Systems of linear equations

We start with some linear algebra. Let K be a field. We consider a system of linear homogeneous equations over K,

$$f_{11}t_1 + \ldots + f_{1n}t_n = 0\,,$$

$$f_{21}t_1 + \ldots + f_{2n}t_n = 0\,,$$

$$\vdots$$

$$f_{m1}t_1 + \ldots + f_{mn}t_n = 0\,,$$

where the f_{ij} are elements in K. The solution set to this system of homogeneous equations is a vector space V over K (a subvector space of K^n), its dimension is $n - \mathrm{rk}(A)$, where $A = (f_{ij})_{ij}$ is the matrix given by these elements. Additional elements $f_1, \ldots, f_m \in K$ give rise to the system of inhomogeneous linear equations,

$$f_{11}t_1 + \ldots + f_{1n}t_n = f_1\,,$$

$$f_{21}t_1 + \ldots + f_{2n}t_n = f_2\,,$$

$$\vdots$$

$$f_{m1}t_1 + \ldots + f_{mn}t_n = f_m\,.$$

The solution set T of this inhomogeneous system may be empty, but nevertheless it is tightly related to the solution space of the homogeneous system. First of all, there exists an action

$$V \times T \longrightarrow T,\ (v,t) \longmapsto v+t,$$

because the sum of a solution of the homogeneous system and a solution of the inhomogeneous system is again a solution of the inhomogeneous system. This action is a group action of the group $(V, +, 0)$ on the set T. Moreover, if we fix one solution $t_0 \in T$ (supposing that at least one solution exists), then there exists a bijection

$$V \longrightarrow T,\ v \longmapsto v + t_0.$$

This means that the group V acts simply transitive on T, and so T can be identified with the vector space V, however not in a canonical way.

Suppose now that X is a geometric object (a topological space, a manifold, a variety, the spectrum of a ring) and that instead of elements in the field K we have functions

$$f_{ij}\colon X \longrightarrow K$$

on X (which are continuous, or differentiable, or algebraic). We form the matrix of functions $A = (f_{ij})_{ij}$, which yields for every point $P \in X$ a matrix $A(P)$ over K. Then we get from these data the space

$$V = \left\{ (P, t_1, \ldots, t_n) | \, A(P) \begin{pmatrix} t_1 \\ \\ t_n \end{pmatrix} = 0 \right\} \subseteq X \times K^n$$

together with the projection to X. For a fixed point $P \in X$, the fiber V_P of V over P is the solution space to the corresponding homogeneous system of linear equations given by inserting P. In particular, all fibers of the map

$$V \longrightarrow X,$$

are vector spaces (maybe of non-constant dimension). This vector space structures yield an addition

$$V \times_X V \longrightarrow V,\ (P; t_1, \ldots, t_n; s_1, \ldots, s_n) \longmapsto (P; t_1 + s_1, \ldots, t_n + s_n)$$

(only points in the same fiber can be added). The mapping

$$X \longrightarrow V,\ P \longmapsto (P; 0, \ldots, 0)$$

is called the zero-section.

Suppose now that there are additionally functions

$$f_1, \ldots, f_m\colon X \longrightarrow K$$

given. Then we can form the set

$$T = \left\{ (P, t_1, \ldots, t_n) | \, A(P) \begin{pmatrix} t_1 \\ \\ t_n \end{pmatrix} = \begin{pmatrix} f_1(P) \\ \\ f_n(P) \end{pmatrix} \right\} \subseteq X \times K^n$$

with the projection to X. Again, every fiber T_P of T over a point $P \in X$ is the solution set to the system of inhomogeneous linear equations which arises by inserting P. The actions of the fibers V_P on T_P (coming from linear algebra) extend to an action[1]

$$V \times_X T \longrightarrow T,\ (P; t_1, \ldots, t_n; s_1, \ldots, s_n) \longmapsto (P; t_1 + s_1, \ldots, t_n + s_n).$$

Also, if a (continuous, differentiable, algebraic) map

$$s\colon X \longrightarrow T$$

with $s(P) \in T_P$ exists, then we can construct an (continuous, differentiable, algebraic) isomorphism between V and T. However, different from the situation in linear algebra (which corresponds to the situation where X is just one point), such a section does rarely exist.

[1] $V \times_X T$ is the fiber product of $V \to X$ and $T \to X$.

