longrightarrow V$$

fulfilling several natural conditions. In general, as a group may act on a set, a group scheme may act on another scheme. We give the precise definition.

DEFINITION 2.1. Let (G, α, n) denote a group scheme over a scheme X and let

$$T \longrightarrow X$$

denote a scheme over X. A morphism

$$\beta : G \times_X T \longrightarrow T$$

is called a *group scheme action* of G on T, if the diagram

$$\begin{array}{ccc} G \times_X G \times_X T & \stackrel{\mathrm{id}_G \times \beta}{\longrightarrow} & G \times_X T \\ \downarrow & & \downarrow \\ G \times_X T & \stackrel{\beta}{\longrightarrow} & T \end{array}$$

commutes and if the composition

$$T \stackrel{\cong}{\longrightarrow} X \times_X T \stackrel{n \times \mathrm{Id}_T}{\longrightarrow} G \times_X T \stackrel{\beta}{\longrightarrow} T$$

is the identity on T.

The multiplication of a group scheme may be considered as an operation of the group scheme on itself. These are in some sense the easiest group operations. The next easiest case is the situation of an operation which looks locally like the group acting on itself. This leads to the following natural definition.

DEFINITION 2.2. Let G denote a group scheme over a scheme X. A scheme $T \to X$ together with a group scheme action

$$\beta : G \times_X T \longrightarrow T$$

is called a geometric (Zariski)-*torsor* for G (or a *G-principal fiber bundle* or a *principal homogeneous space*) if there exists an open covering $X = \bigcup_{i \in I} U_i$ and isomorphisms

$$\varphi_i : T|_{U_i} \longrightarrow G|_{U_i}$$

such that the diagrams (we set $U = U_i$ and $\varphi = \varphi_i$)

$$\begin{array}{ccc} G|_U \times_U T|_U & \stackrel{\beta}{\longrightarrow} & T|_U \\ \downarrow & & \downarrow \\ G|_U \times_U G|_U & \stackrel{\beta}{\longrightarrow} & G|_U \end{array}$$

commute.

Torsors of vector bundles

We look now at the torsors of vector bundles. They can be classified in the following way.

PROPOSITION 2.3. *Let X denote a Noetherian separated scheme and let*

$$p : V \longrightarrow X$$

denote a geometric vector bundle on X with sheaf of sections $\mathcal{S}$. Then there exists a correspondence between first cohomology classes $c \in H^1(X, \mathcal{S})$ and geometric V-torsors.

Proof. We will describe this correspondence. Let T denote a V-torsor. Then there exists by definition an open covering $X = \bigcup_{i \in I} U_i$ such that there exists isomorphisms

$$\varphi_i : T|_{U_i} \longrightarrow V|_{U_i}$$

which are compatible with the action of $V|_{U_i}$ on itself. The isomorphisms φ_i induce automorphisms

$$\psi_{ij} = \varphi_j \circ \varphi_i^{-1} : V|_{U_i \cap U_j} \longrightarrow V|_{U_i \cap U_j}.$$

These automorphisms are compatible with the action of V on itself, and this means that they are of the form

$$\psi_{ij} = \mathrm{Id}_V |_{U_i \cap U_j} + s_{ij}$$

with suitable sections $s_{ij} \in \Gamma(U_i \cap U_j, \mathcal{S})$. This family defines a Cech-cocycle for the covering and gives therefore a cohomology class in $H^1(X, \mathcal{S})$. For the reverse direction, suppose that the cohomology class $c \in H^1(X, \mathcal{S})$ is represented by a Cech-cocycle $s_{ij} \in \Gamma(U_i \cap U_j, \mathcal{S})$ for an open covering $X = \bigcup_{i \in I} U_i$. Set T_iÂ $:= V|_{U_i}$. We take the morphisms

$$\psi_{ij} : T_i|_{U_i \cap U_j} = V|_{U_i \cap U_j} \longrightarrow V|_{U_i \cap U_j} = T_j|_{U_i \cap U_j}$$

given by ψ_{ij}Â $:= \mathrm{Id}_V |_{U_i \cap U_j} + s_{ij}$ to glue the T_i together to a scheme T over X. This is possible since the cocycle condition guarantees the glueing condition for schemes (EGA I, 0, 4.1.7). The action of V_i on T_i by itself glues also together to give an action on T. $\square$

It follows immediately that for an affine scheme there are no non-trivial torsor for any vector bundle. There will however be in general many non-trivial torsors on the punctured spectrum (and on a projective variety). This is already true if we take the affine line $\mathbb{A}^1_X$ over X (corresponding to the structure sheaf) as vector bundle and consider the $\mathbb{A}^1_X$-torsors. These are in particular an interesting class of schemes for a two-dimensional ring.

