Forcing algebras and closure operations

Let R denote a commutative ring and let $I = (f_1, \ldots, f_n)$ be an ideal. Let $f \in R$ and let

$$B = R[T_1, \ldots, T_n]/(f_1T_1 + \ldots + f_nT_n - f)$$

be the corresponding forcing algebra and

$$\varphi : \text{Spec}\,(B) \longrightarrow \text{Spec}\,(R)$$

the corresponding spectrum morphism. How are properties of φ (or or the R-algebra B) related to certain ideal closure operations? We start with some examples. The element f belongs to the ideal I if and only if we can write $f = r_1f_1 + \ldots + r_nf_n$. By the universal property of the forcing algebra this means that there exists an R-algebra-homomorphism

$$B \longrightarrow R,$$

hence $f \in I$ holds if and only if φ admits a scheme section. This is also equivalent to

$$R \longrightarrow B$$

admitting an R-module section or B being a pure algebra (so for forcing algebras properties might be equivalent which are not equivalent for arbitrary algebras).

The radical of an ideal

Now we look at the radical of the ideal I,

$$\text{rad}\,(I) = \{f \in R |\, f^k \in I \text{ for some } k\}\,.$$

The importance of the radical comes mainly from Hilbert's Nullstellensatz, saying that for algebras of finite type over an algebraically closed field there is a natural bijection between radical ideals and closed algebraic zero-sets. So geometrically one can see from an ideal only its radical. As this is quite a coarse closure operation we should expect that this corresponds to a quite coarse property of the morphism φ as well. Indeed, it is true that $f \in \text{rad}\,(I)$ if and only if φ is surjective. This is true since the radical of an ideal is the intersection of all prime ideals in which it is contained. Hence an element f belongs to the radical if and only if for all residue class homomorphisms

$$\varphi : R \longrightarrow \kappa(\mathfrak{p})$$

where I is sent to 0, also f is sent to 0. But this means for the forcing equation that whenever the equation degenerates to 0, then also the inhomogeneous part becomes zero, and so there will always be a solution to the inhomogeneous equation. Exercise: Define the radical of a submodule inside a module.

Integral closure of an ideal

Another closure operation is integral closure. It is defined by

$$\overline{I} = \{ f \in R |\, f^k + a_1 f^{k-1} + \ldots + a_{k-1} f + a_k = 0 \text{ for some } k \text{ and } a_i \in I^i \} \,.$$

This notion is important for describing the integral closure of the blow up of the ideal I. Another characterization is that there exists a $z \in R$, not contained in any minimal prime ideal of R, such that $zf^n \in I^n$ holds for all n. Another equivalent property - the valuative criterion - is that for all ring homomorphisms

$$\theta : R \longrightarrow D$$

to a discrete valuation domain D (assume that R is noetherian) the containment $\theta(f) \in \theta(I)D$ holds. The characterization of the integral closure in terms of forcing algebras requires some notions from topology. A continuous map

$$\varphi : X \longrightarrow Y$$

between topological spaces X and Y is called a *submersion*, if it is surjective and if Y carries the image topology (quotient topology) under this map. This means that a subset $W \subseteq Y$ is open if and only if its preimage $\varphi^{-1}(W)$ is open. Since the spectrum of a ring endowed with the Zarisiki topology is a topological space, this notion can be applied to the spectrum morphism of a ring homomorphism. With this notion we can state that $f \in \overline{I}$ if and only if the forcing morphism

$$\varphi : \operatorname{Spec}(B) \longrightarrow \operatorname{Spec}(R)$$

is a universal submersion (universal means here that for any ring change $R \to R'$ to a noetherian ring R', the resulting homomorphism $R' \to B'$ still has this property). The relation between these two notions stem from the fact that also for universal submersions there exist a criterion in terms of discrete valuation domains. For a morphism

$$Z \longrightarrow \operatorname{Spec}(D)$$

(D a discrete valuation domain) to be a submersion means that above the only chain of prime ideals in $\operatorname{Spec}(D)$, namely $(0) \subset \mathfrak{m}_D$, there exists a chain of prime ideals $\mathfrak{p}' \subseteq \mathfrak{q}'$ in $\operatorname{Spec}(D)$ lying over this chain. So this property is stronger that lying over (which means surjective) but weaker than the going-down or going-up property (in the presence of surjectivity). If we are dealing only with algebras of finite type over the complex numbers C, then we may also consider the corresponding complex spaces with their natural topology induced from the euklidean topology of $\mathbb{C}^n$. Then universal submersive with respect to the Zariski topology is the same as submersive in the complex topology (the target space needs to be normal).

EXAMPLE 3.1. Let K be a field and consider $R = K[X]$. Since this is a principal ideal domain, the only interesting forcing algebras (if we are only interested in the local behavior around (X)) are of the form $K[X,T]/(X^nT - X^m)$.

For $m \geq n$ this $K[X]$-algebra admits a section (corresponding to the fact that $X^m \in (X^n)$), and if $n \geq 1$ there exists an affine line over the maximal ideal (X). So now assume $m < n$. If $m = 0$, then we have a hyperbola mapping to an affine line, with the fiber over (X) being empty, corresponding to the fact that 1 does not belong to the radical of (X^n) for $n \geq 1$. So assume finally $1 \leq m < n$. Then X^m belongs to the radical of (X^n), but not to its integral closure (which is the identical closure on a one-dimensional regular ring). We can write the forcing equation as $X^nT - X^m = X^m(X^{n-m}T - 1)$. So the spectrum of the forcing algebra consists of a (thickend) line over (X) and of a hyperbola. The forcing morphism is surjective, but it is not a submersion. For example, the preimage of (X) is a connected component hence open, but this single point is not open.