These objects T have new and sometimes difficult global properties which we try to understand in these lectures. We will work mainly in an algebraic setting and restrict to the situation where just one equation

$$f_1T_1 + \ldots + f_nT_n = f$$

is given. Then in the homogeneous case ($f = 0$) the fibers are vector spaces of dimension $n-1$ or n, and the later holds exactly for the points $P \in X$ where $f_1(P) = \ldots = f_n(P) = 0$. In the inhomogeneous case the fibers are either empty or of dimension $n-1$ or n. We give some typical examples.

Example 1.1. We consider the line $X = \mathbb{A}^1_K$ (or $X = K, \mathbb{R}, \mathbb{C}$ etc.) with the (identical) function x. For $f_1 = x$ and $f = 0$, i.e. for the equation $xt = 0$, the geometric object V consists of a horizontal line (corresponding to the zero-solution) and a vertical line over $x = 0$. So all fibers except one are zero-dimensional vector spaces. For the equation $xt = 1$, T is a hyperbola, and all fibers are zero-dimensional with the exception that the fiber over $x = 0$ is empty.

For the equation $0t = 0$, V is just the affine cylinder over the base line. For the equation $0t = x$, T consists of one vertical line, almost all fibers are empty.

Example 1.2. Let X denote a plane ($K^2, \mathbb{R}^2, \mathbb{A}^2_K$) with coordinate functions x and y. We consider a linear equation of type

$$x^a t_1 + y^b t_2 = x^c y^d \, .$$

The fiber of the solution set T over a point $\neq (0,0)$ is onedimensional, whereas the fiber over $(0,0)$ has dimension two (for $a, b, c, d \geq 1$). Many properties of T depend on these four exponents.

In (most of) these example we can observe the following behavior. On an open subset, the dimension of the fibers is constant and equals $n-1$, whereas the fiber over some special points degenerates to an n-dimensional solution set (or becomes empty).

Forcing algebras

We describe now the algebraic setting of systems of linear equations depending on a base space. For a commutative ring R, its spectrum $X = \operatorname{Spec}(R)$ is a topological space on which the ring elements can be considered as functions. The value of $f \in R$ at a prime ideal $P \in \operatorname{Spec}(R)$ is just the image of f under the morphism $R \to R/P \to \kappa(P) = Q(R/P)$. In this interpretation, a ring element is a function with values in different fields. Suppose that R contains a field K. Then an element $f \in R$ gives rise to the ring homomorphism

$$K[Y] \longrightarrow R,\ Y \longmapsto f,$$

which itself gives rise to a scheme morphism

$$\mathrm{Spec}\,(R) \longrightarrow \mathrm{Spec}\,(K[Y]) \cong \mathbb{A}^1_K .$$

This is another way to consider f as a function on $\mathrm{Spec}\,(R)$.

The following object was introduced by M. Hochster in 1994 in his work on solid closure.

DEFINITION 1.3. Let R be a commutative ring and let $f_1, \ldots, f_n$ and f be elements in R. Then the R-algebra

$$R[T_1, \ldots, T_n]/(f_1T_1 + \ldots + f_nT_n - f)$$

is called the *forcing algebra* of these elements (or these data).

The forcing algebra B forces f to lie inside the extended ideal $(f_1, \ldots, f_n)B$ (hence the name) For every R-algebra S such that $f \in (f_1, \ldots, f_n)S$ there exists a (non unique) ring homomorphism $B \to S$ by sending T_i to the coefficient $s_i \in S$ in an expression $f = s_1f_1 + \ldots + s_nf_n$.

The forcing algebra induces the spectrum morphism

$$\mathrm{Spec}\,(B) \longrightarrow \mathrm{Spec}\,(R).$$

Over a point $P \in X = \mathrm{Spec}\,(R)$, the fiber of this morphism is given by

$$\mathrm{Spec}\,(B \otimes_R \kappa(P)) \, ,$$

and we can write

$$B \otimes_R \kappa(P) = \kappa(P)[T_1, \ldots, T_n]/(f_1(P)T_1 + \ldots + f_n(P)T_n - f(P)) \, ,$$

where $f_i(P)$ means the evaluation of the f_i in the residue class field. Hence the $\kappa(P)$-points in the fiber are exactly the solutions to the inhomogeneous linear equation $f_1(P)T_1 + \ldots + f_n(P)T_n = f(P)$. In particular, all the fibers are (empty or) affine spaces.