EXAMPLE 2.4. Let $(R, \mathfrak{m})$ denote a two-dimensional local noetherian domain and let f and g be two parameters in R, i.e. elements which generate the maximal ideal $\mathfrak{m}$ up to radical. Then the punctured spectrum is

$$U = D(\mathfrak{m}) = D(f, g) = D(f) \cup D(g)$$

and every cohomology class $c \in H^1(U, \mathcal{O}_X)$ can be represented by a Cech cohomology class

$$c = \frac{h}{f^i g^j}$$

with some $h \in R$ $(i, j \geq 1)$. If R is normal then the cohomology class is 0 if and only if $h \in (f^i, g^j)$. To see this, we work with $i = j = 1$, which does not make a difference as powers of powers are again parameters. We note that under the normality assumption the syzygy module $\operatorname{Syz}(f, g)$ is free of rank one ($(g, -f)$ is a generator). Then we look at the short exact sequence on U,

$$0 \longrightarrow \mathcal{O}_U \cong \operatorname{Syz}(f, g) \longrightarrow \mathcal{O}_U^2 \xrightarrow{f,g} \mathcal{O}_U \longrightarrow 0$$

and its corresponding long exact sequence of cohomology,

$$0 \longrightarrow R \longrightarrow R^2 \xrightarrow{f,g} R \xrightarrow{\delta} H^1(U, \mathcal{O}) \longrightarrow \ldots .$$

Here, the connecting homomorphisms δ for an element $h \in R$ works in the following way. On both open subsets $D(f)$ and $D(g)$ we get the local representatives $(\frac{h}{f}, 0)$ and $(0, \frac{h}{g})$ and their difference, considered in $\Gamma(D(fg), \mathcal{O}_U)$, defines the cohomology class. This difference is just $\frac{h}{fg}$. By the exactness of the long cohomology sequence, $c = \delta(h) = \frac{h}{fg} = 0$ if and only if h comes from the left, which is true if and only if h belongs to the ideal generated by (f, g). If we want to realize the geometric torsor corresponding to such a cohomology class, we start with two affine lines over $D(f)$ and $D(g)$, which we write as $\operatorname{Spec}(R_f[V])$ and $\operatorname{Spec}(R_g[W])$. According to Proposition 2.3 these have to be glued with the identification $W = V + \frac{h}{fg}$.

Forcing algebras and induced torsors

We have seen in the first lecture that the spectrum of the algebra

$$A = R[S_1, \ldots, S_n]/(f_1 S_1 + \ldots + f_n S_n)$$

is, when restricted to $U = D(f_1, \ldots, f_n)$, a model for the geometric syzygy bundle (and in general a syzygy group scheme). Now we make the transition from homogeneous linear equations to inhomogeneous linear equations with the following definition.

Definition 2.5. Let R be a commutative ring and let $f_1, \ldots, f_n$ and f be elements in R. Then the R-algebra

$$R[T_1, \ldots, T_n]/(f_1 T_1 + \ldots + f_n T_n - f)$$

is called the *forcing algebra* of these elements (or these data).