EXAMPLE 3.2. Let K be a field and let $R = K[X, Y]$ be the polynomial ring in two variables. We consider the ideal $I = (X^2, Y)$ and the element X. This element belongs to the radical of this ideal, hence the forcing morphism

$$\operatorname{Spec}(K[X, Y, T_1, T_2]/(X^2T_1 + YT_2 + X) \longrightarrow \operatorname{Spec}(K[X, Y])$$

is surjective. We claim that it is not a submersion. For this we look at the reduction modulo Y. In $K[X, Y]/(Y) \cong K[X]$ the ideal becomes (X^2) which does not contain X. Hence by the valuative criterion for integral closure, X does not belong to the integral closure of the ideal. One can also say that the chain $V(X, Y) \subset V(Y)$ in the affine plane does not have a lift (as a chain) to the spectrum of the forcing algebra. For the ideal $I = (X^2, Y^2)$ and the element XY the situation looks different. Let

$$\theta : K[X, Y] \longrightarrow D$$

be a ring homomorphism to a discrete valuation domain D. If X or Y is mapped to 0, then also XY is mapped to 0 and hence belongs to the extendend ideal. So assume that $\theta(X) = u\pi^r$ and $\theta(Y) = v\pi^s$, where π is a local parameter of D and u and v are units. Then $\theta(XY) = uv\pi^{r+s}$ and the exponent is at least the minimum of $2r$ and $2s$, hence $\theta(XY) \in (\pi^{2r}, \pi^{2s}) = (\theta(X^2), \theta(Y^2))D$. Hence XY belongs to the integral closure of (X^2, Y^2) and the forcing morphism

$$\operatorname{Spec}(K[X, Y, T_1, T_2]/(X^2T_1 + Y^2T_2 + XY) \longrightarrow \operatorname{Spec}(K[X, Y])$$

is a universal submersion.

Continuous closure

Suppose now that $R = \mathbb{C}[X_1, \ldots, X_k]$. Then every polynomial $f \in R$ can be considered as a continuous function

$$f : \mathbb{C}^k \longrightarrow \mathbb{C},\ (x_1, \ldots, x_k) \longmapsto f(x_1, \ldots, x_k)$$

in the complex topology. If $I = (f_1, \ldots, f_n)$ is an ideal and $f \in R$ is an element, we say that f belongs to the *continuous closure* of I, if there exist continuous functions

$$g_1, \ldots, g_k : \mathbb{C}^k \longrightarrow \mathbb{C}$$

such that

$$f = \sum_{i=1}^{n} g_i f_i$$

(identity of functions) (the same definition works for $\mathbb{C}$-algebras of finite type). It is not at all clear at once that there may exist polynomials $f \notin I$ but inside the continuous closure of I. For $\mathbb{C}[X]$ it is easy to show that the continuous closure is (like the integral closure) just the ideal itself. We also remark that when we would only allow holomorphic functions $g_1, \ldots, g_k$ then we could not get something larger. However, with continuous functions we can for example write

$$X^2Y^2 = g_1X^3 + g_2Y^3 \, .$$

Continuous closure is always inside the integral closure and hence also inside the radical. The element XY does not belong to the continuous closure of (X^2, Y^2), though it belongs to the integral closure of I. In terms of forcing algebras, an element f belongs to the continuous closure if and only if the complex forcing mapping

$$\varphi_{\mathbb{C}} : \text{Spec}\,(B)_{\mathbb{C}} \longrightarrow \text{Spec}\,(R)_{\mathbb{C}}$$

(between the corresponding complex spaces) admits a continuous section. The closure operations we have considered so far can be characterized by some property of the forcing algebra. However, they can not be characterized by a property of the corresponding torsor alone. For example, for $R = K[X, Y]$, we may write

$$\frac{1}{XY} = \frac{X}{X^2Y} = \frac{XY}{X^2Y^2} = \frac{X^2Y^2}{X^3Y^3},$$

so the torsors given by the forcing algebras

$$R[T_1, T_2]/(XT_1 + YT_2 + 1),\ R[T_1, T_2]/(X^2T_1 + YT_2 + X),$$

$$R[T_1, T_2]/(X^2T_1 + Y^2T_2 + XY) \text{ and } R[T_1, T_2]/(X^3T_1 + Y^3T_2 + X^2Y^2)$$

are all the same (the restriction over $D(X, Y)$), but there global properties are quite different. We have a non-surjection, a surjective non submersion, a submersion which does not admit (for $K = \mathbb{C}$) a continuous section and a map which admits a continuous section. In the next lecture we will look at closure operations which can be characterized by a property of the torsor alone, so they only depend on the cohomology class of the syzygzy sheaf. These closure operations are plus closure, tight closure and solid closure. In particular we will be interested in a two-dimensional base ring (typically normal with an isolated singularity) and the question when a torsor over the punctured spectrum is an affine scheme. This is directly related to the question whether an element belongs to the tight closure (solid closure over a field of characteristic zero).