Forcing algebras and closure operations

Let R denote a commutative ring and let $I = (f_1, \ldots, f_n)$ be an ideal. Let $f \in R$ and let

$$B = R[T_1, \ldots, T_n]/(f_1T_1 + \ldots + f_nT_n - f)$$

be the corresponding forcing algebra and

$$\varphi\colon \ \mathrm{Spec}\,(B) \longrightarrow \mathrm{Spec}\,(R)$$

the corresponding spectrum morphism. How are properties of φ (or of the R-algebra B) related to certain ideal closure operations?

We start with some examples. The element f belongs to the ideal I if and only if we can write $f = r_1f_1 + \ldots + r_nf_n$. By the universal property of the forcing algebra this means that there exists an R-algebra-homomorphism

$$B \longrightarrow R,$$

hence $f \in I$ holds if and only if φ admits a scheme section. This is also equivalent to

$$R \longrightarrow B$$

admitting an R-module section or B being a pure R-algebra (so for forcing algebras properties might be equivalent which are not equivalent for arbitrary algebras).

We have a look at the radical of the ideal I,

$$\operatorname{rad}(I) = \left\{ f \in R \mid f^k \in I \text{ for some } k \right\} .$$

As this is quite a coarse closure operation we should expect that this corresponds to a quite coarse property of the morphism φ as well. Indeed, it is true that $f \in \operatorname{rad}(I)$ if and only if φ is surjective. This and the interpretation of other closure operations in terms of forcing algebras will be discussed in the tutorial session and in the next lectures.

Geometric vector bundles

We have seen that the fibers of the spectrum of a forcing algebra are (empty or) affine spaces. However, this is not only fiberwisely true, but more generally: If we localize the forcing algebra at f_i we get

$$(R[T_1, \ldots, T_n]/(f_1T_1 + \ldots + f_nT_n - f))_{f_i} \cong R_{f_i}[T_1, \ldots, T_{i-1}, T_{i+1}, \ldots, T_n],$$

since we can write

$$T_i = - \sum_{j \neq i} \frac{f_j}{f_i} T_j + \frac{f}{f_i} .$$

So over every $D(f_i)$ the spectrum of the forcing algebra is an $(n-1)$-dimensional affine space over the base. So locally, restricted to $D(f_i)$, we have isomorphisms

$$T|_{D(f_i)} \cong D(f_i) \times \mathbb{A}^{n-1} .$$

On the intersections $D(f_i) \cap D(f_j)$ we get two identifications with affine space, and the transition morphisms are linear if $f = 0$, but only affine-linear in general (because of the translation with $\frac{f}{f_i}$).

So the forcing algebra has locally the form $R_{f_i}[T_1, \ldots, T_{i-1}, T_{i+1}, \ldots, T_n]$ and its spectrum $\operatorname{Spec}(B)$ has locally the form $D(f_i) \times \mathbb{A}_K^{n-1}$. This description holds on the union $U = \bigcup_{i=1}^n D(f_i)$. Moreover, in the homogeneous case (f=0) the transition mappings are linear. Hence $V|_U$ is a geometric vector bundle according to the following definition.[2]

[2] V itself, the spectrum of a homogeneous forcing algebra, is not a geometric vector bundle, unless the ideal is the unit ideal. There is however an addition $V \times_X V \to V$ and a zero-section $X \to V$ which makes V into a group scheme over X.

Definition 1.4. Let X denote a scheme. A scheme

$$p\colon\ V \longrightarrow X$$

is called a *geometric vector bundle* of rank r over X if there exists an open covering $X = \bigcup_{i \in I} U_i$ and U_i-isomorphisms

$$\psi_i\colon\ U_i \times \mathbb{A}^r = \mathbb{A}^r_{U_i} \longrightarrow V|_{U_i} = p^{-1}(U_i)$$

such that for every open affine subset $U \subseteq U_i \cap U_j$ the transition mappings

$$\psi_j^{-1} \circ \psi_i\colon\ \mathbb{A}^r_{U_i}|_U \longrightarrow \mathbb{A}^r_{U_j}|_U$$

are linear automorphisms, i.e. they are induced by an automorphism of the polynomial ring $\Gamma(U, \mathcal{O}_X)[T_1, \ldots, T_r]$ given by $T_i \mapsto \sum_{j=1}^r a_{ij} T_j$.