The forcing algebra B forces f to lie inside the extended ideal $(f_1, \ldots, f_n)B$. For every R-algebra S such that $f \in (f_1, \ldots, f_n)S$ there exists a (non unique) ring homomorphism $B \to S$ by sending T_i to the coefficient $s_i \in S$ in an expression $f = s_1 f_1 + \ldots + s_n f_n$. The forcing algebra induces the spectrum morphism

$$\mathrm{Spec}\,(B) \longrightarrow \mathrm{Spec}\,(R).$$

Over a point $x \in X = \mathrm{Spec}\,(R)$, the fiber of this morphism is given by

$$\mathrm{Spec}\,(B \otimes_R \kappa(x))\,,$$

and we can write

$$B \otimes_R \kappa(x) = \kappa(x)[T_1, \ldots, T_n]/(f_1(x)T_1 + \ldots + f_n(x)T_n - f(x))\,,$$

where $f_i(x)$ means the evaluation of f_i in the residue class field. Hence the $\kappa(x)$-points in the fiber are exactly the solution to the inhomogeneous linear equation $f_1(x)T_1 + \ldots + f_n(x)T_n = f(x)$. In particular, all the fibers are affine spaces. If we localize the forcing algebra at f_i we get

$$\begin{aligned} & (R[T_1, \ldots, T_n]/(f_1T_1 + \ldots + f_nT_n - f))_{f_i} \\ \cong \;& R_{f_i}[T_1, \ldots, T_{i-1}, T_{i+1}, \ldots, T_n], \end{aligned}$$

since we can write

$$T_i = -\sum_{j \neq i} \frac{fj}{f_i} T_j + \frac{f}{f_i}\,.$$

So over every $D(f_i)$ the spectrum of the forcing algebra is an $(n-1)$-dimensional affine space over the base. On the intersetions $D(f_i) \cap D(f_j)$ we get (as in the first lecture) two identifications with affine space, but the transition morphisms are now not linear anymore, only affine-linear (because of the translation with $\frac{f}{f_i}$).

Proposition 2.6. *Let R denote a commutative ring, $I = (f_1, \ldots, f_n)$ an ideal with the syzygy group scheme given by $G = \mathrm{Spec}\,(R[S_1, \ldots, S_n]/(f_1S_1 + \ldots + f_nS_n))$. Let $f \in R$ be another element and let*

$$Z = \mathrm{Spec}\,(R[T_1, \ldots, T_n]/(f_1T_1 + \ldots + f_nT_n - f))$$

be the spectrum of the corresponding forcing algebra. Then there is a natural action of G on Z. The restriction of this action to the open subset $U = D(I)$ makes $Z|_U$ to a torsor for the vector bundle $G|_U = \mathrm{Syz}\,(f_1, \ldots, f_n)|_U$.

Proof. The action is induced by the co-operation which sends $T_i \mapsto S_i + T_i$. In terms of points this is just the mapping which sends a syzygy $(s_1, \ldots, s_n)$ and a solution $t_1, \ldots, t_n$ of the forcing equation to the new solution $(s_1 + t_1, \ldots, s_n + t_n)$ of the forcing equation. For the second statement let $V = G|_U$ denote the syzygy bundle over U. We may consider the situation on $U_i = D(f_i)$, where we have the isomorphism

$$V|_{U_i} \longrightarrow T_{U_i},\ (s_1, \ldots, s_n) \longmapsto (s_1, \ldots, s_{i-1}, s_i + \frac{f}{f_i}, s_{i+1}, \ldots, s_n).$$

With these isomorphisms the natural diagram

$$\begin{array}{ccc} V|_{U_i} \times_{U_i} V|_{U_i} & \longrightarrow & V|_{U_i} \\ \downarrow & & \downarrow \\ V|_{U_i} \times_{U_i} T|_{U_i} & \longrightarrow & T|_{U_i} \end{array}$$

commutes, so locally the natural action of the vector bundle on the restricted spectrum of the forcing algebra is isomorphic to the addition of the vector bundle on itself. $\square$

As T_U is a V-torsor, and as every V-torsor is represented by a unique cohomology class, there should be natural cohomology class coming from the forcing data. To see this, let R be a noetherian ring and $I = (f_1, \ldots, f_n)$ be an ideal. Then on $U = D(I)$ we have the short exact sequence

$$0 \longrightarrow \mathrm{Syz}\,(f_1, \ldots, f_n) \longrightarrow \mathcal{O}_U^n \longrightarrow \mathcal{O}_U \longrightarrow 0\,.$$

An element $f \in R$ defines an element $f \in \Gamma(U, \mathcal{O}_U)$ and hence a cohomology class $\delta(f) \in H^1(U, \mathrm{Syz}\,(f_1, \ldots, f_n))$. Hence f defines in fact a $\mathrm{Syz}\,(f_1, \ldots, f_n)$-torsor over U. We will see that this torsor is induced by the forcing algebra given by $f_1, \ldots, f_n$ and f.

Theorem 2.7. *Let R denote a noetherian ring, let $I = (f_1, \ldots, f_n)$ denote an ideal and let $f \in R$ be another element. Let $c \in H^1(D(I), \mathrm{Syz}\,(f_1, \ldots, f_n))$ be the corresponding cohomology class and let $B = R[T_1, \ldots, T_n]/(f_1T_1 + \ldots + f_nT_n - f)$ denote the forcing algebra for these data. Then the scheme $\mathrm{Spec}\,(B)|_{D(I)}$ together with the natural action of the syzygy bundle on it is isomorphic to the torsor given by c.*

Proof. We compute the cohomology class $\delta(f) \in \mathrm{Syz}\,(f_1, \ldots, f_n)$ and the cohomology class given by the forcing algebra. For the first computation we look at the short exact sequence

$$0 \longrightarrow \mathrm{Syz}\,(f_1, \ldots, f_n) \longrightarrow \mathcal{O}_U^n \longrightarrow \mathcal{O}_U \longrightarrow 0\,.$$

On $D(f_i)$, the element f is the image of $(0, \ldots, 0, \frac{f}{f_i}, 0, \ldots, 0)$ (the non-zero entry is at the ith place). The cohomology class is therefore represented by the family of differences

$$(0, \ldots, 0, \frac{f}{f_i}, 0, \ldots, 0, -\frac{f}{f_j}, 0, \ldots, 0) \in \Gamma(D(f_i) \cap D(f_j), \mathrm{Syz}\,(f_1, \ldots, f_n))\,.$$

On the other hand, there are isomorphisms

$$V|_{D(f_i)} \longrightarrow T|_{D(f_i)},\ (s_1, \ldots, s_n) \longmapsto (s_1, \ldots, s_{i-1}, s_i + \frac{f}{f_i}, s_{i+1}, \ldots, s_n).$$

The difference of two such isomorphisms on $D(f_if_j)$ is the same as before. $\square$

EXAMPLE 2.8. We continue with Example 2.4, so let $(R, \mathfrak{m})$ denote a two-dimensional normal local noetherian domain and let f and g be two parameters in R. The torsor given by a cohomology class $c = \frac{h}{fg} \in H^1(U, \mathcal{O}_X)$ can be realized by the forcing algebra

$$R[T_1, T_2]/(fT_1 + gT_2 - h)\,.$$

Note that different forcing algebra may give the same torsor, because the torsor depends only on the spectrum of the forcing algebra restricted to the punctured spectrum of R. For example, the cohomology class $\frac{1}{fg} = \frac{fg}{f^2g^2}$ defines one torsor, but the two quotients yield the two forcing algebras $R[T_1, T_2]/(fT_1 + gT_2 + 1)$ and $R[T_1, T_2]/(f^2T_1 + g^2T_2 + fg)$, which are quite different. The fiber over the maximal ideal of the first one is empty, whereas the fiber over the maximal ideal of the second one is a plane. If R is regular, say $R = K[X, Y]$ (or the localization of this at (X, Y) or the corresponding power series ring) then the first cohomology classes are linear combinations of $\frac{1}{x^iy^j}$, $i, j \geq 1$. They are realized by the forcing algebras $K[X, Y]/(X^iT_1 + Y^jT_2 - 1)$. Since the fiber over the maximal ideal is empty, the spectrum of the forcing algebra equals the torsor. Or, the other way round, the torsor is itself an affine scheme.

It is a difficult question when a torsor is an affine scheme. In the next two lecture we will deal with global properties of torsors and forcing algebras and how these properties are related to closure operations of ideals. Exercise for Sunday: Show that f belongs to the radical of the ideal $(f_1, \ldots, f_n)$ if and only if the spectrum morphism

$$\mathrm{Spec}\,(R[T_1, \ldots, T_n]/(f_1T_1 + \ldots + f_nT_n - f)) \longrightarrow \mathrm{Spec}\,(R)$$

is surjective.