Here we can restrict always to affine open coverings. If X is separated then the intersection of two affine open subschemes is again affine and then it is enough to check the condition on the intersection. The trivial bundle of rank r is the r-dimensional affine space $\mathbb{A}^r_X$ over X, and locally every vector bundle looks like this. Many properties of an affine space are enjoyed by general vector bundles. For example, in the affine space we have the natural addition

$$+\colon\ \mathbb{A}^r_U \times_U \mathbb{A}^r_U \longrightarrow \mathbb{A}^r_U,\ (v_1, \ldots, v_r, w_1, \ldots, w_r) \longmapsto (v_1 + w_1, \ldots, v_r + w_r),$$

and this carries over to a vector bundle, that is, we have an addition

$$\alpha\colon\ V \times_X V \longrightarrow V.$$

The reason for this is that the isomorphisms occurring in the definition of a geometric vector bundle are linear, hence the addition on $V|_U$ coming from an isomorphism with some affine space over U is independent of the choosen isomorphism. For the same reason there is a unique closed subscheme of V called the *zero-section* which is locally defined to be $0 \times U \subseteq \mathbb{A}^r_U$. Also, the multiplication by a scalar, i.e. the mapping

$$\cdot\colon\ \mathbb{A}_U \times_U \mathbb{A}^r_U \longrightarrow \mathbb{A}^r_U,\ (s, v_1, \ldots, v_r) \longmapsto (sv_1, \ldots, sv_r),$$

carries over to a scalar multiplication

$$\cdot\colon\ \mathbb{A}_X \times_X V \longrightarrow V.$$

In particular, for every point $P \in X$ the fiber $V_P = V \times_X P$ is an affine space over $\kappa(P)$.

For a geometric vector bundle $p\colon\ V \to X$ and an open subset $U \subseteq X$ one sets

$$\Gamma(U, V) = \{s : U \to V|_U \,|\, p \circ s = \mathrm{id}_U\}\ ,$$

so this is the set of sections in V over U. This gives in fact for every scheme over X a set-valued sheaf. Because of the observations just mentioned, these sections can also be added and multiplied by elements in the structure sheaf, and so we get for every vector bundle a locally free sheaf, which is free on the open subsets where the vector bundle is trivial.

DEFINITION 1.5. A coherent $\mathcal{O}_X$-module $\mathcal{F}$ on a scheme X is called *locally free* of rank r, if there exists an open covering $X = \bigcup_{i \in I} U_i$ and $\mathcal{O}_{U_i}$-module-isomorphisms $\mathcal{F}|_{U_i} \cong \mathcal{O}^r|_{U_i}$ for every $i \in I$.

Vector bundles and locally free sheaves are essentially the same objects.

THEOREM 1.6. *Let X denote a scheme. Then the category of locally free sheaves on X and the category of geometric vector bundles on X are equivalent. A geometric vector bundle $V \to X$ corresponds to the sheaf of its sections, and a locally free sheaf $\mathcal{F}$ corresponds to the (relative) Spectrum of the symmetric algebra of the dual module $\mathcal{F}^*$.*

The free sheaf of rank r corresponds to the affine space $\mathbb{A}^r_X$ over X.

The global section in

$$V = \operatorname{Spec}(R[S_1, \ldots, S_n]/(f_1 S_1 + \ldots + f_n S_n) \longrightarrow \operatorname{Spec}(R)$$

are just the tuples $(s_1, \ldots, s_n)$ such that $\sum_{i=1}^n s_i f_i = 0$. So these are just the syzygies for the ideal generators, and they form the syzygy module. We denote this by $\operatorname{Syz}(f_1, \ldots, f_n)$. The sheaf of sections in V is also the sheafification of this syzygy module. The restriction of this sheaf to $U = D(f_1, \ldots, f_n)$ is locally free.

As the solution vector space of a system of homogeneous linear equations acts on the solution set of a system of inhomogeneous linear equations, the spectrum of a homogeneous forcing algebra acts on the spectrum of an inhomogeneous forcing algebra. This action is given by

$$\operatorname{Spec}(A) \times_{\operatorname{Spec}(R)} \operatorname{Spec}(B) \longrightarrow \operatorname{Spec}(B),\ (s_1, \ldots, s_n, t_1, \ldots, t_n) \longmapsto (s_1 + t_1, \ldots, s_n + t_n).$$

On the ring level this map is induced by $T_i \mapsto S_i + T_i$. This action induces an action of the vector bundle $V|_U$ on $\operatorname{Spec}(B)|_U$, and endowed with this action $\operatorname{Spec}(B)|_U$ becomes a torsor.